A SYMPLECTIC VIEW OF STABILITY FOR TRAVELING WAVES IN ACTIVATOR-INHIBITOR SYSTEMS

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ABSTRACT

Paul Cornwell: A symplectic view of stability for traveling waves in activator-inhibitor systems
(Under the direction of Christopher K.R.T. Jones)

This thesis concerns the stability of traveling pulses for reaction-diffusion equations of skew-gradient (a.k.a activator-inhibitor) type. The centerpiece of this investigation is a homotopy invariant called the Maslov index which is assigned to curves of Lagrangian planes. The Maslov index has been used in recent years to count positive eigenvalues for self-adjoint Schrödinger operators. Such operators arise, for instance, from linearizing a gradient reaction-diffusion equation about a steady state. In that case, positive eigenvalues correspond to unstable modes.

In this work, we focus on two aspects of the Maslov index as a tool in the stability analysis of nonlinear waves. First, we show why and how the Maslov index is useful for traveling pulses in skew-gradient systems, for which the associated linear operator is *not* self-adjoint. This leads naturally to a discussion of the famous Evans function, the classic eigenvalue-hunting tool for steady states of semilinear parabolic equations. A major component of this work is unifying the Evans function theory with that of the Maslov index.

Second, we address the issue of calculating the Maslov index, which is intimately tied to its utility. The key insight is that the relevant curve of Lagrangian planes is everywhere tangent to an invariant manifold for the traveling wave ODE. The Maslov index is then encoded in the twisting of this manifold as the wave moves through phase space. We carry out the calculation for fast traveling pulses in a doubly-diffusive FitzHugh-Nagumo system. The calculation is made possible by the timescale separation of this system, which allows us to track the invariant manifold of interest using techniques from geometric singular perturbation theory. Combining the calculation with the stability framework established in the first part, we conclude that the pulses are stable.

To Brett

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CHAPTER 1

Introduction

When modeling a physical system, a critical concern is whether a distinguished state is stable to perturbations in initial conditions. Indeed, this determines whether the state–which is usually described as the solution to a differential equation–is physically realizable or not. This dissertation concerns the use of dynamical systems techniques to determine the stability of traveling waves for reaction-diffusion equations. Traveling waves are solutions of evolutionary PDE which move at a constant speed c while maintaining their shape. They are ubiquitous in applications, as they often model the transmission of information in one direction. Examples abound in mathematical biology [65], chemical reactions (e.g. flame fronts in combustion) [80], nonlinear optics [1], conservation laws in gas dynamics [35, 74], fluid dynamics [43], and more.

As we will see below, the stability analysis of a traveling wave leads to an eigenvalue problem for an unbounded operator acting on a function space. We will show how the Maslov index of symplectic geometry can be used to detect unstable eigenvalues for this operator. The Maslov index of a traveling wave is determined by the twisting of an invariant manifold (containing the wave) of the traveling wave ODE. It therefore gives an *intrinsic* criterion for the (in)stability of the wave in question. Such a result is the paragon of stability analysis, as it gives *structural* reasons for why some states are stable (i.e. observable) and others are not.

The Maslov index has been used extensively in recent years to study the spectra of self-adjoint Schrödinger operators, including those obtained by linearizing reaction-diffusion equations about distinguished steady states [4, 7, 16, 18, 26, 30, 41, 42, 47, 48, 49]. This project was born out of the need to address two major issues plaguing the application of the Maslov index in the stability analysis of nonlinear waves. First, the existing theory deals primarily with self-adjoint operators. In the context of reaction-diffusion equations, this necessitates that the nonlinearity be a gradient. This is an undesirable condition, since gradient reaction-diffusion equations do not exhibit much of the interesting behavior (such as pattern formation) that makes these equations important to the scientific community. Second, actual calculations of the Maslov index remain elusive. Obviously, the ability to calculate the index is intimately tied to its utility. In this thesis, we will develop a geometric technique for calculating the Maslov index in singularly perturbed reaction-diffusion equations.

1.1 Reaction-diffusion equations: a dynamics approach to stability

We study reaction-diffusion equations of the form

$$u_t = Du_{xx} + f(u), \quad u \in \mathbb{R}^n, x \in \mathbb{R}, \tag{1.1}$$

where D is a positive, diagonal matrix of *diffusion coefficients*. A *traveling wave* $\hat{u}(z)$ is a solution to (1.1) of one variable $z = x - ct$ which decays exponentially to end states $u_\pm$ as $z \to \pm\infty$. If $u_+ = u_-$, then $\hat{u}$ is called a *pulse*. Otherwise, $\hat{u}$ is called a *front*. Note that a necessary condition for the existence of a traveling front or pulse is $f(u_\pm) = 0$. In this thesis, we will focus on pulses. Without loss of generality, we take $u_+ = u_- = 0$. It is convenient to study traveling waves by rewriting (1.1) in a moving frame, using the chain rule:

$$u_t = Du_{zz} + cu_z + f(u). \tag{1.2}$$

The traveling wave $\hat{u}$ can now be viewed as a "fixed point" of (1.2), and the right-hand side of (1.2) as a "vector field" on some function space. It is known [39] that the operator $Du_{zz} + cu_z$ generates an analytic semigroup on the Banach space (with sup norm)

$$BU(\mathbb{R}, \mathbb{R}^n) = \{u : \mathbb{R} \to \mathbb{R}^n : u \text{ is bounded and uniformly continuous}\}, \tag{1.3}$$

and hence the Cauchy problem (1.2) with initial condition $u(z, 0) = u_0(z)$ can be solved for small t via the variation of constants formula. We are thus justified in taking the view that (1.2) defines a flow on $BU(\mathbb{R}, \mathbb{R}^n)$.

Both the definition of stability and the strategy for assessing it push the analogy with fixed points of vector fields further, so it is worth devoting some time to understanding the simpler case. To that end, suppose that

$$x'(t) = F(x) \tag{1.4}$$

defines a vector field on some open set $U \subseteq \mathbb{R}^n$, and let $F(p) = 0$ for some point p. We say that p is *stable* if trajectories of (1.4) through points sufficiently close to p stay close to p for all time. If, in addition, there is a neighborhood V of p such that the trajectory through any point in V approaches p as $t \to \infty$, then p is called *asymptotically stable* (or *attracting*). The stability properties of p are obtained by *linearizing* (1.4) about p to obtain a constant coefficient, linear system

$$y'(t) = DF(p)y(t). \tag{1.5}$$

$DF(p)$ is an $n \times n$ matrix, so it has n complex eigenvalues, counting multiplicity. Now partition the spectrum $\sigma(DF(p))$ of $DF(p)$ into three sets σ_-, σ_0, and σ_+, consisting of the eigenvalues of $DF(p)$ of negative, zero, and positive real part respectively. To each of these sets one can associate a subspace of all generalized eigenvectors for the constituent eigenvalues. We call these the stable, center, and unstable subspaces respectively and denote them $V^s(p), V^c(p)$, and $V^u(p)$. The solution operator for (1.5) is given by $\exp(DF(p)t)$, so it is clear that each of $V^{s/c/u}(p)$ is invariant under the flow of (1.5).

It can be shown (e.g. [20]) using Gronwall's inequality and Taylor's theorem that p is asymptotically stable if $\sigma(DF(p)) = \sigma_-$. More generally, each subspace defined above is tangent at p to a local invariant manifold for (1.4). The local stable and unstable manifolds, $W^s_{\text{loc}}(p)$ and $W^u_{\text{loc}}(p)$, are defined as follows in a neighborhood $N \subset U$ of p:

$$\begin{aligned} W^s_{\text{loc}}(p) &= \{y \in N : y \cdot t \in N \text{ for all } t \geq 0, \text{ and } \lim_{t\to\infty} y \cdot t = p\} \\ W^u_{\text{loc}}(p) &= \{y \in N : y \cdot t \in N \text{ for all } t \leq 0, \text{ and } \lim_{t\to-\infty} y \cdot t = p\} \end{aligned}. \tag{1.6}$$

Here, $\cdot$ refers to the action of the flow of (1.4). It is standard (e.g. [20, 36]) that $W^s_{\text{loc}}(p)$ ($W^u_{\text{loc}}(p)$) is positively (negatively) invariant, invariant relative to the neighborhood N, and unique. Furthermore, these manifolds are as smooth as the function F. If $\sigma_0 \neq \{\}$, then there is also a (non-unique) local center manifold which is tangent to $V^c(p)$ at p. The dynamics on the center manifold can be more complicated than on its stable and unstable counterparts. All of the invariant manifolds just described can be iterated in forward and backwards time to generate global versions, $W^s(p)$, $W^u(p)$, and $W^c(p)$. The global versions are not necessarily embedded manifolds [36, §1.4], but this detail is not important for our purposes. We now define stability of the traveling wave $\hat{u}$.

Definition 1.1. *The traveling wave $\hat{u}(z)$ is* ***asymptotically stable*** *relative to (1.2) if there is a neighborhood $V \subset BU(\mathbb{R}, \mathbb{R}^n)$ of $\hat{u}(z)$ such that if $u(z,t)$ solves (1.2) with $u(z,0) \in V$, then*

$$||\hat{u}(z+k) - u(z,t)||_\infty \to 0$$

as $t \to \infty$ for some $k \in \mathbb{R}$.

The inclusion of the phase shift k is necessary due to the *translation invariance* of (1.2); any translate of $\hat{u}$ is also a traveling pulse solution. As with fixed points, the strategy for proving that $\hat{u}$ is stable is to linearize (1.2) about this solution. This time, linearizing leads not to a matrix, but to an operator

$$L := D\partial_z^2 + c\partial_z + f'(\hat{u}) \tag{1.7}$$

acting on $BU(\mathbb{R}, \mathbb{R}^n)$. As expected, the goal is to determine the spectrum $\sigma(L)$ of L. This set consists of two parts. First, there is the set $\sigma_n(L)$ of isolated eigenvalues of L of finite multiplicity. The rest of $\sigma(L)$ is called the *essential spectrum*, $\sigma_{\text{ess}}(L) = \sigma(L) \setminus \sigma_n(L)$. By differentiating the traveling wave equation

$$0 = D\hat{u}_{zz} + c\hat{u}_z + f(\hat{u}) \tag{1.8}$$

with respect to z, one sees immediately that $L\hat{u}'(z) = 0$. Since $\hat{u}$ and its derivatives decay exponentially as $z \to \pm\infty$, we see that $0 \in \sigma_n(L)$ with eigenfunction $\hat{u}'(z)$. Thus the situation is already more complicated than for fixed points due to the presence of a center direction (corresponding to the translation invariance). Moreover, $\sigma(L)$ is an infinite set, so in principle it could accumulate near the imaginary axis. The existence of a *spectral gap* for matrices is implicitly used when proving the invariant manifold theorems referenced above, so it is important to check that such a gap exists for $\sigma(L)$ as well. Taking these concerns into consideration, we have the following theorem.

Theorem 1.1.1. *Suppose that the operator L in (1.7) satisfies*

1. *There exists $\beta < 0$ such that $\sigma(L) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda < \beta\}$.*

2. *0 is a simple eigenvalue.*

Then $\hat{u}$ is stable in the sense of Definition 1.1.

This theorem can be proved by constructing appropriate invariant manifolds in $BU(\mathbb{R}, \mathbb{R}^n)$. More precisely, assuming the above hypotheses on $\sigma(L)$, the wave $\hat{u}(z)$ and its translates form a center manifold, which is itself an attractor. For the details, the reader is referred to [6, 39].

We close this section by pointing out that the existence of a traveling wave can often be proved using dynamical systems techniques. By introducing the variable $v = u_z$, the traveling wave ODE (1.8) becomes a first order system

$$\begin{pmatrix} u \\ v \end{pmatrix}_z = \begin{pmatrix} v \\ D^{-1}(-cv - f(u)) \end{pmatrix}. \tag{1.9}$$

The end state $0 = (0, 0) \in \mathbb{R}^{2n}$ is now a fixed point for this system (or $(u_\pm, 0)$ more generally). The traveling wave corresponds to a *homoclinic orbit* $\varphi(z) = (\hat{u}(z), \hat{u}_z(z))$ for (1.9), which means that it approaches $(0, 0)$ exponentially quickly as $z \to \pm\infty$. One important fact about $\varphi(z)$ is that it lies in the intersection $W^s(0) \cap W^u(0)$ of stable and unstable manifolds for the rest state. The definition we will use for the Maslov index of the traveling wave is actually applicable to homoclinic orbits in general. The calculation of the index will rely heavily on the fact that the wave is contained in the unstable manifold $W^u(0)$.

1.2 Short primer on symplectic linear algebra

The Maslov index has its roots in symplectic geometry, so we now take some time to discuss the basics of that field. For more background, the reader is referred to [27, 38, 62]. Let $\mathbb{V}$ be a $2n$-dimensional real vector space with basis $\{e_i\}_{i=1}^{2n}$. Let $\langle \cdot, \cdot \rangle$ denote an inner product on $\mathbb{V}$. A linear map $J : \mathbb{V} \to \mathbb{V}$ is called a *complex structure* if it satisfies

$$\begin{aligned} J^2 &= -I_{2n} \\ J^T &= -J \end{aligned}, \tag{1.10}$$

where I_{2n} is the identity on $\mathbb{V}$. Note that the first condition implies that J is invertible. We remark as well that the second condition in (1.10) is not typically included in the definition of a complex structure, but it is necessary here since we wish to work with a fixed inner product. We next introduce the bilinear form

$$\omega(a,b) := \langle a, Jb\rangle. \tag{1.11}$$

It follows immediately from (1.10) and (1.11) that ω is *skew-symmetric*, *nondegenerate*, and *bilinear*. It is therefore a *symplectic form*. Now let $W \subset \mathbb{V}$ be a fixed subspace. Define the symplectic complement of W to be

$$W^{\omega} = \{u \in \mathbb{V} : \omega(u,w) = 0 \text{ for all } w \in W\}. \tag{1.12}$$

We say that the subspace W is:

1. *symplectic* if $W \cap W^{\omega} = \{0\}$.

2. *isotropic* if $W \subseteq W^{\omega}$.

3. *coisotropic* if $W^{\omega} \subseteq W$.

4. *Lagrangian* if $W^{\omega} = W$.

Observe that a Lagrangian subspace W must be n-dimensional, since ω is nondegenerate (and thus $\dim W = \dim W^{\omega}$). It follows that the set $\Lambda(n)$ of Lagrangian planes is contained in $\mathrm{Gr}_n(\mathbb{V})$, the Grassmannian of n-dimensional subspaces of $\mathbb{V}$. In [4, §1], it is explained how $\Lambda(n)$ can be realized as a homogeneous space $\Lambda(n) = U(n)/O(n)$, where $U(n), O(n) \subset \mathrm{GL}(n,\mathbb{C})$ are the unitary and orthogonal groups respectively. This proves that $\Lambda(n)$ is actually a closed submanifold of $\mathrm{Gr}_n(\mathbb{V})$ of dimension $n^2 - n(n-1)/2 = n(n+1)/2$. Moreover, considering a long exact sequence of homotopy groups for the above fiber bundle proves that $\pi_1(\Lambda(n)) = \mathbb{Z}$ for all $n \in \mathbb{N}$ [4, §1.3].

The fundamental group of $\Lambda(n)$ is the critical piece of topological information for our purposes. It suggests that one can define an integer index for curves in this space. This index is the Maslov index, which will be discussed in Chapter 3.

1.3 Intuition from the scalar case: Sturm-Liouville theory

The intuition for the mathematical techniques used in this thesis comes from Sturm-Liouville theory, which applies to the eigenvalue problem for steady states of (1.1) with $n = 1$. For simplicity, consider the stability of a time-independent solution $\hat{u}(x)$ to the scalar equation

$$u_t = u_{xx} + f(u) \tag{1.13}$$

on the interval $[0, 1]$, subject to Dirichlet boundary conditions

$$\hat{u}(0) = \hat{u}(1) = 0. \tag{1.14}$$

The stability of $\hat{u}$ is again determined by the spectrum of the operator L defined in (1.7), which this time acts on $H^2([0,1]) \cap H^1_0([0,1])$. The following result can be found in a number of sources, for example [53, 76].

Theorem 1.3.1. *The eigenvalues of L are real and simple and can be enumerated in a strictly descending order*

$$\lambda_0 > \lambda_1 > \lambda_2 > \ldots, \qquad \lim_{n\to\infty} \lambda_n = -\infty. \tag{1.15}$$

Furthermore, any eigenvector p_j corresponding to λ_j has exactly j zeros in the open interval $(0, 1)$.

Dynamical systems offers a nice, geometric proof of this theorem. The first step is to convert the eigenvalue equation $Lp = \lambda p$ to a first-order system, which is done by introducing the variable $q = p_x$:

$$\begin{pmatrix} p \\ q \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \lambda - f'(\hat{u}) & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{1.16}$$

By uniqueness of solutions to ODE, no nontrivial solution to (1.16) can pass through $(0, 0)$. Consequently, the solution operator of (1.16) induces an action on $\mathbb{R}P^1 = S^1$. For any $\lambda \in \mathbb{R}$, one can therefore check for an eigenvalue by flowing the subspace corresponding to the Dirichlet data at $x = 0$ (i.e. $\theta = \pi/2$) along the circle to check if it ends at $(2k + 1)\pi/2$ at $x = 1$.

The key to proving the above theorem is that this angle function is *monotone* in λ. In other words, as λ decreases, the solutions $(p, p_x)(\lambda, 0)$ to (1.16) satisfying $(p, p_x)(\lambda, 0) \in \{0\} \times \mathbb{R}$ wind more and more around the origin as x evolves from 0 to 1. Such an oscillation theorem is at the heart of the Maslov index analysis, except that the winding is occurring in a larger manifold. In fact, the Maslov index can be used to prove the above theorem in the scalar case. The issue of monotonicity in λ is more subtle for systems, and we will see that it is something that distinguishes the analysis of this thesis from previous studies involving the Maslov index.

The reader will recall that $\lambda = 0$ is an eigenvalue for the operator obtained by linearizing about a pulse solution, and the derivative of the solution is a corresponding eigenfunction. This will not be the case for the Dirichlet problem above, since $(\hat{u}_x, \hat{u}_{xx})$ does not satisfy the same boundary conditions. However, the Sturm Oscillation and Comparison theorems [76] imply that the number of zeros of $\hat{u}_x$ is within one of the number of zeros of any solution to (1.16) satisfying $p(0) = 0$. Furthermore, the zeros of $\hat{u}$ and $\hat{u}_x$ are intertwined, which follows from the Mean Value Theorem and a phase plane analysis for the steady state ODE

$$0 = u_{xx} + f(u). \tag{1.17}$$

We therefore obtain the following corollary to Theorem 1.3.1.

Corollary 1.3.1. *If $\hat{u}(x)$ has n zeros in the interval $[0,1]$, then there are $n-1$ or $n-2$ positive eigenvalues of L. In particular, if $\hat{u}(x)$ has an interior zero, then it must be unstable.*

Thus we can obtain spectral information (the number of unstable eigenvalues) from a *qualitative* property of the solution (the number of zeros). The beauty of the Maslov index analysis is that we will be able to generalize this result to systems of equations. Instead of counting zeros, the number of unstable eigenvalues will be encoded in the number of twists made by an invariant manifold of the traveling wave ODE.

1.4 Activator-inhibitor systems and Turing patterns

The previous section contained the mathematical motivation for this thesis. In this section, we describe the physical motivation which spurs us to consider skew-gradient (or activator-inhibitor) systems. In his lone published work in biology [78], A.M. Turing described a simple mechanism for pattern formation in biological/chemical systems. Counter-intuitively, the interplay of diffusion and a stable reaction can destabilize a uniform state, which spurs the growth of periodic patterns of certain wavelengths.

In the context of a chemical reaction modeled by (1.1) (assuming $n = 2$ for simplicity), the uniform state corresponds to a stable fixed point u_* for the kinetics equation

$$u_t = f(u). \tag{1.18}$$

Using a simple linear stability analysis, Turing showed how the presence of diffusion (which is typically thought to having a stabilizing effect) can cause this state to become unstable and lead to pattern formation. However, such patterns require that the elements of D–called the diffusivities–are vastly different from each other. Although reaction-diffusion models for chemical interactions are oversimplifying, Turing's discovery has been very influential in mathematical biology [65].

A particularly interesting theme of the research into Turing patterns has been the difficulty of observing or producing them in experiment. This was not done successfully until several decades after Turing's original paper was published [14, 58]. One possible explanation for this is the difficulty of finding two reagents whose diffusivities are sufficiently different as to support diffusion-driven instability. This seems to provide evidence in favor of the conjecture that any spatially periodic patterns must be unstable (hence unobservable) if the diffusivities of the chemicals involved are the same or similar.

So what does this have to do with the Maslov index? The Maslov index has been used to study the spectra of *self-adjoint* operators. In order for L in (1.7) to be self-adjoint, the nonlinearity f must be a gradient. This condition, however, is known to be incompatible with the conditions for a Turing instability. In the case $n = 2$, this is easy to see, since Turing's condition requires that the off-diagonal terms in $f'(u_*)$ have opposite signs [65]. Systems that are susceptible to Turing bifurcations are called *activator-inhibitor*, meaning that the presence of one species stimulates the growth of both, whereas the presence of the other impedes the growth of both species.

In this thesis, we study a broader class of equations called *skew-gradient* [83, 84], which will encompass many activator-inhibitor systems. More precisely, we study systems of the form

$$u_t = u_{xx} + QSf(u), \qquad u \in \mathbb{R}^n, \tag{1.19}$$

where $f(u) = \nabla F(u)$ is a gradient, S is positive and diagonal, and Q is diagonal with entries of ± 1. Some aspects of the analysis will be restricted to the critical case $n = 2$ studied by Turing. However, it is clear that most of the results contained herein are valid in any dimension. As stated above, we study traveling pulses (as opposed to spatially periodic patterns) in this work. We suggest some further directions concerning periodic solutions at the end of the dissertation.

1.5 Overview of dissertation

The rest of this dissertation is organized as follows. In Chapter 2, we set up the eigenvalue problem and identify the symplectic structure that underlies the rest of the analysis. In Chapter 3, we discuss the various formulations of the Maslov index that we will use and define the Maslov index of the traveling wave. In Chapter 4, we discuss the symplectic Evans function $D(\lambda)$ and prove that the parity of the Maslov index determines (in part) the sign of $D'(0)$. In Chapter 5, we prove the existence of fast traveling pulses and fronts for a doubly diffusive FitzHugh-Nagumo system using geometric singular perturbation theory. In Chapter 6, we show how the Maslov index can be used to prove that the aforementioned waves are stable. Additionally, we prove a more general result that the Maslov index gives a lower bound on the Morse index for traveling waves of skew-gradient systems. In Chapter 7, we describe a method for calculating the Maslov index in singularly perturbed systems. We apply this method to the FitzHugh-Nagumo system and prove that the traveling waves of Chapter 5 are nonlinearly stable. Finally, in Chapter 8 we summarize our results and discuss possible future directions.

CHAPTER 2

Framework for stability

In this chapter, we set up the eigenvalue problem for the most general equations studied in this thesis, namely skew-gradient systems (1.19). We point out that all results obtained in this chapter apply both to the FitzHugh-Nagumo and to the two-component activator-inhibitor systems studied later. Although this thesis focuses on the stability of pulses, the Maslov index can be used to study eigenvalue problems with different boundary conditions as well.

Assume that (1.19) possesses a traveling wave solution $\hat{u}(z)$, which decays exponentially to 0 as $z \to \pm\infty$. As mentioned in the previous chapter, this implies that $QSf(0) = f(0) = 0$. In light of the discussion of Turing patterns, we make the additional assumption that

$$\text{there exists } \beta < 0 \text{ such that the } n \text{ eigenvalues } \nu_i \text{ of } QSf'(0) \text{ satisfy } \operatorname{Re} \nu_i < \beta. \tag{2.1}$$

In other words, we assume that 0 is a stable equilibrium of the associated kinetics equation. Without loss of generality, we take

$$c < 0, \tag{2.2}$$

which means that the wave moves to the left. We stress that the goal is to understand the spectrum of the operator

$$L := \partial_z^2 + c\partial_z + QSf'(\hat{u}(z)), \tag{2.3}$$

acting on $BU(\mathbb{R}, \mathbb{R}^n)$.

For skew-gradient systems, we will convert the traveling wave ODE (1.9) to a first-order system by setting $u_z = Sv$, which gives

$$\begin{pmatrix} u \\ v \end{pmatrix}_z = \begin{pmatrix} Sv \\ -cv - Qf(u) \end{pmatrix}. \tag{2.4}$$

2.1 Eigenvalue equation and essential spectrum

In this section, we state the definitions pertinent to the spectrum of the operator L. First, we say that $\lambda \in \mathbb{C}$ is an eigenvalue for L if there exists a solution $p \in BU(\mathbb{R}, \mathbb{C}^n)$ to the equation

$$Lp = \lambda p. \tag{2.5}$$

The set of isolated eigenvalues of L of finite multiplicity is denoted $\sigma_n(L)$. Note the importance of the boundary conditions for determining eigenvalues. For any value of λ, the eigenvalue equation (2.5) can always be solved for a fixed initial condition. However, a generic solution will blow up at both ∞ and $-\infty$, so not every $\lambda \in \mathbb{C}$ is an eigenvalue. This motivates the strategy of tracking boundary data for potential eigenfunctions, which is really the crux of the approach that we will take to hunt for eigenvalues. Now, comparing with (2.3), setting $p_z = Sq$ converts (2.5) to the first order system

$$\begin{pmatrix} p \\ q \end{pmatrix}' = \begin{pmatrix} 0 & S \\ \lambda S^{-1} - Qf'(\hat{u}) & -cI \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.6}$$

It is common to abbreviate (2.6) as

$$Y'(z) = A(\lambda, z)Y(z), \tag{2.7}$$

with $Y \in \mathbb{C}^{2n}$ and $A(\lambda, z) \in M_n(\mathbb{C}^{2n})$. Assumption (2.1) guarantees that $\hat{u}$ approaches 0 exponentially, and thus there is a well-defined matrix

$$A_\infty(\lambda) = \lim_{z \to \pm\infty} A(\lambda, z), \tag{2.8}$$

and this limit is also achieved exponentially quickly. The eigenvalues of L comprise only part of the spectrum; the rest is the essential spectrum $\sigma_{\text{ess}}(L)$. For operators of the form (2.3), it is known (Lemma 3.1.10 of [53]) that the essential spectrum is given by

$$\sigma_{\text{ess}}(L) = \{\lambda \in \mathbb{C} : A_\infty(\lambda) \text{ has an eigenvalue } \mu \in i\mathbb{R}\}. \tag{2.9}$$

This characterization is unsurprising, since (2.6) is a relatively compact perturbation of the autonomous equation

$$Y'(z) = A_\infty(\lambda)Y(z). \tag{2.10}$$

The dynamics of the latter equation are easy to describe provided that $A_\infty(\lambda)$ has no eigenvalues of zero real part; they are characterized by exponential growth or decay of solutions at both $\pm\infty$. The characterization (2.9) then follows from Weyl's essential spectrum theorem, cf. [53, Theorem 2.2.6].

We claim that $\sigma_{\text{ess}}(L)$ is contained in the half-plane

$$H = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \beta\}. \tag{2.11}$$

Indeed, a simple calculation using (2.6) shows that the eigenvalues of $A_\infty(\lambda)$ are given by

$$\mu_j(\lambda) = \frac{1}{2}\left(-c \pm \sqrt{c^2 + 4(\lambda - \nu_i)}\right), \tag{2.12}$$

with ν_i from (2.1). We need to show that $A_\infty(\lambda)$ has no purely imaginary eigenvalues if $\operatorname{Re}\lambda \geq \beta$, which is clearly equivalent to showing that $\operatorname{Re}\sqrt{c^2 + 4(\lambda - \nu_i)} \neq -c$ for such λ. The formula

$$\operatorname{Re}\sqrt{a + bi} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2 + b^2} + a} \tag{2.13}$$

and the fact that $\operatorname{Re}(c^2 + 4(\lambda - \nu_i)) > 0$ from (2.1) together imply that

$$\operatorname{Re}\sqrt{c^2 + 4(\lambda - \nu_i)} \geq \sqrt{\operatorname{Re}(c^2 + 4(\lambda - \nu_i))} > \sqrt{c^2} = -c, \tag{2.14}$$

as desired. This calculation actually proves that $A_\infty(\lambda)$ has exactly n eigenvalues of positive real part and n eigenvalues of negative real part for $\lambda \in (\mathbb{C} \setminus H)$. We label these $\mu_i(\lambda)$ in order of increasing real part and observe that

$$\operatorname{Re}\mu_1(\lambda) \leq \cdots \leq \operatorname{Re}\mu_n(\lambda) < 0 < -c < \operatorname{Re}\mu_{n+1}(\lambda) \leq \cdots \leq \operatorname{Re}\mu_{2n}(\lambda). \tag{2.15}$$

Furthermore, one sees from (2.12) that for each $1 \leq i \leq n$ we have

$$\mu_i(\lambda) + \mu_{i+n}(\lambda) = -c. \tag{2.16}$$

Having shown that the essential spectrum is bounded away from $i\mathbb{R}$ in the left half-plane, Theorem 1.1.1 asserts that we can focus on unstable eigenvalues of L.

2.2 Stable and unstable bundles

Away from $\sigma_{\text{ess}}(L)$, the dynamics of (2.6) are dominated by (2.10) and are thus easy to understand. In the jargon, for $\lambda \in \mathbb{C} \setminus H$, we say that (2.6) admits *exponential dichotomies* [67, 68] on $\mathbb{R}^+$ and $\mathbb{R}^-$ with the same Morse indices. These are projections which allow us to extract solutions to (2.6) which decay in backward and/or forward time. See [71, §3.2-3.3] for more details. We thus define the *stable* and *unstable bundles*

$$\begin{aligned} E^s(\lambda, z) &= \{\xi(z) \in \mathbb{C}^{2n} : \xi \text{ solves (2.6) and } \xi \to 0 \text{ as } z \to \infty\} \\ E^u(\lambda, z) &= \{\xi(z) \in \mathbb{C}^{2n} : \xi \text{ solves (2.6) and } \xi \to 0 \text{ as } z \to -\infty\} \end{aligned}. \tag{2.17}$$

Observe first that these are linear spaces, as they consist of solutions to (2.6). Also, the dimensions of these spaces match the dimensions of the corresponding stable and unstable subspaces for $A_\infty(\lambda)$. In light of the preceding discussion, it then follows from (2.1) that $E^{s/u}(\lambda, z)$ are each n-dimensional for $\lambda \notin H$. The theory of exponential dichotomies implies that the decay of the solutions in $E^{s/u}(\lambda, z)$ is exponential. Consequently, the growth of any unbounded solution is exponential as well.

The smoothness of solutions to (2.6) in z follows from basic ODE theory. It is also true, albeit less obvious, that these spaces vary analytically in λ. The difficult part of that proof rests on work of Kato [54, II.4.2], who showed that the stable and unstable subspaces for $A_\infty(\lambda)$ (as well as bases thereof) can be chosen in an analytic fashion. Since these spaces will play an important role later, we assign them the notation $S(\lambda)$ and $U(\lambda)$. The analyticity is key for utilizing the Evans function in the next section.

The stable and unstable bundles can also be defined in a geometric way, à la [2, §3]. Since $A(\lambda, z)$ has an exponentially quickly-attained limit for all $\lambda \in \mathbb{C}$, equation (2.6) can be "compactified" by the introduction of the variable

$$\xi = \frac{e^{2\kappa z} - 1}{e^{2\kappa z} + 1}. \tag{2.18}$$

The (unimportant) constant κ depends on the spectral gap for the eigenvalues of $QSf'(0)$. This change of variables allows us to treat (2.6) as an autonomous equation on $\mathbb{C}^{2n+1}$, where ξ increases between -1 and 1. The planes $\xi = \pm 1$ are invariant, as these correspond to the asymptotic states $z \to \pm\infty$. One can then show that $E^u(\lambda, z)$ is the unstable manifold of the point $(Y, \xi) = (0, -1)$, and $E^s(\lambda, z)$ is the stable manifold of the point $(Y, \xi) = (0, +1)$. For the details on the compactification, we refer the reader to [2]. One byproduct of this construction is that any solution in $E^s(\lambda, z)$ (resp. $E^u(\lambda, z)$) must be asymptotically tangent to $S(\lambda)$ (resp. $U(\lambda)$) as $z \to \infty$ (resp. $z \to -\infty$).

Using the stable and unstable bundles, we have the following characterization of an eigenvalue:

$$\lambda \in \sigma(L) \cap (\mathbb{C} \setminus H) \iff E^s(\lambda, z) \cap E^u(\lambda, z) \neq \{0\} \text{ for some (and hence all) } z \in \mathbb{R}. \tag{2.19}$$

Intersections between the bundles can be detected using the *Evans function*, which will be defined in the next section. In order to define the Evans function, we will first need to develop a means for tracking the evolution of the stable and unstable bundles. This will be accomplished through the use of Plücker coordinates.

We close this section with a comment on terminology. The word "bundle" is used above because $E^{s/u}(\lambda, z)$ are used in [2] to define complex vector bundles over a topological sphere defined by the parameters λ and z. Although we adopt the terminology, our perspective will typically be to consider $E^{s/u}(\lambda, z)$ instead as two-parameter curves in the Grassmannian $\mathrm{Gr}_n(\mathbb{V})$, with $\mathbb{V} = \mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$. It will be clear from context whether $E^{s/u}(\lambda, z)$ refers to a point in a Grassmannian or a vector subspace.

2.3 Plücker coordinates and the induced equation on exterior powers

It follows from basic ODE theory that (2.6) induces a flow on Grassmannians of all orders, but it is not obvious how to write down this flow. To bridge the gap from linear differential equations to Grassmannians, a useful tool is the Plücker embedding, which allows one to realize points in Grassmannians as elements of the exterior algebra of $\mathbb{C}^{2n}$ (or $\mathbb{R}^{2n}$). It is then easy to write down the equation induced on exterior powers of $\mathbb{C}^n$ by (2.6).

We will begin with the latter task. System (2.6) induces an equation

$$Z' = A^{(n)}(\lambda, z) Z, \tag{2.20}$$

on $\bigwedge^n \mathbb{C}^{2n}$. Explicitly, $A^{(n)}(\lambda, z)$ is the unique endomorphism of $\bigwedge^n \mathbb{C}^{2n}$ satisfying

$$A^{(n)}(\lambda, z)(Y_1 \wedge \cdots \wedge Y_n) = A(\lambda, z)Y_1 \wedge \cdots \wedge Y_n + \cdots + Y_1 \wedge \cdots \wedge A(\lambda, z)Y_n \tag{2.21}$$

for any $Y_i \in \mathbb{C}^{2n}$. It follows that if Y_i are solutions to (2.6), then $Y_1 \wedge \cdots \wedge Y_n$ is a solution to (2.20). The eigenvalues of the matrix $A_\infty^{(n)}(\lambda) = \lim_{z\to\pm\infty} A^{(n)}(\lambda, z)$ are the sums of n-tuples of eigenvalues of $A_\infty(\lambda)$, so for any $\lambda \notin H$ we have a simple eigenvalue of largest (positive) real part and a simple eigenvalue of least (negative) real part. We next choose corresponding eigenvectors $\zeta_u(\lambda) \in \bigwedge^n \mathbb{C}^{2n}$ and $\zeta_s(\lambda) \in \bigwedge^n \mathbb{C}^{2n}$. Since these vectors span one-dimensional subspaces of $\bigwedge^n \mathbb{C}^{2n}$, we can therefore find solutions $\tilde{E}^s(\lambda, z)$ and $\tilde{E}^u(\lambda, z)$ to (2.20) satisfying

$$\begin{aligned} \lim_{z\to\infty} \tilde{E}^s(\lambda, z) &= 0 \\ \lim_{z\to-\infty} \tilde{E}^u(\lambda, z) &= 0 \end{aligned}. \tag{2.22}$$

Moreover, these multi-vectors are unique up to a scalar multiple, and they approach $\zeta_s(\lambda)$ and $\zeta_u(\lambda)$ respectively in the appropriate limits.

We claim that the multi-vectors $\tilde{E}^{s/u}(\lambda, z)$ encode the corresponding subspaces $E^{s/u}(\lambda, z)$. The way to see this is via the Plücker embedding [37, 79]. Let $\mathbb{V} = \mathbb{C}^{2n}$ or $\mathbb{R}^{2n}$ with basis $\{e_i\}$. This choice induces a basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ of $\bigwedge^n \mathbb{V}$, which is indexed over all combinations $\binom{2n}{n}$. (It is combinations and not permutations because the wedge product is alternating.) For any $W = \mathrm{sp}\{w_1, \ldots, w_n\} \subset \mathbb{V}$, we then have a map $\tilde{j} : \mathrm{Gr}_n(\mathbb{V}) \to \bigwedge^n \mathbb{V}$ defined by

$$\tilde{j}(W) = w_1 \wedge \cdots \wedge w_n. \tag{2.23}$$

For this map to be well-defined, it would have to be independent of the choice of basis of W. In fact, it is not, because changing the basis of W changes the n-vector $j(W)$ by a nonzero multiple [79]. However, by projectivizing we can obtain a well-defined map

$$\begin{aligned} j : \mathrm{Gr}_n(\mathbb{V}) &\to \mathbb{P}\left(\bigwedge^n \mathbb{V}\right) \\ W = \mathrm{sp}\{w_1, \ldots, w_n\} &\mapsto [w_1 \wedge \cdots \wedge w_n]. \end{aligned} \tag{2.24}$$

The Plücker embedding thus provides a means for realizing vector spaces as elements of the exterior algebra. To see that $j(E^{s/u}(\lambda, z)) = [\tilde{E}^{s/u}(\lambda, z)]$, one simply has to pick bases of each bundle for a given λ and then use (2.21) and (2.17). Note that we are not claiming that these bases can be chosen in an analytic way, which indeed they cannot in general [35].

We are now prepared to discuss the Evans function. However, before doing so, we dig a little deeper into the Plücker embedding in the special case $n = 2$, $\mathbb{V} = \mathbb{R}^4$. We focus on this case because this is the setting of the Maslov index calculation which comprises a large portion of this dissertation. In this case, there are $6 = \binom{4}{2}$ basis vectors for $\bigwedge^2 \mathbb{R}^4$. Let $W = \text{sp}\{u, v\}$, with $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$. By mapping W into $\mathbb{P}\left(\bigwedge^2 \mathbb{R}^4\right)$ via j, we obtain the Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ of W. Using the definition of the wedge product, one checks that

$$p_{ij} = \begin{vmatrix} u_i & v_i \\ u_j & v_j \end{vmatrix}. \tag{2.25}$$

As the name "Plücker embedding" suggests, the map j is injective for spaces of any dimension. However, it is not surjective. Given a form $\alpha \in \bigwedge^k \mathbb{V}$, we have that $[\alpha]$ is in the image of j if and only if $\alpha \wedge \alpha = 0$. From the perspective of algebraic geometry, the case $k = 2, n = 4$ is interesting because this wedge condition reduces to the single *Grassmannian condition*

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \tag{2.26}$$

Using the Plücker coordinates, it is possible to write down a matrix for the derivation (2.20). In particular, one can calculate directly that $\frac{d}{dz}(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) = 0$, meaning that the set of bivectors which correspond to *two*-dimensional subspaces of $\mathbb{R}^4$ is invariant. This is to be expected, since (2.6) respects linear independence of solutions. We will forgo writing down the matrix for the equation on Plücker coordinates until we specialize to the FitzHugh-Nagumo system.

2.4 The Evans function

In this section we develop the Evans function [2, 29, 53, 71], which is the paradigmatic eigenvalue-hunting tool for traveling waves of semilinear parabolic PDE. The idea is to use the criteria (2.19) for $\lambda \in \mathbb{C} \setminus H$ to be an eigenvalue of L. Suppose that we could find bases

$$E^s(\lambda, z) = \mathrm{sp}\{u_1(\lambda, z), \ldots, u_n(\lambda, z)\}, \quad E^u(\lambda, z) = \mathrm{sp}\{u_{n+1}(\lambda, z), \ldots, u_{2n}(\lambda, z)\} \tag{2.27}$$

of the stable and unstable bundles. A natural way to check for an intersection $E^s(\lambda, z) \cap E^u(\lambda, z)$ would then be to evaluate

$$\det\left[u_1, \ldots, u_{2n}\right]. \tag{2.28}$$

Values of λ for which this determinant vanishes would then correspond to eigenvalues of L. However, the issue of finding analytically varying bases of the stable and unstable bundles is subtle, so it is preferable to use the exterior algebra framework developed in the previous section. Notice that $\tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z) \in \bigwedge^{2n} \mathbb{C}^{2n}$, which is one dimensional. Furthermore, by the definition of the wedge product [79, §8.4], we see that

$$E^s(\lambda, z) \cap E^u(\lambda, z) \neq \{0\} \iff \tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z) = 0. \tag{2.29}$$

Since the $\tilde{E}^{s/u}(\lambda, z)$ were chosen analytically in λ, the wedge product will be analytic as well. To remove the dependence on z, we use the fact that $\tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z)$ solves

$$W'(z) = \mathrm{Trace}(A(\lambda, z))W(z), \tag{2.30}$$

which is the equation induced by (2.6) on $\bigwedge^{2n} \mathbb{C}^{2n}$ [2, §4.C]. Since $\mathrm{Trace}(A(\lambda, z)) = -nc$, it follows that

$$\tilde{D}(\lambda) := e^{ncz} \tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z) \tag{2.31}$$

is independent of z. This is the Evans function, as defined in [2]. Mostly for notational convenience, we prefer to work with a complex-valued function, as opposed to $\tilde{D}(\lambda)$, which takes values in $\bigwedge^{2n} \mathbb{C}^{2n}$. To that end, observe that $\bigwedge^{2n} \mathbb{C}^{2n}$ is spanned by $\mathrm{vol}^* := e_1 \wedge \cdots \wedge e_{2n}$, the basis element induced by the chosen basis of $\mathbb{C}^{2n}$.

Definition 2.1. *The* ***Evans function*** $D(\lambda)$ *is defined by*

$$D(\lambda)\mathrm{vol}^* = e^{ncz} \tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z). \tag{2.32}$$

It can be shown that $D(\lambda)$ has the following properties:

1. $D(\lambda)$ is analytic on an open domain containing $\mathbb{C} \setminus H$.

2. $D(\lambda) \in \mathbb{R}$ if $\lambda \in \mathbb{R}$.

3. $D(\lambda) = 0$ if and only if $\lambda \in \sigma(L)$. Furthermore, the algebraic multiplicity of λ as an eigenvalue of L is equal to the order of λ as a root of D.

Most of these properties follow immediately from the construction of the Evans function. Only the last part–namely that the algebraic multiplicity of λ as an eigenvalue is equal to its order as a root–is difficult. This is proved in [2, §6]. For good overviews of the Evans function and its properties, the reader can also consult [71, §4.1] or [53, §9.1-9.3].

2.5 Symplectic structure of the eigenvalue equation

To this point, none of the material in this chapter has relied on the special structure of skew-gradient systems. In this section, we will show that the eigenvalue equations for skew-gradient systems have an underlying symplectic structure. We will develop this without any reference to a Hamiltonian, which one would have to work harder to unmask.

Consider (2.6) for real $\lambda > \beta$. Note that $E^{s/u}(\lambda, z)$ are real vector spaces in this case, since they consist of solutions to an ODE with real coefficients. To identify the symplectic structure, we introduce the matrix

$$J = \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix}. \tag{2.33}$$

Since $Q^2 = I$ and $Q^* = Q$, it is clear that J satisfies the conditions (1.10) for being a complex structure. As in Chapter 1, the bilinear form

$$\omega(a, b) := \langle a, Jb \rangle. \tag{2.34}$$

is therefore symplectic. The following theorem underpins all of the analysis of this thesis.

Theorem 2.5.1. *Let Y_1, Y_2 be any two solutions of (2.6) for fixed $\lambda \in \mathbb{R}$. Then*

$$\frac{d}{dz}\omega(Y_1, Y_2) = -c\,\omega(Y_1, Y_2). \tag{2.35}$$

In particular, if $\omega(Y_1(z_0), Y_2(z_0)) = 0$ for some $z_0 \in \mathbb{R}$, then $\omega(Y_1, Y_2) \equiv 0$. More generally, the symplectic form

$$\Omega := e^{cz}\omega \tag{2.36}$$

is constant in z on any two solutions of (2.6).

Proof. A direct computation gives that

$$\begin{aligned} \frac{d}{dz}\omega(Y_1, Y_2) &= \omega(Y_1, A(\lambda, z)Y_2) + \omega(A(\lambda, z)Y_1, Y_2) \\ &= \langle Y_1, JA(\lambda, z)Y_2\rangle + \langle A(\lambda, z)Y_1, JY_2\rangle \\ &= \langle Y_1, \left[JA + A^T J\right] Y_2\rangle. \end{aligned} \tag{2.37}$$

In light of (2.34), we therefore need to show that

$$JA + A^T J = -cJ. \tag{2.38}$$

Recalling that S and Q are diagonal and that $(f'(\hat{u}))^T = F''(\hat{u})^T = F''(\hat{u}) = f'(\hat{u})$, we compute

$$\begin{aligned} JA + A^T J &= \begin{pmatrix} \lambda Q S^{-1} - f'(\hat{u}) & -cQ \\ 0 & -QS \end{pmatrix} + \begin{pmatrix} -\lambda S^{-1} Q + f'(\hat{u}) & 0 \\ cQ & SQ \end{pmatrix} \\ &= -c \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix} = -cJ. \end{aligned} \tag{2.39}$$

For the second part, we see that

$$\frac{d}{dz}\Omega(Y_1, Y_2) = e^{cz}\left(c\omega(Y_1, Y_2) + \frac{d}{dz}\omega(Y_1, Y_2)\right) = 0. \tag{2.40}$$

□

For fixed $\lambda \in \mathbb{R}$, we have mentioned the standard result that (2.6) induces a flow on $\mathrm{Gr}_k(\mathbb{R}^{2n})$ for any k. The following is then a consequence of the preceding theorem.

Corollary 2.5.1. *The set of ω-Lagrangian planes $\Lambda(n)$ is an invariant manifold for the equation induced by (2.6) on $\mathrm{Gr}_n(\mathbb{R}^{2n})$.*

As explained above, eigenvalues are found by looking for intersections of the sets $E^{s/u}(\lambda, z)$. To make use of Corollary 2.5.1, it is therefore critical that the stable and unstable bundles are actually Lagrangian subspaces. We show now that this is indeed the case.

Theorem 2.5.2. *For all $\lambda \in \mathbb{R} \cap (\mathbb{C} \setminus H)$ and $z \in \mathbb{R}$, the subspaces $E^u(\lambda, z)$ and $E^s(\lambda, z)$ are Lagrangian.*

Proof. First, it is clear that ω and Ω define the same set of Lagrangian planes. By Theorem 2.5.1, we just need to show that $\Omega(Y_1, Y_2) = 0$ for some value of z, given $Y_1, Y_2 \in E^{s/u}(\lambda, z)$. We begin with $E^s(\lambda, z)$. By definition, $Y_1, Y_2 \in E^s(\lambda, z)$ decay to 0 as $z \to \infty$. Since $c < 0$, it is easy to see that

$$\Omega(Y_1, Y_2) = \lim_{z \to \infty} e^{cz} \omega(Y_1, Y_2) = 0. \tag{2.41}$$

Now consider $Y_1, Y_2 \in E^u(\lambda, z)$. The decay of these solutions at $-\infty$ will be faster than e^{-cz}, by (2.15) and Theorem 3.1 of [71]. It follows that

$$\Omega(Y_1, Y_2) = \lim_{z \to -\infty} e^{cz} \omega(Y_1, Y_2) = \lim_{z \to -\infty} \omega(e^{cz} Y_1, Y_2) = 0. \tag{2.42}$$

This completes the proof. $\square$

The upshot of this section is that the stable and unstable bundles define two-parameter curves of Lagrangian planes. To exploit this fact, we will consider the Maslov index associated to the unstable bundle. The Maslov index and its properties are the subject of the next chapter.

CHAPTER 3

The Maslov index

Recall from Chapter 1 that the fundamental group of $\Lambda(n)$ is infinite cyclic for all $n \in \mathbb{N}$. The homotopy class of a loop in this space is therefore like a winding number. Intuitively, the duality between winding numbers and intersection numbers should allow us to identify the homotopy class of a loop as an intersection count with a codimension one set in $\Lambda(n)$. This is indeed the case, as was shown by Arnol'd [4]. In fact, Arnol'd extended this definition to non-closed curves under certain assumptions. These assumptions were relaxed considerably in [70], and the intersection number discussed therein is the Maslov index that we will employ. We remark that the application of the Maslov index described in this thesis goes back to the work of Maslov [61] and the seminal papers of Arnol'd [4, 5].

We will develop the Maslov index in steps. In the next section, we define it abstractly for a curve of Lagrangian planes and a fixed reference plane. Next, we discuss the Maslov index for a pair of curves. Finally, we specialize to the nonlinear wave setting and define the Maslov index of the homoclinic orbit. In each case, the important construction is that of the *crossing form* [70], which assigns an integer to each intersection point determining its contribution to the Maslov index.

3.1 Maslov index of a path of Lagrangian planes

To start, fix a Lagrangian plane $V \in \Lambda(n)$ and define the *train* of V to be

$$\Sigma(V) = \{V' \in \Lambda(n) : V \cap V' \neq \{0\}\}. \tag{3.1}$$

There is a natural partition of this set into submanifolds of $\Lambda(n)$ given by

$$\Sigma(V) = \bigcup_{k=1}^{n} \Sigma_k(V), \qquad \Sigma_k(V) = \{V' \in \Lambda(n) : \dim(V \cap V') = k\}. \tag{3.2}$$

In particular, the set $\Sigma_1(V)$ is dense in $\Sigma(V)$, and it is a two-sided, codimension one submanifold of $\Lambda(n)$ (cf. [70, §2]). In [4], the Maslov index of a loop α is defined as the number of signed intersections of α with $\Sigma_1(V)$. A homotopy argument is used to ensure that all intersections with $\Sigma(V)$ are actually with $\Sigma_1(V)$, and hence this definition makes sense. More generally, for a curve $\gamma : [a, b] \to \Lambda(n)$, it is shown ([4, §2.2]) that the same index is well-defined, provided that $\gamma(a), \gamma(b) \notin \Sigma(V)$ and that all intersections with $\Sigma(V)$ are one-dimensional and transverse. Both the assumptions of transversality at the endpoints and of only one-dimensional crossings were dispensed of in [70]. The key was to make robust the notion of intersections with $\Sigma(V)$, which was accomplished through the introduction of the "crossing form."

Now let $\gamma : [a, b] \to \Lambda(n)$ be a smooth curve. The tangent space to $\Lambda(n)$ at any point $\gamma(t)$ can be identified with the space of quadratic forms on $\gamma(t)$, cf. [27, §1.6]. This allows one to define a quadratic form that determines whether $\gamma(t)$ is transverse to $\Sigma(V)$ at a given intersection; this quadratic form is the crossing form. Specifically, suppose that $\gamma(t^*) \in \Sigma(V)$ for some $t^* \in [a, b]$. It can be checked from (2.34) that the plane $J \cdot \gamma(t^*)$ is orthogonal to $\gamma(t^*)$, with J as in (2.33). Furthermore, any other Lagrangian plane W transverse to $J \cdot \gamma(t^*)$ can be written uniquely as the graph of a linear operator $B_W : \gamma(t^*) \to J \cdot \gamma(t^*)$ [27]. This includes $\gamma(t)$ for $|t - t^*| < \delta \ll 1$. Writing $B_{\gamma(t)} = B(t)$, it follows that the curve $v + B(t)v \in \gamma(t)$ for all $v \in \gamma(t^*)$. The crossing form is then defined by

$$\Gamma(\gamma, V, t^*)(v) = \frac{d}{dt}\omega(v, B(t)v)|_{t=t^*}. \tag{3.3}$$

The form is defined on the intersection $\gamma(t^*) \cap V$. It is shown in Theorem 1.1 of [70] that this definition is independent of the choice $J \cdot \gamma(t^*)$; any other *Lagrangian complement* of $\gamma(t^*)$ would produce the same crossing form. The crossing form is quadratic, so it has a well-defined signature. For a quadratic form Q, we use the notation $\text{sign}(Q)$ for its signature. We also write $n_+(Q)$ and $n_-(Q)$ for the positive and negative indices of inertia of Q (see [79, p. 187]), so that

$$\text{sign}(Q) = n_+(Q) - n_-(Q). \tag{3.4}$$

Roughly speaking, $\text{sign}(\Gamma(\gamma, V, t^*))$ gives the dimension and the direction of the intersection $\gamma(t^*) \cap V$. Reminiscent of Morse theory, a value t^* such that $\gamma(t^*) \cap V \neq \{0\}$ is called a *conjugate point* or *crossing*. A crossing is called *regular* if the associated form Γ is nondegenerate. For a curve with only regular crossings, one can define the Maslov index as follows.

Definition 3.1. *Let $\gamma : [a, b] \to \Lambda(n)$ and $V \in \Lambda(n)$ such that $\gamma(t)$ has only regular crossings with the train of V. The* ***Maslov index*** *is then given by*

$$\mu(\gamma, V) = -n_-(\Gamma(\gamma, V, a)) + \sum_{t^* \in (a,b)} \operatorname{sign} \Gamma(\gamma, V, t^*) + n_+(\Gamma(\gamma, V, b)), \tag{3.5}$$

where the sum is taken over all interior conjugate points.

Remark 3.1. *The reader will notice that the Maslov index defined in [70] has a different endpoint convention than Definition 3.1. Instead, they take $(1/2)\text{sign}(\Gamma)$ as the contribution at both a and b. This is merely convention, provided that one is careful to make sure that the additivity property (see Proposition 3.1.1 below) holds. Our convention follows [23, 41] to ensure that the Maslov index is always an integer.*

The crucial fact about the Maslov index is that it is a homotopy invariant. In [70, §2], it is shown that it enjoys several other properties, a few of which we list in the following proposition.

Proposition 3.1.1. *Let $\gamma : [a, b] \to \Lambda(n)$ be a curve with only regular crossings with the train of a Lagrangian plane V. Then*

(i) (Additivity by concatenation) For any $c \in (a, b)$, $\mu(\gamma, V) = \mu(\gamma|_{[a,c]}, V) + \mu(\gamma|_{[c,b]}, V)$.

(ii) (Homotopy invariance) Two paths $\gamma_{1,2} : [a, b] \to \Lambda(n)$ with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ are homotopic with fixed endpoints if and only if $\mu(\gamma_1, V) = \mu(\gamma_2, V)$.

(iii) If $\dim(\gamma(t) \cap V) =$ constant, then $\mu(\gamma, V) = 0$.

3.2 Maslov index of a pair of curves

In the previous section, we considered one curve of Lagrangian planes and showed how to count intersections with a fixed reference plane. Alternatively, one could consider two curves of Lagrangian planes and count how many times they intersect each other. This theory is also developed in [70]; see §3. Let $\gamma_{1,2} : [a, b] \to \Lambda(n)$ be two curves of Lagrangian planes with a common domain. As before, we say that t^* is a *conjugate point* if $\gamma_1(t^*) \cap \gamma_2(t^*) \neq \{0\}$. At such a point, define the *relative crossing form*

$$\Gamma(\gamma_1, \gamma_2, t^*) = \Gamma(\gamma_1, \gamma_2(t^*), t^*) - \Gamma(\gamma_2, \gamma_1(t^*), t^*) \tag{3.6}$$

on the intersection $\gamma_1(t^*) \cap \gamma_2(t^*)$. For clarity, we emphasize that the right-hand side of (3.6) consists of two crossing forms: the first considers $\gamma_1(t)$ as varying and $\gamma(t^*)$ as fixed; the second treats $\gamma_2(t)$ as moving and $\gamma_1(t^*)$ as fixed. One then applies (3.3) to calculate each of the two forms. Again, the crossing t^* is called *regular* if Γ in (3.6) is nondegenerate.

For a pair of curves with only regular crossings, one can define the *relative Maslov index* by

$$\mu(\gamma_1, \gamma_2) = -n_-(\Gamma(\gamma_1, \gamma_2, a)) + \sum_{t^* \in (a,b)} \operatorname{sign} \Gamma(\gamma_1, \gamma_2, t^*) + n_+(\Gamma(\gamma_1, \gamma_2, b)), \tag{3.7}$$

where the sum is taken over interior intersections of γ_1 and γ_2. We point out now that it is easy to show that regular crossings are isolated, so that both the sums in (3.5) and (3.7) are finite. Also, it is clear that (3.7) coincides with (3.5) in the case where $\gamma_2 =$ constant. Accordingly, most of the properties of the Maslov index in [70, §2] carry over to the two-curve case without much trouble. However, in moving from paths to pairs of curves, one must be careful about the homotopy axiom. The following is proved in Corollary 3.3 of [70].

Proposition 3.2.1. *Let γ_1 and γ_2 be curves of Lagrangian planes with common domain $[a, b]$. If $\gamma_1(a) \cap \gamma_2(a) = \gamma_1(b) \cap \gamma_2(b) = \{0\}$, then $\mu(\gamma_1, \gamma_2)$ is a homotopy invariant, provided that the homotopy respects the stated condition on the endpoints.*

3.3 Maslov index of the traveling wave

We are now ready to specialize to the problem at hand and define the Maslov index of the traveling wave. The definition below is due to Chen and Hu [18]. To motivate things, recall from Theorem 1.3.1 that the number of unstable modes for the Sturm-Liouville operator is determined by the number of zeros of the steady state whose stability is being analyzed (or, to be more precise, the number of critical points it has). The key to making that connection is the fact that the derivative of the steady state solves the eigenvalue problem (ignoring boundary conditions) when $\lambda = 0$. This is neither a coincidence, nor a feature unique to the scalar case. In general, one sees that (2.6) with $\lambda = 0$ is the *variational equation* for the traveling wave ODE

$$\begin{pmatrix} u \\ v \end{pmatrix}_z = \begin{pmatrix} Sv \\ -cv - Qf(u) \end{pmatrix}. \tag{3.8}$$

(This system is obtained by setting $u_z = v$ and writing the steady-state equation for (1.19) as a first-order system.)

We will say more about the importance of the nonlinear problem later. For now, it suffices to say that the Maslov index of interest *should* be with $\lambda = 0$ fixed, since the eigenvalue problem in that case is related to the wave itself (i.e. not dependent on the spectral parameter). To nail down the curve and reference plane, remember the "shooting approach" to the Sturm theorem, wherein the subspace satisfying the left boundary data is flowed forward in time (x). Counting zeros of the derivative corresponds to counting crossings with the reference plane $\{0\} \times \mathbb{R}$, which can also be thought of as the space of boundary data for the right endpoint. This suggests that the curve of interest should be the unstable bundle $E^u(0, z)$, and the reference plane should be $V^s(0) = S(0)$, the stable subspace of $A_\infty(0)$, which corresponds to the right boundary data.

For technical reasons, this choice of curve and reference plane are untenable. Indeed, the domain of the curve $E^u(0, z)$ is $\mathbb{R}$, in contrast to the Dirichlet problem in which it was a compact interval. It is therefore possible to have an infinite number of conjugate points, and hence an infinite Maslov index. Moreover, we already know that the derivative of the wave $\varphi'(z) = (\hat{u}', S\hat{u}'') \in E^s(0, z)$ for all z, since it is bounded as $z \to \infty$. This implies that there will be a conjugate point at $z = +\infty$. Even worse, since this conjugate point is reached in infinite time, the crossing form will vanish in the limit, thus the contribution to the Maslov index will be impossible to determine. To remedy this, Chen and Hu [18, §1] instead pulled back $V^s(0)$ slightly along φ and used $E^s(0, \tau)$, $\tau \gg 1$ as a reference plane. The domain of the curve is truncated as well to $(-\infty, \tau]$, which forces a conjugate point at the right end point; φ' (at least) is in the intersection $E^u(0, \tau) \cap E^s(0, \tau)$. The only requirement on τ is that

$$V^u(0) \cap E^s(0, z) = \{0\} \text{ for all } z \geq \tau. \tag{3.9}$$

One then arrives at the following definition.

Definition 3.2. *Let τ satisfy (3.9). The **Maslov index** of φ is given by*

$$\operatorname{Maslov}(\varphi) := \sum_{z^* \in (-\infty, \tau)} \operatorname{sign} \Gamma(E^u, E^s(0, \tau), z^*) + n_+(\Gamma(E^u, E^s(0, \tau), \tau)), \tag{3.10}$$

where the sum is taken over all interior crossings of $E^u(0, z)$ with Σ, the train of $E^s(0, \tau)$.

For the Maslov index to be well-defined, it must be independent of τ. It is perhaps surprising that this would be the case. Indeed, it is quite possible that conjugate points would be added or erased by moving τ, which has the effect of shortening/lengthening the curve $E^u(0,z)$. However, the Maslov index counts *signed* intersections, so while extra conjugate points may appear, they would have to do so in a way so that the signed count does not change.

To prove that this definition makes sense, let $\tau' > \tau_0 \geq \tau$ satisfy (3.9). As $z \to -\infty$, we know that $E^u(0,z) \to U(0) = V^u(0)$, the unstable subspace of $A_\infty(0)$. Now, consider a trapezoid in $z\tau$-space bounded by the following four curve segments:

$$\begin{aligned}
\gamma_1 &= \{(z,\tau) : -\infty \leq z \leq \tau_0, \tau = \tau_0\} \\
\gamma_2 &= \{(z,\tau) : z = \tau, \tau_0 \leq z \leq \tau'\} \\
\gamma_3 &= \{(z,\tau) : -\infty \leq z \leq \tau', \tau = \tau'\} \\
\gamma_4 &= \{(z,\tau) : z = -\infty, \tau_0 \leq \tau \leq \tau'\}
\end{aligned}. \tag{3.11}$$

(Note that we include the point at $-\infty$ because $E^u(0,z)$ is well-behaved in that limit.) Call the boundary curve of the trapezoid γ, equipped with a counter-clockwise orientation. Map this trapezoid into $\Lambda(n) \times \Lambda(n)$ via the assignment

$$(z,\tau) \mapsto (E^u(0,z), E^s(0,\tau)). \tag{3.12}$$

It is clear that the trapezoid can be continuously shrunk down to the vertex where γ_1 and γ_4 meet. It then follows from Proposition 3.2.1 that the Maslov index of the pair of curves given by the image of γ is zero, since $E^u(0,-\infty) = V^u(0)$, which is transverse to $E^s(0,\tau)$ for $\tau_0 \leq \tau \leq \tau'$, by (3.9). From (3.7) and Definition 3.2, it is clear that the Maslov indices of the edges corresponding to γ_1 and γ_3 are Maslov(φ) for $\tau = \tau_0$ and $\tau = \tau'$ respectively (assuming z increases in each case). Furthermore, since $\dim(E^u(0,z) \cap E^s(0,z))$ is constant in z, the Maslov index corresponding to the edges γ_2 and γ_4 are both zero, by Proposition 3.1.1(iii). It then follows from Proposition 3.1.1(i) that the two edges corresponding to Maslov(φ) have opposite Maslov indices, which completes the proof, since the orientation is reversed for $\tau = \tau'$.

We now know that Maslov(φ) is well-defined and independent of τ (provided τ is large enough).

This will be very valuable later, as we will need to refine the value τ several times in the Maslov Box argument. Although we will consider several curves in this thesis, any mention of *the* Maslov index will always mean Maslov(φ).

3.4 The crossing form in z

To use Definition 3.1, we will need to know how to compute the crossing form. This is the content of the following theorem, which is found in [23, §5].

Theorem 3.4.1. *Fix τ as in (3.9). Let z^* be a conjugate time for the curve $E^u(0,z)$ with respect to the reference plane $E^s(0,\tau)$. Then for $\xi \in E^u(0,z^*) \cap E^s(0,\tau)$, the crossing form is given by*

$$\Gamma(E^u(0,\cdot), E^s(0,\tau), z^*)(\xi) = \omega(\xi, A(0,z^*)\xi). \tag{3.13}$$

Proof. Take $W \in \Lambda(n)$ such that $E^u(0,z^*) \oplus W = \mathbb{R}^{2n}$. As above, any other Lagrangian subspace transverse to W can be written uniquely as the graph of a linear operator $B : E^u(0,z^*) \to W$. In particular, for $|z - z^*| < \delta \ll 1$,

$$E^u(0,z) = \{v + \psi(z)v : v \in E^u(0,z^*)\} \tag{3.14}$$

with $\psi(z) : E^u(0,z^*) \to W$ smooth in z. For any $\xi \in E^u(0,z^*) \cap E^s(0,\tau)$, we therefore have a curve $w(z) \in W$ defined by $\xi + w(z) \in E^u(0,z)$, or, equivalently, $w(z) = \psi(z)\xi$. Furthermore, we have $\psi(z^*) = 0$. It is shown in [70, p. 3] that the form

$$Q(\xi) = \frac{d}{dz}\Big|_{z=z^*} \omega(\xi, w(z)) \tag{3.15}$$

is independent of the choice of W and defines the crossing form. To show that (3.13) holds, it will be helpful to consider the evolution operator $\Phi(\zeta, z)$ for (2.6) with $\lambda = 0$. Φ satisfies $\Phi(\zeta,\zeta) = \mathrm{Id}$ and $\Phi(z^*, z) \cdot E^u(0,z^*) = E^u(0,z)$. (Here, $\cdot$ refers to the induced action of $\Phi(z^*, z)$ on an n-dimensional subspace.) Notice that (3.14) defines a curve $\gamma(z) \in E^u(0,z^*)$ by the formula

$$\xi + \psi(z)\xi = \Phi(z^*, z)\gamma(z). \tag{3.16}$$

From above, $\gamma(z^*) = \xi$. We are now ready to compute:

$$\begin{aligned}
\frac{d}{dz}\Big|_{z=z^*} \omega(\xi, w(z)) &= \frac{d}{dz}\Big|_{z=z^*} \omega(\xi, \Phi(z^*, z)\gamma(z) - \xi) \\
&= \frac{d}{dz}\Big|_{z=z^*} \omega\left(\xi, \Phi(z^*, z)\gamma(z)\right) \\
&= \omega\left(\xi, A(0, z^*)\gamma(z^*)\right) + \omega\left(\xi, \gamma'(z^*)\right) \\
&= \omega\left(\xi, A(0, z^*)\xi\right) + \lim_{z\to z^*} \frac{1}{z - z^*}\omega\left(\xi, \gamma(z) - \gamma(z^*)\right) \\
&= \omega\left(\xi, A(0, z^*)\xi\right).
\end{aligned} \tag{3.17}$$

The last equality follows since $\gamma(z) \in E^u(z^*)$ for all z, which is a Lagrangian plane containing ξ. This completes the proof. $\square$

Notice that the same proof goes through *mutatis mutandis* for $\lambda \neq 0$. Since we will later consider the unstable bundle for $\lambda \geq 0$, we record the crossing form for such curves here.

Theorem 3.4.2. *Consider the curve $z \mapsto E^u(\lambda, z)$, for fixed λ. Assume that for a reference plane V, there exists a value $z = z^*$ such that $E^u(\lambda, z^*) \cap V \neq \{0\}$. Then the crossing form for $E^u(\lambda, \cdot)$ with respect to V is given by*

$$\Gamma(E^u(\lambda, \cdot), V, z^*)(\zeta) = \omega(\zeta, A(\lambda, z^*)\zeta), \tag{3.18}$$

restricted to the intersection $E^u(\lambda, z^*) \cap V$.

CHAPTER 4

Stability index for two-component activator inhibitor systems

The goal of this chapter is to prove the main result of [23], which shows how the Maslov index can be used to determine the sign of the derivative of the Evans function at $\lambda = 0$. Although this result will hold for general skew-gradient systems, we will focus on a two-component activator-inhibitor system

$$\begin{aligned} u_t &= u_{xx} + f(u) - \sigma v \\ v_t &= v_{xx} + g(v) + \alpha u, \end{aligned} \tag{4.1}$$

where $f, g \in C^2(\mathbb{R})$, and $\alpha, \sigma > 0$ are real constants. As before we assume that (4.1) possesses a traveling wave $\varphi = (\hat{u}, \hat{v})$, which has speed $c < 0$ and approaches $(0, 0)$ as $z \to \pm\infty$. In an abuse of notation, we will use φ for both the wave $\varphi = (\hat{u}, \hat{v})$, and for the corresponding homoclinic orbit $\varphi = (\hat{u}, \hat{v}, \hat{u}'/\sigma, \hat{v}'/\alpha)$ in the traveling wave ODE

$$\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}_z = \begin{pmatrix} \sigma w \\ \alpha y \\ -cw - \frac{f(u)}{\sigma} + v \\ -cy - u - \frac{g(v)}{\alpha} \end{pmatrix}. \tag{4.2}$$

Calling the nonlinearity in (4.1) $F(U)$, assumption (2.1) this time takes the form

$$\begin{aligned} \operatorname{Trace}(DF(0)) &= f'(0) + g'(0) < 0 \\ \operatorname{Det}(DF(0)) &= f'(0)g'(0) + \sigma\alpha > 0. \end{aligned} \tag{4.3}$$

The eigenvalue equation for the linearization about the wave is given by

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_z = \begin{pmatrix} 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \alpha \\ \frac{\lambda}{\sigma} - \frac{f'(\hat{u})}{\sigma} & 1 & -c & 0 \\ -1 & \frac{\lambda}{\alpha} - \frac{g'(\hat{v})}{\alpha} & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}. \tag{4.4}$$

We again abbreviate this system $Y'(z) = A(\lambda, z)Y(z)$, with asymptotic matrix $A_\infty(\lambda)$. In this section, we make a few additional assumptions about φ.

(A1) The tails of φ are monotone, as opposed to oscillatory. From (2.12) and (4.3), we see that this is equivalent to assuming that ν_1 and ν_2 are real. Additionally, we assume that ν_1 and ν_2 are simple and thus satisfy

$$\nu_1 < \nu_2 < 0. \tag{4.5}$$

(A2) Assumption (A1) guarantees that the eigenvalues $\mu_i(0)$ of $A_\infty(0)$ satisfy

$$\mu_1(0) < \mu_2(0) < 0 < \mu_3(0) < \mu_4(0), \tag{4.6}$$

so that the leading eigenvalues $\mu_2(0)$ and $\mu_3(0)$ are real and simple. We assume that the exponential decay rate of φ (as a homoclinic orbit) is given by $\mu_2(0)$ in forwards time and by $\mu_3(0)$ in backwards time. This assumption is generic, c.f. [40, §2.1].

(A3) φ is *transversely constructed*. This means that $(\varphi(z), c) \in \mathbb{R}^5$ is given by the transverse intersection of the center-unstable and center-stable manifolds of the fixed point $(0, 0, 0, 0, c)$ for (4.2) with the equation $c' = 0$ appended.

Assumptions (A1) and (A2) are mostly for notational convenience. Our results could be obtained in the case where the eigenvalues are complex or if the wave is in a so-called orbit-flip configuration [40]. However, some of the results use the decay properties of the wave, so the arguments would require a different presentation. On the other hand, if $\nu_1 = \nu_2$, then $\mu_1(0) = \mu_2(0)$ and $\mu_3(0) = \mu_4(0)$. Having eigenvalues of multiplicity greater than one causes trouble for analytically picking bases of $E^{s/u}(\lambda, z)$ [11], so we assume that is not the case. Vis-à-vis σ and α, having $\nu_1 \neq \nu_2$ is clearly generic. Finally, assumption (A3) is known to be equivalent to the simplicity of the translation eigenvalue $\lambda = 0$ [3, pp. 57-60]. Since we wish to use the Maslov index to say something about sign$D'(0)$, this is a natural assumption to make.

4.1 Evans function parity argument

Before jumping into the details of the argument, it is worthwhile first to discuss why one would care about the sign of $D'(0)$. Due to translation invariance, it will always be the case (at least for autonomous systems) that $\lambda = 0$ is an eigenvalue with eigenfunction $\varphi'(z)$. Recall from Chapter 2 that the Evans function is real-valued for $\lambda \in \mathbb{R}$. The sign of $D(\lambda)$ for real $\lambda \gg 1$ can also be determined easily, using the fact that (4.4) is essentially autonomous for large enough λ. Typically, $D(\lambda)$ is constructed so that $D(\lambda) > 0$ for $\lambda \gg 1$. It follows that if $D'(0) < 0$, then L must have a positive eigenvalue. Likewise, if $D'(0) > 0$ and other information is known about $\sigma(L)$, then it is sometimes possible to conclude stability. This parity argument has been used many times in the stability analysis of nonlinear waves, e.g. [3, 9, 10, 35, 44, 52, 66, 69, 81].

We will close this section by fixing the sign of $D(\lambda)$, $\lambda \gg 1$ for (4.4). First, it follows from (A1) and (2.12) that for all $\lambda > \beta$, the eigenvalues of $A_\infty(\lambda)$ satisfy

$$\mu_1(\lambda) < \mu_2(\lambda) < 0 < -c < \mu_3(\lambda) < \mu_4(\lambda). \tag{4.7}$$

As in §2.3, we can find $\zeta_{s/u}(\lambda) \in \bigwedge^2 \mathbb{R}^4$ and $\tilde{E}^{s/u}(\lambda, z) \in \bigwedge^2 \mathbb{R}^4$ such that

$$\begin{aligned} e^{-(\mu_1(\lambda)+\mu_2(\lambda))z}\tilde{E}^s(\lambda, z) &\to \zeta_s(\lambda) \text{ as } z \to \infty \\ e^{-(\mu_3(\lambda)+\mu_4(\lambda))z}\tilde{E}^u(\lambda, z) &\to \zeta_u(\lambda) \text{ as } z \to -\infty. \end{aligned} \tag{4.8}$$

Again, $\tilde{E}^s(\lambda, z)$ and $\tilde{E}^u(\lambda, z)$ are the only solutions of the equation induced by (4.4) on $\bigwedge^2 \mathbb{R}^4$ which are bounded as $z \to \infty$ and as $z \to -\infty$ respectively. Since $\zeta_{s/u}(\lambda)$ correspond to the (non-intersecting) stable and unstable subspaces of $A_\infty(\lambda)$, the quantity $\zeta_s(\lambda) \wedge \zeta_u(\lambda)$ will be a nonzero multiple of the volume element vol* on $\mathbb{R}^4$. We can fix the orientation by choosing multiples of $\zeta_{s/u}(0)$ such that

$$\zeta_s(0) \wedge \zeta_u(0) = \rho \, \mathrm{vol}^*, \qquad \rho > 0. \tag{4.9}$$

Under this assumption, it is known (c.f. Lemma 4.2 of [84] and Lemma 4.2 of [3]) that

$$D(\lambda) > 0 \text{ for } \lambda \gg 1. \tag{4.10}$$

4.2 The symplectic Evans function

In this section, we show how to exploit the symplectic structure of (4.4) by rewriting $D(\lambda)$ in terms of the symplectic form Ω. To do so, we will need bases for the stable and unstable subspaces. Since the eigenvalues $\mu_i(\lambda)$ are all real and simple for $\lambda > \beta$, it is well known (e.g. [69, p. 56]) that there exist solutions $u_i(\lambda, z)$ to (4.4) satisfying

$$\begin{aligned} \lim_{z\to\infty} e^{-\mu_i(\lambda)z} u_i(\lambda, z) &= \eta_i(\lambda), \qquad i = 1, 2 \\ \lim_{z\to-\infty} e^{-\mu_i(\lambda)z} u_i(\lambda, z) &= \eta_i(\lambda), \qquad i = 3, 4, \end{aligned} \tag{4.11}$$

where $\eta_i(\lambda)$ is a nonzero eigenvector of $A_\infty(\lambda)$ corresponding to eigenvalue $\mu_i(\lambda)$. Furthermore, the u_i are analytic in λ, and the limits are achieved uniformly on compact subsets of (β, ∞). By (2.21), (2.22), and (4.11), it must be the case that

$$\begin{aligned} u_1(\lambda, z) \wedge u_2(\lambda, z) &= \tilde{E}^s(\lambda, z) \\ u_3(\lambda, z) \wedge u_4(\lambda, z) &= \tilde{E}^u(\lambda, z) \end{aligned}, \tag{4.12}$$

and hence that

$$\eta_1(\lambda) \wedge \eta_2(\lambda) \wedge \eta_3(\lambda) \wedge \eta_4(\lambda) = K\, \zeta_s(\lambda) \wedge \zeta_u(\lambda), \tag{4.13}$$

for some $K \neq 0$. By (A2), we can take

$$\varphi'(z) = u_2(0, z) = u_3(0, z), \tag{4.14}$$

since $\varphi' \in E^s(0, z) \cap E^u(0, z)$. By rescaling $u_{1/4}(\lambda, z)$ if necessary, we can take $K = 1$ in (4.13). Using Definition 2.1, we therefore have

$$\begin{aligned} D(\lambda)\text{vol}^* &= e^{2cz} \tilde{E}^s(\lambda, z) \wedge \tilde{E}^u(\lambda, z) \\ &= e^{2cz} \det\left[u_1(\lambda, z), u_2(\lambda, z), u_3(\lambda, z), u_4(\lambda, z)\right] \text{vol}^* \end{aligned}. \tag{4.15}$$

We are now ready to bring in the symplectic form. This idea was pioneered in [9, 10] for systems of Hamiltonian PDEs with a multi-symplectic structure, and the following formula first appeared in [15]. The slight difference in our formula and that of Chardard-Bridges' is due to the fact that the symplectic form is different in activator-inhibitor systems.

Theorem 4.2.1. *Let $a_1, a_2, b_1, b_2 \in \mathbb{R}^4$. Then*

$$\det[a_1, a_2, b_1, b_2] = -\det \begin{bmatrix} \omega(a_1, b_1) & \omega(a_1, b_2) \\ \omega(a_2, b_1) & \omega(a_2, b_2) \end{bmatrix} + \omega(a_1, a_2)\omega(b_1, b_2). \tag{4.16}$$

This formula is proved for arbitrary (even) dimension in [15] using the Leibniz formula for determinants. However, in this low-dimensional case it is easy enough to verify using brute force. Notice that the second term in (4.16) disappears if either $\mathrm{sp}\{a_1, a_2\}$ or $\mathrm{sp}\{b_1, b_2\}$ is a Lagrangian plane. Combining (4.15) and (4.16), we arrive at the symplectic Evans function.

Corollary 4.2.1. *The symplectic Evans function is*

$$D(\lambda) = -e^{2cz} \det \begin{bmatrix} \omega(u_1, u_3) & \omega(u_1, u_4) \\ \omega(u_2, u_3) & \omega(u_2, u_4) \end{bmatrix}. \tag{4.17}$$

In this form, it is easy to see the $z-$independence of D. Distributing one copy of e^{cz} to each row of the matrix in (4.17), one can replace $e^{cz}\omega$ with Ω in each entry and consult Theorem 2.5.1.

Let us now consider the case $\lambda = 0$. First, due to translation invariance, the derivative of the traveling wave $\varphi'(z)$ is a zero-eigenfunction for L, hence $D(0)$ should be zero. Indeed, from Theorem 2.5.2 and (4.14), it follows that each entry of the matrix in (4.17) is zero, with the possible exception of $\omega(u_1, u_4)$. Thus $D(0) = 0$, as expected. Corollary 4.2.1 can also be used to show that $D'(0) = 0$ if the stable and unstable bundles have a two-dimensional intersection (i.e. they are tangent to each other). In this case, the matrix in (4.17) is the zero matrix for $\lambda = 0$, and an application of the product rule shows that $D'(0) = 0$.

The main result of this paper involves $D'(0)$, so we start by calculating this using Jacobi's formula. The second part of this calculation is inspired by the proof of Theorem 1.11 in [69], and similar calculations are carried out in [10].

Lemma 4.2.1. *The quantity $D'(0)$ is given by*

$$D'(0) = \Omega(u_1, u_4) \int_{-\infty}^{\infty} e^{cz} \left(\frac{(\hat{u}')^2}{\sigma} - \frac{(\hat{v}')^2}{\alpha} \right) dz. \tag{4.18}$$

Before giving the proof, a few comments are in order. First, the fact that the wave is transversely constructed implies that both terms in the product in (4.18) are nonzero. The term $\Omega(u_1, u_4)$–named the Lazutkin-Treschev invariant in [15]–carries information about the intersection of the stable and unstable manifolds; if it were zero, then $E^u(0, z)$ and $E^s(0, z)$ would have a two-dimensional intersection [15, pp. 84-85]. The integral, on the other hand, is solely dependent on the wave itself. In [21] it is shown that this encodes the *deficiency* of the eigenvalue. More precisely, it vanishes if and only if the algebraic multiplicity of 0 as an eigenvalue of L is greater than the geometric multiplicity. While the integral is calculable if the wave is known, the Lazutkin-Treschev invariant is more difficult to determine. Indeed the decay rates of the u_i and the $z-$independence of Ω can be used to show that $e^{-\mu_4(0)z}u_4(0, z)$ converges to a multiple of $\eta_4(0)$ in forward time. However, the orientation of this vector (i.e. whether that multiple is positive or negative) is difficult to ascertain, and it is what determines the sign of $D'(0)$. We will see that the Maslov index can be used to circumvent this difficulty. Now for the proof of the Lemma.

Proof. Denote by $\Sigma(\lambda, z)$ the matrix in (4.17), and let $\Sigma(\lambda, z)^{\#}$ be its adjugate (i.e. the transpose of its cofactor matrix). By the Jacobi formula [60, §8.3], we have

$$\begin{aligned}
D'(0) &= -e^{2cz}\,\mathrm{Trace}(\Sigma^{\#}\Sigma_\lambda)|_{\lambda=0}. \\
&= -\mathrm{Trace}\left(\begin{bmatrix} \Omega(u_2,u_4) & -\Omega(u_1,u_4) \\ -\Omega(u_2,u_3) & \Omega(u_1,u_3) \end{bmatrix} \begin{bmatrix} \partial_\lambda\Omega(u_1,u_2) & \partial_\lambda\Omega(u_1,u_4) \\ \partial_\lambda\Omega(u_2,u_3) & \partial_\lambda\Omega(u_2,u_4) \end{bmatrix} \right) |_{\lambda=0} \\
&= -\mathrm{Trace}\left(\begin{bmatrix} 0 & -\Omega(u_1,u_4) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_\lambda\Omega(u_1,u_2) & \partial_\lambda\Omega(u_1,u_4) \\ \partial_\lambda\Omega(u_2,u_3) & \partial_\lambda\Omega(u_2,u_4) \end{bmatrix} \right) |_{\lambda=0} \\
&= \Omega(u_1,u_4)\partial_\lambda\Omega(u_2,u_3)|_{\lambda=0}.
\end{aligned} \tag{4.19}$$

The vanishing terms in the third equality are due to the fact that $u_2 = u_3 = \varphi'$ when $\lambda = 0$, and $E^{u/s}(0, z)$ are both Lagrangian planes for all z. It remains to calculate $\partial_\lambda\Omega(u_2, u_3)$. Since

$\mu_2(\lambda) + \mu_3(\lambda) \equiv -c$, we can write $\Omega(u_2, u_3) = \omega(U, V)$, where $U = e^{-\mu_2(\lambda)z}u_2$ and $V = e^{-\mu_3(\lambda)z}u_3$, from which it follows that $\omega(U, V)$ is $z-$independent, and $\partial_\lambda \Omega(u_2, u_3) = \partial_\lambda \omega(U, V)$. Furthermore, U and V satisfy the equations

$$\begin{aligned} U_z &= (A(\lambda, z) - \mu_2(\lambda))\, U \\ V_z &= (A(\lambda, z) - \mu_3(\lambda))\, V. \end{aligned} \tag{4.20}$$

Taking derivatives in λ, we have that

$$\begin{aligned} U_{\lambda z} &= (A(\lambda, z) - \mu_2(\lambda))\, U_\lambda + (A_\lambda - \mu_2'(\lambda))U \\ V_{\lambda z} &= (A(\lambda, z) - \mu_3(\lambda))\, V_\lambda + (A_\lambda - \mu_3'(\lambda))V, \end{aligned} \tag{4.21}$$

where

$$A_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 & 0 \end{pmatrix} \tag{4.22}$$

is independent of z. Finally, since $\partial_z\, \omega(U, V) = 0$, $\partial_z \partial_\lambda\, \omega(U, V) = 0$ as well, so

$$\partial_z\, \omega(U_\lambda, V) = -\partial_z\, \omega(U, V_\lambda). \tag{4.23}$$

Using (4.21) and (4.20), we calculate that

$$\begin{aligned} \partial_z\, \omega(U_\lambda, V) &= \omega\left(A(\lambda, z)U_\lambda - \mu_2(\lambda)U_\lambda + A_\lambda U - \mu_2'(\lambda)U, V\right) \\ &\quad + \omega(U_\lambda, (A(\lambda, z) - \mu_3(\lambda))\, V) \\ &= -c\, \omega(U_\lambda, V) - (\mu_2(\lambda) + \mu_3(\lambda))\omega(U_\lambda, V) + \omega(A_\lambda U, V) - \mu_2'(\lambda)\, \omega(U, V) \\ &= \omega(A_\lambda U, V) - \mu_2'(\lambda)\, \omega(U, V). \end{aligned} \tag{4.24}$$

In the second equality, we used (2.38) to conclude that

$$\omega(A(\lambda, z)v_1, v_2) + \omega(v_1, A(\lambda, z)v_2) = -c\, \omega(v_1, v_2) \tag{4.25}$$

for any z, λ and any vectors $v_i \in \mathbb{R}^4$. This yields the third equality in conjunction with the identity $\mu_2(\lambda)+\mu_3(\lambda)\equiv -c$. If we evaluate this expression at $\lambda=0$, whence $U=e^{-\mu_2(0)z}\varphi'(z)$, $V=e^{-\mu_3(0)z}\varphi'(z)$, and $\omega(U,V)=\Omega(\varphi',\varphi')=0$, we end up with

$$\partial_z\,\omega(U_\lambda,V)(0,z)=e^{cz}\omega(A_\lambda\varphi',\varphi')=-e^{cz}\left(\frac{(\hat{u}')^2}{\sigma}-\frac{(\hat{v}')^2}{\alpha}\right). \tag{4.26}$$

To complete the proof, we use the Fundamental Theorem of Calculus and (4.23), à la [69]. For any large $R,S>0$, we have

$$\begin{aligned}\omega(U_\lambda,V)(0,0)&=\omega(U_\lambda,V)(0,R)+\int_0^R e^{cz}\left(\frac{(\hat{u}')^2}{\sigma}-\frac{(\hat{v}')^2}{\alpha}\right)dz\\ \omega(U,V_\lambda)(0,0)&=\omega(U,V_\lambda)(0,-S)+\int_{-S}^0 e^{cz}\left(\frac{(\hat{u}')^2}{\sigma}-\frac{(\hat{v}')^2}{\alpha}\right)dz\end{aligned}. \tag{4.27}$$

Adding these equations and taking $R,S\to\infty$ gives the desired result, provided that the boundary terms vanish in the limit. Since the limits in (4.11) are achieved uniformly on compact subsets of $(-\beta,\infty)$, we know that the limits

$$\begin{gathered}\lim_{z\to-\infty}U_\lambda(\lambda,z)\\ \lim_{z\to\infty}V_\lambda(\lambda,z)\end{gathered} \tag{4.28}$$

exist. Furthermore, for $\lambda=0$, it is clear that $V=e^{-\mu_3(0)z}\varphi'\to 0$ as $z\to\infty$ and that $U=e^{-\mu_2(0)z}\to 0$ as $z\to-\infty$, so the boundary terms vanish by the linearity of ω, giving the result. $\square$

4.3 The detection form and main result

In this section, we prove the following theorem, which is the core result of [23].

Theorem 4.3.1. *Define* Maslov(φ) *for* $\varphi=(\hat{u}(z),\hat{v}(z))$ *as in Definition 3.1. Then*

$$(-1)^{\mathrm{Maslov}(\varphi)}=\operatorname{sign}\Omega(u_1(0,z),u_4(0,z)). \tag{4.29}$$

Comparing with Lemma 4.2.1, one could then use Maslov(φ) to determine $D'(0)$, if the sign of the Melnikov integral in (4.18) were known. In principle, this should not be a problem, since proving the existence of the wave is prior to proving its stability. We point out that this result is the analog of the result proved in [15] for Hamiltonian systems. The strategy in that work was to use a definition of the Maslov index due to Souriau [75], in which the index is defined for elements in the universal cover of $U(n)$. By contrast, our proof is more elementary and works directly with the crossing form (3.18).

The reader will recall that Maslov(φ) is determined by the curve $E^u(0,z)$, defined for $z \in (-\infty, \tau]$, and the reference plane $E^s(0,\tau)$. We will prove Theorem 4.3.1 through a close examination of the conjugate point $z = \tau$. Indeed, $z = \tau$ is a conjugate point because

$$E^u(0,\tau) \cap E^s(0,\tau) = \mathrm{sp}\{\varphi'(\tau)\}. \tag{4.30}$$

In other words, this conjugate point encodes the translation invariance. It is therefore not surprising that it should be the distinguished z-value used to connect the Maslov index and Evans function. Now, to calculate the Maslov index, we must have a way of finding the other conjugate points. This is accomplished through the introduction of the *detection form* $\pi \in \left(\bigwedge^2 \mathbb{R}^4\right)^*$, defined by

$$\pi(w_1 \wedge w_2) = \det\left[e^{-\mu_1(0)\tau}u_1(0,\tau), e^{-\mu_2(0)\tau}u_2(0,\tau), w_1, w_2\right]. \tag{4.31}$$

π is called the detection form because it is 0 precisely when the plane $W = \mathrm{sp}\{w_1, w_2\}$ intersects $E^s(0,\tau)$ non-trivially. Thus it detects conjugate points for a curve of Lagrangian planes. This form is traditionally called the dual to the characterizing two-vector $w_1 \wedge w_2$ for W, see [24, pp. 97-98]. We next define a function $\beta : \mathbb{R} \to \mathbb{R}$, which evaluates π on $E^u(0,z)$. Explicitly, we have

$$\beta(z) = e^{-(\mu_1+\mu_2)\tau-(\mu_3+\mu_4)z}\det[u_1(\tau), u_2(\tau), u_3(z), u_4(z)]. \tag{4.32}$$

(This is unrelated to the β giving the spectral gap for $F'(0)$–for the rest of this section β refers to the function defined above.) We henceforth suppress the dependence of u_i, μ_i on λ, since we take $\lambda = 0$ for this calculation. For brevity, we also set $M(z) = -(\mu_1+\mu_2)\tau - (\mu_3+\mu_4)z$. Recall that $u_2 = u_3 = \varphi'$ for $\lambda = 0$, so we see immediately that $\beta(\tau) = 0$, since columns two and three are both $\varphi'(\tau)$.

Now, we can use (4.16) to rewrite β as

$$\beta(z) = -e^{M(z)} \det \begin{bmatrix} \omega(u_1(\tau), u_3(z)) & \omega(u_1(\tau), u_4(z)) \\ \omega(u_2(\tau), u_3(z)) & \omega(u_2(\tau), u_4(z)) \end{bmatrix}. \tag{4.33}$$

The next ingredient is $\beta'(\tau)$, whose sign we claim will help determine the sign of $D'(0)$. Since $\beta(\tau) = 0$, we see that

$$\beta'(\tau) = -e^{M(\tau)} \frac{d}{dz} \left[\omega(u_1(\tau), u_3(z))\omega(u_2(\tau), u_4(z)) - \omega(u_1(\tau), u_4(z))\omega(u_2(\tau), u_3(z))\right]\Big|_{z=\tau}. \tag{4.34}$$

Before jumping into the product rule expansion, recall that $u_2(\tau) = u_3(\tau) = \varphi'(\tau)$, and hence $\mathrm{sp}\{u_i(\tau), u_j(\tau)\}$ is Lagrangian for $(i,j) = (1,2), (1,3), (2,4), (3,4)$, with $\omega(u_2, u_3) = 0$ as well. It follows that the only surviving term is $-\omega(u_1(\tau), u_4(\tau))\omega(u_2(\tau), u_3'(\tau))$. Since $M(\tau) = -(\mu_1 + \mu_2 + \mu_3 + \mu_4)\tau = 2c\tau$, we conclude that

$$\beta'(\tau) = \Omega(u_1, u_4)\Omega\left(\varphi'(\tau), \varphi''(\tau)\right). \tag{4.35}$$

The relation to (4.18) is now apparent. Noticing that $\varphi'' = A(0,z)\varphi'$, the second term in (4.35) is the crossing form for the conjugate point $z = \tau$, scaled by a positive factor $e^{c\tau}$. We will show that the sign of $\Omega(u_1, u_4)$ can be determined from the Maslov index, regardless of the sign of the crossing at $z = \tau$. The tie that binds the two is $\beta(z)$. First, from (4.32) we can see that β is asymptotically constant as $z \to -\infty$. Indeed, if τ is large enough, then (4.9) and (4.13) imply that

$$\lim_{z\to-\infty} \beta(z) \approx \det\left[\eta_1, \eta_2, \eta_3, \eta_4\right] = \rho > 0. \tag{4.36}$$

Thus $\beta(z) > 0$ for large, negative z, provided τ is large enough. By definition, zeros of β correspond to conjugate points for the curve $E^u(0,z)$. Heuristically, the sign of $\beta'(\tau)$ is positive if there are an odd number of conjugate points (excluding τ) and negative if there are an even number of conjugate points. (See Figure 4.1 below.) Since the Maslov index, roughly speaking, counts the number of conjugate points, its parity should therefore determine the sign of $\beta'(\tau)$.

To make the preceding precise, we must know a few things about zeros of β and the Maslov

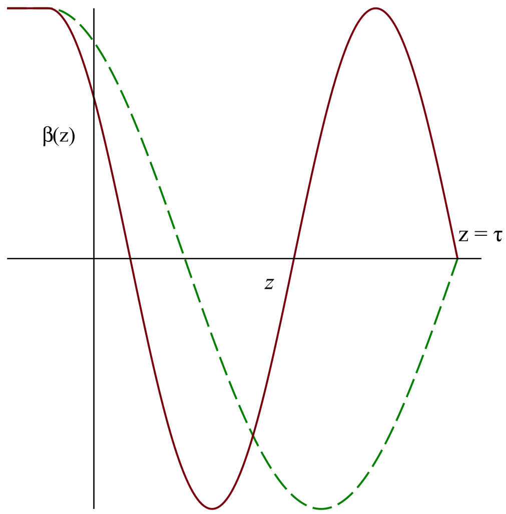


Figure 4.1: Graph of $\beta(z)$: sign of $\beta'(\tau)$ depends on whether there were an odd (dashed green) or even (solid red) number of prior crossings.

index. First, since an intersection of $E^u(0,z)$ with $E^s(0,\tau)$ can be one- or two-dimensional, the contribution to the Maslov index at any (interior) conjugate point is $-2,-1,0,1$, or 2. Since the parity of the index is unchanged if the contribution is even, we need β to cross through the $z-$axis if and only if the crossing is one-dimensional. Obviously, we also need β to have finitely many zeros for this to make sense. The latter is true if we assume that there are only regular crossings, which is an assumption needed to define the Maslov index for non-loops in the first place; see [70, §2]. It turns out that the assumption of regularity is also sufficient for the former. This is the content of the next two lemmas.

Lemma 4.3.1. *If z^* is a conjugate point such that the intersection $E^u(0,z^*) \cap E^s(0,\tau) = \mathrm{sp}\{\xi\}$ is one-dimensional, then the crossing is regular if and only if $\beta'(z^*) \neq 0$.*

Proof. Let $\xi = \beta_1 u_1(\tau) + \beta_2 u_2(\tau)$ be a vector in the intersection. Let ν be a second basis vector for $E^u(0,z^*)$. As noted in [15, §4], $\omega(u_i(\tau),\nu) \neq 0$ $(i = 1,2)$, else we would have $\nu \in E^s(0,z^*)$, violating the assumption that the intersection is one-dimensional. Changing to the basis $\{\xi,\nu\}$ of $E^u(0,z^*)$

would introduce a nonzero multiple in the expression for $\beta(\tau)$, which we call B. It follows that

$$\begin{aligned}
\beta'(z^*) &= Be^{M(z^*)}\left\{\det[u_1(\tau), u_2(\tau), A(0,z^*)\xi, \nu] + \det[u_1(\tau), u_2(\tau), \xi, A(0,z^*)\nu]\right\} \\
&= Be^{M(z^*)}\det[u_1(\tau), u_2(\tau), A(0,z^*)\xi, \nu] \\
&= -Be^{M(z^*)}\det\begin{bmatrix} \omega(u_1(\tau), A(0,z^*)\xi) & \omega(u_1(\tau),\nu) \\ \omega(u_2(\tau), A(0,z^*)\xi) & \omega(u_2(\tau),\nu) \end{bmatrix}.
\end{aligned} \tag{4.37}$$

Since $\mathrm{sp}\{\xi,\nu\}$ is a Lagrangian plane, we have

$$0 = \omega(\xi,\nu) = \beta_1\omega(u_1(\tau),\nu) + \beta_2\omega(u_2(\tau),\nu). \tag{4.38}$$

Without loss of generality, we can assume $\beta_2 \neq 0$, and hence $\omega(u_2(\tau),\nu) = -\dfrac{\beta_1}{\beta_2}\omega(u_1(\tau),\nu)$. Returning to (4.37), we see that

$$\begin{aligned}
\beta'(z^*) &= -Be^{M(z^*)}\left\{-\frac{\beta_1}{\beta_2}\omega(u_1(\tau), A(0,z^*)\xi)\omega(u_1(\tau),\nu) - \omega(u_2(\tau), A(0,z^*)\xi)\omega(u_1(\tau),\nu)\right\} \\
&= \frac{B}{\beta_2}e^{M(z^*)}\omega(u_1(\tau),\nu)\omega(\beta_1u_1(\tau)+\beta_2u_2(\tau), A(0,z^*)\xi) \\
&= \frac{B}{\beta_2}\omega(u_1(\tau),\nu)e^{M(z^*)}\omega(\xi, A(0,z^*)\xi).
\end{aligned} \tag{4.39}$$

Comparing with (3.13), it is now clear that the crossing is regular if and only if $\beta'(z^*) \neq 0$. $\square$

Lemma 4.3.2. *If z^* is a conjugate point such that the intersection $E^u(0,z^*) \cap E^s(0,\tau)$ is two-dimensional, then the following are true:*

1. *$\beta'(z^*) = 0$.*

2. *$\beta''(z^*) \neq 0 \iff$ the crossing at τ is regular.*

Proof. Immediately we see that

$$\begin{aligned}
\beta'(z^*) &= e^{M(z^*)}\left\{\det[u_1(\tau), u_2(\tau), A(0,z^*)u_3(z^*), u_4(z^*)] + \det[u_1(\tau), u_2(\tau), u_3(z^*), A(0,z^*)u_4(z^*)]\right\} \\
&= 0,
\end{aligned} \tag{4.40}$$

since there is linear dependence in the first, second and fourth (resp. first, second and third) columns in the matrix on the left (resp. right). In a similar way, the second derivative is seen to be

$$\beta''(z^*) = 2e^{M(z^*)} \det[u_1(\tau), u_2(\tau), A(0, z^*)u_3(z^*), A(0, z^*)u_4(z^*)]. \tag{4.41}$$

Next, since the crossing is two-dimensional, we have $\text{sp}\{u_1(\tau), u_2(\tau)\} = \text{sp}\{u_3(z^*), u_4(z^*)\}$, so by some change of basis in the first two columns of (4.41), we end up with

$$\begin{aligned} \beta''(z^*) &= 2Be^{M(z^*)} \det\left[u_3(z^*), u_4(z^*), A(0, z^*)u_3(z^*), A(0, z^*)u_4(z^*)\right] \\ &= -2Be^{M(z^*)} \det \begin{bmatrix} \omega(u_3(z^*), A(0, z^*)u_3(z^*)) & \omega(u_3(z^*), A(0, z^*)u_4(z^*)) \\ \omega(u_4(z^*), A(0, z^*)u_3(z^*)) & \omega(u_4(z^*), A(0, z^*)u_4(z^*)) \end{bmatrix}, \end{aligned} \tag{4.42}$$

using (4.16). The symplectic version of the matrix in (4.42) is exactly the matrix of the crossing form Γ in the basis $\{u_3(z^*), u_4(z^*)\}$ for $E^s(0, \tau) \cap E^u(0, z^*)$. To say that the crossing is regular then is to say that this matrix does not have zero as an eigenvalue. Since the determinant of this matrix is the product of the eigenvalues, the Lemma follows. $\square$

These lemmas allow us to conclude the following: consider the curve $\gamma(z)$, which is $E^u(0, z)$ restricted to an interval $(-\infty, \tau - \epsilon)$ containing all conjugate points prior to τ. Then

$$\mu(\gamma, E^s(0, \tau)) \text{ is even } \iff \beta'(\tau) < 0. \tag{4.43}$$

In other words, since $\beta(\tau) = 0$, the direction in which $\beta(z)$ crosses through 0 at τ is completely determined by how many times $\beta(z)$ passed through the $z-$axis prior to τ.

To calculate Maslov(φ), one would need to know the direction of the final crossing, i.e. the sign of $\omega(\varphi'(\tau), \varphi''(\tau))$. However, this is *not* needed to prove Theorem 4.3.1. First, assume that Maslov(φ) is even. There are now two possibilities regarding the final crossing at $z = \tau$. If $\omega(\varphi'(\tau), \varphi''(\tau)) > 0$, then this crossing contributes $+1$ to the index, which means that there were an odd number of weighted crossings prior to τ: odd $+1$ = even. (In the above notation, $\mu(\gamma, E^s(0, \tau))$ is odd.) Thus $\beta'(\tau) > 0$, from which we conclude that $\Omega(u_1, u_4) > 0$, using (4.35).

On the other hand, if $\omega(\varphi'(\tau), \varphi''(\tau)) < 0$, then there must be an even number of weighted crossings prior to τ, since the last crossing contributes 0 to the count (being a negative crossing).

This implies that $\beta'(\tau) < 0$. Again using (4.35), we see that $\Omega(u_1, u_4) > 0$, showing that its sign does not depend on the direction of the final crossing. An analogous argument shows that $\Omega(u_1, u_4) < 0$ if and only if $\mathrm{Maslov}(\varphi)$ is odd. This completes the proof of Theorem 4.3.1.

CHAPTER 5

Existence of fast traveling waves for a FitzHugh-Nagumo system

In this section, we take a break from the Maslov index and begin our close examination of the doubly-diffusive FitzHugh-Nagumo system

$$
\begin{aligned}
u_t &= u_{xx} + f(u) - v \\
v_t &= v_{xx} + \epsilon(u - \gamma v).
\end{aligned}
\tag{5.1}
$$

Here, $u, v \in \mathbb{R}$, and $x, t \in \mathbb{R}$ are space and time respectively. The function f is the "bistable" nonlinearity $f(u) = u(1-u)(u-a)$, where $0 < a < 1/2$ is constant. We take $\epsilon > 0$ to be very small, making this a singular perturbation problem. We also choose $\gamma > 0$ small enough so that $(0, 0)$ is the only fixed point of the associated kinetics equation

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} f(u) - v \\ \epsilon(u - \gamma v) \end{pmatrix}. \tag{5.2}
$$

We call (5.1) the "doubly-diffusive" FitzHugh-Nagumo equation because most of the research into (5.1) considers the case where v does not diffuse or the diffusion coefficient is a small parameter, e.g. [2, 12, 33, 51, 44, 66, 81, 82]. For the case of equal or similar diffusivities, standing waves (i.e. those with $c = 0$) were shown to exist and be stable in [19]. Using variational techniques, traveling waves were shown to exist for (5.1) in [17]. Notably, the issue of the stability of these waves remained open. We will prove that the waves are stable in Chapter 7.

The goal of this chapter is to give another existence proof for the fast traveling waves of (5.1) using geometric singular perturbation theory. The value in this proof is that the construction of the pulse provides the means for assessing its stability using the Maslov index. Along the way, we will also prove that (5.1) possesses fast traveling *fronts* in different parameter regimes. We could use the techniques in this thesis to prove that those fronts are also stable, but we will not pursue that here.

We begin by giving a short overview of geometric singular perturbation theory.

5.1 Overview of geometric singular perturbation theory

Geometric singular perturbation theory (GSP) concerns the dynamics of vector fields with multiple timescales. The basic equations are of the form

$$\begin{aligned} x' &= f(x, y, \epsilon) \\ y' &= \epsilon g(x, y, \epsilon) \end{aligned}, \tag{5.3}$$

where $' = \frac{d}{dt}, x \in \mathbb{R}^n, y \in \mathbb{R}^l$, and $\epsilon > 0$ is a small parameter. The *geometric* theory of such equations is due to Fenichel [31, 32], and good introductions to the topic can be found in [46, 56]. By setting $\tau = \epsilon t$ and $\dot{} = \frac{d}{d\tau}$, one can rewrite (5.1) on the slow timescale as

$$\begin{aligned} \epsilon \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= g(x, y, \epsilon) \end{aligned}. \tag{5.4}$$

As long as $\epsilon \neq 0$, these systems are equivalent. However, the limits $\epsilon \to 0$ in (5.3) and in (5.4) yield very different systems. The former is the n-dimensional system

$$\begin{aligned} x' &= f(x, y, 0) \\ y' &= 0 \end{aligned}, \tag{5.5}$$

in which y plays the role of a parameter. The latter limit leads to the differential-algebraic system

$$\begin{aligned} 0 &= f(x, y, 0) \\ \dot{y} &= g(x, y, 0). \end{aligned} \tag{5.6}$$

In this system, y is the dynamic variable, and x is a slave to these variables, as it must be ensured that the algebraic condition holds. Broadly speaking, the goal of geometric singular perturbation theory is to reconcile these two reduced systems for $\epsilon > 0$ but small. It is important to note that this is *not* a dimension reduction, such as one would obtain by considering radial solutions of a PDE. Whereas a dimension reduction necessarily sacrifices interesting behavior, GSP allows us to study lower dimensional problems in order to uncover genuinely high-dimensional (i.e. $n + l$) phenomenon.

To set some terminology, it is common to call (5.5) the *layer problem* and (5.6) the *slow* or *reduced problem.*

Intuitively, the dynamics of (5.3) should be dominated by (5.5) when $f(x, y, \epsilon) = O(1)$. It follows that the set

$$M_0 = \{(x, y) \in \mathbb{R}^{n+l} : f(x, y, 0) = 0\} \tag{5.7}$$

should play an important role in the analysis. This is indeed the case, and we call M_0 the *critical manifold.* The use of the term "manifold" will be justified below. Notice the role of the set $f(x, y, 0) = 0$ for the fast and slow subsystems. For (5.5) with $\epsilon = 0$, M_0 is the set of critical points, so the dynamics on this set are trivial. For (5.6), this is the algebraic condition that must hold for the equation to be well-defined.

It is instructive to think of the layer problem as equilibrating to a point on M_0, at which point the slow flow takes over. In order to make this rigorous, one makes the technical assumption that

$$\det D_x f(\hat{x}, \hat{y}, 0) \neq 0 \tag{5.8}$$

at any point $(\hat{x}, \hat{y}) \in M_0$. This condition–called *normal hyperbolicity*–ensures that critical points of the layer problem are hyperbolic. Assuming that f is sufficiently smooth, one sees from the Implicit Function Theorem that M_0 is locally an l-dimensional manifold near points at which (5.8) holds. (Thus we call M_0 the critical *manifold.*) The Implicit Function Theorem also implies that M_0 can (locally) be expressed as the graph of a function of the slow variables.

Although much of the current interest in geometric singular perturbation theory concerns systems in which normal hyperbolicity breaks down, for the FitzHugh-Nagumo equation it will be sufficient to study the normally hyperbolic case. For the rest of this section, we therefore make the following assumptions.

(H1) The critical manifold M_0 is a compact manifold (possibly with boundary) which is normally hyperbolic relative to (5.5).

(H2) M_0 is given as the graph of a smooth function $x = h_0(y)$ for y in some set K. K is a compact, simply connected domain whose boundary is an $(l-1)$-dimensional smooth submanifold.

As a remark, it is quite often the case that the entire set of points defined by (5.7) does not satisfy

(H1) - (H2). Instead, one might consider a compact subset of M_0 on which these assumptions hold. We will still refer to this subset as the critical manifold; this should not be a source of confusion.

We say that a set M is *locally invariant* under the flow of (5.3) if it has a neighborhood V such that no trajectory can leave M without also leaving V. The qualifier "local" allows for the possibility that trajectories escape through the boundary of the set, which indeed is the source of much interesting behavior in singularly perturbed systems. The following theorem is due to Fenichel [31].

Theorem 5.1.1 (Fenichel's First Theorem). *If $\epsilon > 0$ is sufficiently small, then there exists a manifold M_ϵ that lies within $O(\epsilon)$ of M_0 and is diffeomorphic to M_0. Moreover it is locally invariant under the flow of (5.3). Under (H1) - (H2), we can write*

$$M_\epsilon = \{(x, y) : x = h_\epsilon(y)\}, \tag{5.9}$$

with $\lim_{h\to\infty} h_\epsilon(y) = h_0(y)$.

M_ϵ is called the *slow manifold*, in light of the fact that the dynamics on it occur on the slow timescale. M_ϵ behaves like a center manifold, so it consequently shares some of the same properties. Most notably, M_ϵ is generally *not* unique. It is also C^r, including in ϵ, for any $r < +\infty$. The local invariance of M_ϵ and the fact that it is parametrized by y make it easy to write down the equation on the slow manifold. Explicitly, it is given by

$$y' = \epsilon g(h_\epsilon(y), y, \epsilon). \tag{5.10}$$

The advantage of this formulation is that the $\epsilon \to 0$ limit on the slow timescale is now regular, giving a flow

$$\dot{y} = g(h_0(y), y, 0) \tag{5.11}$$

on M_0. One can see that (5.11) gives the leading order approximation to the flow on M_ϵ, a fact that will get used heavily in this thesis. It is clear that Fenichel's result is quite powerful, especially in the case where $l = 1$ or $l = 2$, whence the slow dynamics are easy to understand. For applications of Fenichel's First Theorem to proving the existence of traveling waves, the interested reader can consult [46, §1.3-1.5].

What Fenichel's First Theorem does *not* tell us is how the fast and slow dynamics interact with each other. The key to understanding how this happens is to recall that M_0 defines the set of critical points for (5.5). As such, each point in M_0 has attendant stable and unstable manifolds. Taking the union over points in M_0, one obtains stable and unstable manifolds of M_0. Since M_0 is normally hyperbolic and compact, $D_x f(\hat{x}, \hat{y}, 0)$ has eigenvalues with real part bounded away from the imaginary axis. Consequently, the dimensions of the stable and unstable subspaces are constant over all of M_0. Let m be the dimension of the stable subspace at each point and k the dimension of the unstable subspace. Thus $m + k = n$. It is clear then that $\dim W^s(M_0) = m + l$, and $\dim W^u(M_0) = k + l$. Conveniently, these stable and unstable manifolds survive under the perturbation supplied by ϵ.

Theorem 5.1.2 (Fenichel's Second Theorem)**.** *If $\epsilon > 0$ is sufficiently small, then there exist manifolds $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$ that lie within $O(\epsilon)$ of, and are diffeomorphic to, $W^s(M_0)$ and $W^u(M_0)$ respectively. Moreover they are each locally invariant under the flow of (5.3) and C^r for any $r < +\infty$.*

The points on M_ϵ are typically *not* fixed points, so we should justify the use of the term (un)stable manifold. This is done by characterizing the rate at which solutions are attracted to or repelled from M_ϵ. As with the standard invariant manifold theorems, one picks out a decay rate based on the spectral gap given by the eigenvalues of the linearization about the fixed points in M_ϵ. Hypotheses (H1)-(H2) guarantee that this can be done in a uniform manner. One must be careful, however, with the *local* invariance of M_ϵ.

Theorem 5.1.3 (Decay estimates for (un)stable manifolds of M_ϵ)**.** *There exists a neighborhood D of M_ϵ and constants $\kappa_s > 0$ and $\alpha_s < 0$ so that if $v \in W^s(M_\epsilon)$ and $v \cdot [0, t] \subset D$ with $t > 0$, then*

$$d(v \cdot t, M_\epsilon) \leq \kappa_s e^{\alpha_s t}. \tag{5.12}$$

Likewise, there are $\kappa_u > 0$ and $\alpha_u > 0$ so that if $v \in W^u(M_\epsilon)$ and $v \cdot [0, t] \subset D$ with $t < 0$, then

$$d(v \cdot t, M_\epsilon) \leq \kappa_u e^{\alpha_u t}. \tag{5.13}$$

There is one final fact about $W^{s/u}(M_\epsilon)$ that will aid with the calculation of the Maslov index later on. For $\epsilon = 0$, we have the nice picture of $W^{s/u}(M_\epsilon)$ being built by taking the union of the individual stable and unstable manifolds over all points in M_0. At first glance, it seems impossible that this *fiber* structure would persist for $\epsilon \neq 0$, since the same points are not equilibria for $\epsilon \neq 0$. However, it turns out that the notion of fibering (or *foliation*) can be retained. To state the theorem properly, we define the *forward evolution of a set $A \subset D$ restricted to D* to be

$$A \cdot_D t = \{x \cdot t : x \in A \text{ and } x \cdot [0,t] \subset D\}. \tag{5.14}$$

As before, we have to include the caveat that trajectories can leave the neighborhood D of M_ϵ, at which point any estimates obtained from Theorem 5.1.3 no longer apply. We now have the following result, describing the phenomenon of "Fenichel Fibering."

Theorem 5.1.4 (Fenichel's Third Theorem). *Let $v_\epsilon \in M_\epsilon$ smoothly approach a point $v_0 \in M_0$. Then there is an m-dimensional manifold*

$$W^s(v_\epsilon) \subset W^s(M_\epsilon) \tag{5.15}$$

and an l-dimensional manifold

$$W^u(v_\epsilon) \subset W^u(M_\epsilon), \tag{5.16}$$

lying within $O(\epsilon)$ of, and diffeomorphic to, $W^s(v_0)$ and $W^u(v_0)$ respectively. Moreover, they are C^r (including in v and ϵ) for any $r < +\infty$. The family $\{W^s(v_\epsilon) : v_\epsilon \in M_\epsilon\}$ is invariant in the sense that

$$W^s(v_\epsilon) \cdot_D t \subset W^s(v_\epsilon \cdot t), \tag{5.17}$$

if $v_\epsilon \cdot s \in D$ for all $s \in [0,t]$. Likewise the family $\{W^u(v_\epsilon) : v_\epsilon \in M_\epsilon\}$ is invariant in the sense that

$$W^u(v_\epsilon) \cdot_D t \subset W^s(v_\epsilon \cdot t), \tag{5.18}$$

if $v_\epsilon \cdot s \in D$ for all $s \in [t,0]$.

All three of Fenichel's theorems are proved by making a change of coordinates to accentuate the stable and unstable directions. The most simplified version of the equations–dubbed the *Fenichel coordinates* in [50]–will be used in this chapter to prove the existence of the fast traveling pulses. In particular, they will be used to state the Exchange Lemma [46, 50, 77], which allows one to track the passage of an invariant manifold as it passes by M_ϵ. Instead of stating the abstract form of the Fenichel coordinates here, we will simply apply them directly to the FitzHugh-Nagumo example as the need arises.

5.2 Singular solution for FitzHugh-Nagumo pulses

In this section, we identify the fast-slow structure of the FitzHugh-Nagumo traveling wave ODE and describe the singular solution off which the fast pulses perturb. We begin with the traveling wave ODE, which is written as first-order system by setting $u_z = w$ and $v_z = \epsilon y$:

$$U_z = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}_z = \begin{pmatrix} w \\ \epsilon y \\ -cw - f(u) + v \\ -cy + \gamma v - u \end{pmatrix} = F(U). \tag{5.19}$$

This is a fast-slow system with three fast variables (u, w, y) and one slow variable v. The traveling pulses will be constructed as homoclinic orbits to 0. We will also make use of the linearization of (5.19) around various points, which is given by

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \\ -f'(u) & 1 & -c & 0 \\ -1 & \gamma & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}. \tag{5.20}$$

Using the fact that $f'(0) = -a$, a routine calculation gives that the eigenvalues of the linearization of (5.2) at $(0, 0)$ are

$$\begin{aligned} \eta_1 &= \frac{1}{2}\{-(a + \epsilon\gamma) - \sqrt{(a - \epsilon\gamma)^2 - 4\epsilon}\} \\ \eta_2 &= \frac{1}{2}\{-(a + \epsilon\gamma) + \sqrt{(a - \epsilon\gamma)^2 - 4\epsilon}\} \end{aligned}, \tag{5.21}$$

which are clearly real, distinct, and negative for $\epsilon > 0$ sufficiently small. It follows that (A1) - (A2) from Chapter 4 hold, and hence we have real, distinct eigenvalues $\mu_i(0)$ for the linearization about 0 in (5.19). These eigenvalues satisfy

$$\begin{gathered}\mu_1(0) < \mu_2(0) < 0 < -c < \mu_3(0) < \mu_4(0) \\ \mu_1(0) + \mu_4(0) = \mu_2(0) + \mu_3(0) = -c,\end{gathered} \tag{5.22}$$

cf. (4.7). Explicitly, one can use (2.12) to compute that the eigenvalues are given by

$$\begin{aligned}\mu_1(0) &= -\frac{c}{2} - \frac{1}{2}\sqrt{c^2 + 2(\gamma\epsilon + a) + 2\sqrt{(\gamma\epsilon - a)^2 - 4\epsilon}} \\ \mu_2(0) &= -\frac{c}{2} - \frac{1}{2}\sqrt{c^2 + 2(\gamma\epsilon + a) - 2\sqrt{(\gamma\epsilon - a)^2 - 4\epsilon}} \\ \mu_3(0) &= -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 + 2(\gamma\epsilon + a) - 2\sqrt{(\gamma\epsilon - a)^2 - 4\epsilon}} \\ \mu_4(0) &= -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 + 2(\gamma\epsilon + a) + 2\sqrt{(\gamma\epsilon - a)^2 - 4\epsilon}}\end{aligned}. \tag{5.23}$$

We therefore have two-dimensional stable and unstable manifolds, $W^s(0)$ and $W^u(0)$. Denote by $V^{s/u}(0)$ the associated tangent spaces at 0.

The goal is to construct φ by showing that the stable and unstable manifolds intersect. Ideally, this would be accomplished by showing that the intersection exists when $\epsilon = 0$, and then perturbing to the case $\epsilon > 0$. However, we would need $W^u(0)$ and $W^s(0)$ to intersect transversely when $\epsilon = 0$ to make this argument, as this would ensure that the intersection is not broken when ϵ is "turned on." This is inevitably *not* the case, since two two-dimensional submanifolds of $\mathbb{R}^4$ cannot intersect transversely in a one-dimensional set. To remedy this, we append the equation $c' = 0$ to (5.19) to obtain three-dimensional center-stable and center-unstable manifolds, $W^{cs}(0)$ and $W^{cu}(0)$. The phase space is now $\mathbb{R}^5$, and it follows from [57, p. 144] that the transverse intersection $W^{cu}(0) \pitchfork W^{cs}(0)$ is one-dimensional. Thus if we can prove that this transverse intersection exists when $\epsilon = 0$, it would follow that it persists to the case $\epsilon > 0$, proving the existence of the wave. Ironically, it will be necessary to use information from the perturbed system to conclude that the transverse intersection exists when $\epsilon = 0$. The technical tool that makes this connection is the Exchange Lemma, which will be discussed in §5.4.

Setting $\epsilon = 0$ in (5.19), one arrives at the layer problem

$$\begin{pmatrix} u \\ w \\ y \end{pmatrix}' = \begin{pmatrix} w \\ -cw + v - f(u) \\ -cy + \gamma v - u \end{pmatrix}. \tag{5.24}$$

Comparing with (5.7), the one-dimensional critical manifold is given by

$$M_0 = \{(u, v, w, y) : v = f(u), w = 0, y = \frac{1}{c}(\gamma v - u)\}, \tag{5.25}$$

which is easily seen to be normally hyperbolic as long as $f'(u) \neq 0$. We define $\zeta = \epsilon z$ to be the slow timescale to avoid confusion later with the variable τ. Denoting $\dot{} = \frac{d}{d\zeta}$, one sees that the flow on M_0 is given by

$$\dot{v} = y = \frac{1}{c}(\gamma v - f^{-1}(v)). \tag{5.26}$$

By f^{-1}, we mean the inverse of f restricted to one of three segments of the cubic $v = f(u)$, partitioned by the two zeros of $f'(u)$. Of particular interest are the two outer branches corresponding to the intervals on which $f(u)$ is strictly decreasing. We use the notation M_0^L and M_0^R for the left and right branches respectively.

We are now ready to describe the singular orbit. It will be composed of two slow trajectories–one on each of $M_0^{L/R}$–which are connected by two heteroclinic orbits between the branches. The reader will notice that the waves are very similar to those constructed for the version of (5.1) without diffusion on v; see [44, 51]. In fact, with the y variable tacit, the pictures in (u, v, w)-space are identical (albeit with different labels). A picture of the singular orbit is given in Figure 5.2 below. We will call the equation studied in [44, 51] the '3D system,' in reference to the dimension of phase space for the traveling wave equation.

We begin by describing the heteroclinic connection from M_0^L to M_0^R, called the *fast front* or *first fast jump*. Notice that the equations for u and w decouple from y, so the projection of any solution of (5.24) onto the $uw-$plane will be (part of) a solution to

$$\begin{pmatrix} u \\ w \end{pmatrix}' = \begin{pmatrix} w \\ -cw + v - f(u) \end{pmatrix}. \tag{5.27}$$

This system is considered in the construction of traveling waves for the 3D system. It is shown in [63] that for $v = 0$ and

$$c = c^* := \sqrt{2}(a - 1/2) < 0, \tag{5.28}$$

there exists a heteroclinic orbit connecting the fixed point $(0, 0)$ with the fixed point $(1, 0)$. The explicit solution is given by

$$u(z) = \frac{1}{\left(1 + e^{-\frac{\sqrt{2}}{2}z}\right)}, \tag{5.29}$$

which in turn determines w. One can solve for w as a function of u to get the profile in phase space. It is easy to verify that

$$w(u) = \frac{\sqrt{2}}{2}u(1 - u), \quad 0 \le u \le 1 \tag{5.30}$$

is the profile of the fast jump, together with the fixed points.

To show that the corresponding connection exists for (5.24), consider the linearization of (5.24) about 0:

$$\begin{pmatrix} \delta u \\ \delta w \\ \delta y \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ a & -c^* & 0 \\ -1 & 0 & -c^* \end{pmatrix} \begin{pmatrix} \delta u \\ \delta w \\ \delta y \end{pmatrix}. \tag{5.31}$$

The eigenvalues of the matrix in (5.31) are

$$\mu_1(0) = -a\sqrt{2},\ \mu_3(0) = -c^*,\ \text{and}\ \mu_4(0) = \frac{\sqrt{2}}{2}, \tag{5.32}$$

which one would also obtain by substituting $\epsilon = 0$ and $c = \sqrt{2}(a - 1/2)$ in (5.23). We therefore have a two-dimensional unstable manifold, and we wish to find an intersection with the stable manifold of $p = (1, 0, -1/c^*)$. As noted above, the y direction is invariant–one can see from (5.31) that $[0, 0, 1]^T$ is the $-c-$eigenvector–so the unstable manifold is actually a cylinder over the heteroclinic connection for (5.27). To avoid confusion with the unstable manifold for (5.19), we will call this set

$W^u(0_f)$, where the subscript indicates that this is the origin for the fast subsystem (5.24).

If we now linearize about the landing point p, we obtain

$$\begin{pmatrix} \delta u \\ \delta w \\ \delta y \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 1-a & -c^* & 0 \\ -1 & 0 & -c^* \end{pmatrix} \begin{pmatrix} \delta u \\ \delta w \\ \delta y \end{pmatrix}, \tag{5.33}$$

which still has two unstable and one stable eigenvalues given by

$$\mu_1(p) = -\frac{\sqrt{2}}{2},\ \mu_3(p) = -c,\ \text{and}\ \mu_4(p) = \sqrt{2}(1-a). \tag{5.34}$$

In particular, the δy direction is still invariant and unstable. Now, any trajectory in the cylinder $W^u(0_f)$ (for $0 < u < 1$) must approach the invariant line $\{(u, w, y) : u = 1, w = 0\}$ in forward time. Since this line moves points away from the equilibrium p, there will be some points in $W^u(0_f)$ for which $y \to +\infty$ and others for which $y \to -\infty$. A shooting argument in the cylinder therefore produces an orbit that is bounded, which can only approach p. This orbit coincides with the one-dimensional stable manifold of p.

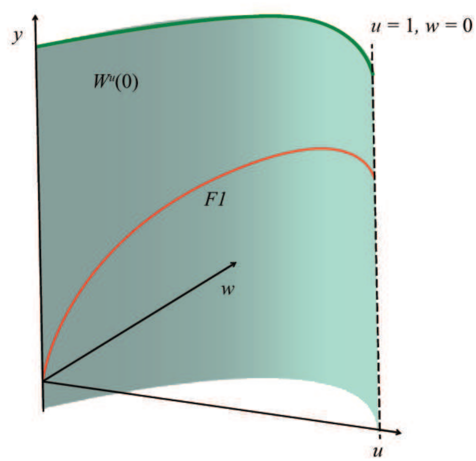


Figure 5.1: Unstable manifold $W^u(0_f)$ near the fast front with v suppressed. The red curve is the fast jump.

Before moving to the slow flow, we point out one important fact about the fast jump. Since $u(z)$ and $w(z)$ are also solutions of the 3D system, they must decay to 0 as $z \to -\infty$ like $e^{\mu_4(0)z}$, since $\mu_4(0)$ is the only unstable eigenvalue for the linearization of (5.27) at 0. On the other hand, y

satisfies

$$y' = -c^* y - u, \tag{5.35}$$

which can be solved explicitly as

$$y(z) = Ke^{-c^* z} - e^{-c^* z} \int_{-\infty}^{z} e^{c^* s} u(s)\, ds. \tag{5.36}$$

The constant

$$K = \int_{-\infty}^{\infty} e^{c^* z} u(z)\, dz = \frac{2\pi}{\sqrt{2}\sin\left(\pi(1-2a)\right)} \tag{5.37}$$

is uniquely determined by the requirement that y is bounded at $\pm\infty$. It is then clear that

$$\lim_{z \to -\infty} e^{c^* z} y(z) = K \neq 0. \tag{5.38}$$

In other words, u and w decay more quickly than y in backwards time, so the traveling front is asymptotically tangent to the $y-$axis. This verifies assumption (A3) from the previous chapter, at least in reverse time.

Having established the existence of a connection between the left and right branches of M_0, the next step is to follow the slow flow up M_0^R. We rescale the independent variable as $\zeta = \epsilon z$ and again set $\epsilon = 0$, to arrive at the reduced problem

$$\dot{v} = y, \quad \left(\dot{} = \frac{d}{d\zeta}\right) \tag{5.39}$$

which is restricted to the critical manifold M_0. Since $y = -1/c^* > 0$ at the landing point, v will increase and move up the graph of the cubic f. Eventually it will reach a point v^*, for which (5.27) with $v = v^*$ and $c = c^*$ has a heteroclinic connection back to M_0^L. Using symmetries of the cubic, it can be shown that

$$v^* = f(2/3(a+1)) := f(u^*). \tag{5.40}$$

(See [12, §3.1] for more details.) We call the point $q := (u^*, v^*, 0, 1/c^*(\gamma v^* - u^*))$ the *jump-off* point for M_0^R. The same shooting argument used for the front proves that the back exists for (5.24) as well. Finally, upon landing on M_0^L at the point $u = 2/3(a - 1/2)$, the slow flow carries us back down to 0, which is a fixed point for both the slow and fast systems.

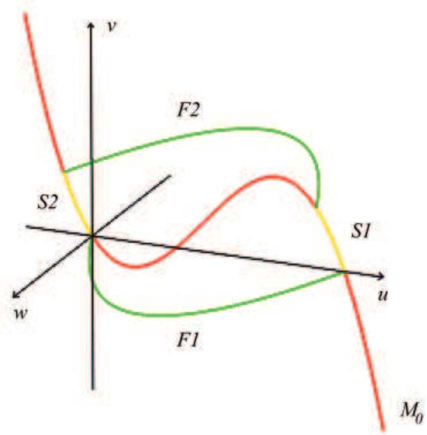


Figure 5.2: Picture of singular orbit, with y suppressed. This orbit is identical to the singular orbit for the '3D system.' See [51].

5.3 Transversality along the front

The mere existence of the singular orbit is not enough to guarantee that a nearby homoclinic orbit will exist for $\epsilon > 0$. The pulse will ultimately be constructed as the transverse intersection of center-stable and center-unstable manifolds. To prove that such an intersection exists, we will appeal to the Exchange Lemma [46, 50, 77], which describes the passage of a shooting manifold near M_ϵ^R. One of the assumptions of the Exchange Lemma is that the shooting manifold transversely intersects $W^s(M_0^R)$ along the fast jump. This section is dedicated to proving that this first transverse intersection exists.

For the rest of this section, any references to (5.19) or its linearization (5.20) assume that the extra equation for c' is included. Since c is a center direction, the fixed point 0 will have a three-dimensional center-unstable manifold $W_\epsilon^{cu}(0)$ and a three-dimensional center-stable manifold $W_\epsilon^{cs}(0)$, whose transverse intersection would be one-dimensional. Notice that the latter is still three-dimensional in the limit $\epsilon = 0$, since the second stable direction becomes a center direction. We include the subscript ϵ to emphasize the dependence of these manifolds on ϵ.

After appending the equation $c' = 0$, the critical manifold is now two-dimensional, parameterized by v and c. If we think of the critical manifold as being the graph of a function $H(v, c)$, it is clear that its tangent space at a generic point P is given by

$$T_P M_0^{R/L} = \mathrm{sp}\left\{\partial_v H, \partial_c H\right\} = \mathrm{sp}\left\{\begin{bmatrix} 1/f'(u) \\ 1 \\ 0 \\ (1/c)(\gamma - 1/f'(u)) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ (-1/c^2)(\gamma v - u) \\ 1 \end{bmatrix}\right\}. \tag{5.41}$$

To highest order, the flow on $M_\epsilon^{L/R}$ for ϵ small or 0 will clearly be tangent to $\partial_v H$, since c is a parameter. This will be important later when we need to select a slow direction. For the sake of completeness, the flow on the critical manifold is now given by

$$\dot{\begin{pmatrix} v \\ c \end{pmatrix}} = \begin{pmatrix} \frac{1}{c}(\gamma v - f^{-1}(v)) \\ 0 \end{pmatrix}. \tag{5.42}$$

Now, recall that $W^{cu}(0)$ and $W^s(M_0^R)$ intersect along the fast jump, which we call

$$q_f(z) = (u(z), 0, w(z), y(z), c^*). \tag{5.43}$$

To prove that this intersection is transverse, it suffices to check any one point along the orbit. A convenient place to check is very close to the landing point $p = (1, 0, 0, -1/c^*, c^*)$. We can therefore take $T_{q_f(z)} W^s(M_0^R)$ to be arbitrarily close to $T_p W^{cs}(p)$. This space is spanned by three vectors: the strong stable direction (which is the direction of the orbit), and the vectors $\partial_v H$ and $\partial_c H$, evaluated at p. $T_{q_f(z)} W^{cu}(0)$, on the other hand, is spanned by $q_f'(z)$, the invariant y direction, and one more vector, which gives the change in $W^{cu}(0)$ as c varies. To find this vector pick any point $q_f(z_0)$ on the fast jump. Since c is a parameter (i.e. there is no flow in this direction), we can find a tangent vector $Y_0 \in T_{q_f(z_0)} W^{cu}(0)$ of the form

$$Y_0 = (*, *, *, *, 1). \tag{5.44}$$

We can then pick a curve $\alpha(c) : (c^* - \delta, c^* + \delta) \to W^{cu}(0)$ such that $\alpha'(c^*) = Y_0$ and $\alpha(c^*) = q_f(z_0)$. By flowing the points on $\alpha(c)$ backwards in z, we obtain a one-parameter family of curves $\Gamma(z, c)$ in

$W^{cu}(0)$. By construction, this family satisfies

$$\partial_c \Gamma(z,c)|_{c=c^*} \in T_{q_f(z)} W^{cu}(0) \tag{5.45}$$

for all $z \in \mathbb{R}$. Of interest then is the direction of the vector $\partial_c \Gamma(z,c)|_{c=c^*}$ as $z \to \infty$ (i.e. as the jump approaches p). This is ascertained by observing that $\partial_c \Gamma|_{c=c^*}$ satisfies the variational equation for (5.19) along $q_f(z)$ with $\epsilon = 0$. Indeed, using the equality of mixed partials, we have

$$\partial_z \left(\partial_c \Gamma(z,c)\right)|_{c=c^*} = DF(\Gamma(z,c)) \cdot \partial_c \Gamma(z,c)|_{c=c^*} = DF(q_f(z)) \cdot \left(\partial_c \Gamma(z,c)\right)|_{c=c^*}. \tag{5.46}$$

Using the notation $\partial_c \Gamma(z,c)|_{c=c^*} = (u_c(z), 0, w_c(z), y_c(z), 1)$, it follows that $W^{cu}(0) \pitchfork W^s(M_0^R)$ if and only if

$$\det \begin{bmatrix} u'(z) & u_c(z) & 0 & \frac{1}{a-1} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ w'(z) & w_c(z) & 0 & 0 & 0 \\ y'(z) & y_c(z) & 1 & \frac{1}{c^*}\left(\gamma - \frac{1}{a-1}\right) & (1/c^*)^2 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} = u'(z)w_c(z) - w'(z)u_c(z) \neq 0 \tag{5.47}$$

for $z \gg 1$. We are now prepared to prove transversality.

Lemma 5.3.1. *The invariant manifolds $W_0^{cu}(0)$ and $W^s(M_0^R)$ intersect transversely along the fast jump $q_f(z)$.*

Proof. Following the preceding discussion, the task is to show that $u'(z)w_c(z) - w'(z)u_c(z) \neq 0$ as $q_f(z)$ approaches the landing point p. Confirming this is a Melnikov-type calculation, which we verify using differential forms (cf. [46, §4.3-4.5] or [51]). Indeed this is natural, since the quantity of interest is $(du \wedge dw)$ applied to the first two columns of the matrix in (5.47). In other words, it is one of the six Plücker coordinates (see §2.3) for the plane spanned by $q_f'(z)$ and $\partial_c \Gamma(z,c)$. This quantity can be tracked by using the equation induced on $\bigwedge^2 \mathbb{R}^4$ by the variational equation for (5.19) along the fast jump (with $\epsilon = 0$). See [23, §3] for more detail. Using differential form notation, we compute

$$
\begin{aligned}
(du \wedge dw)' &= du' \wedge dw + du \wedge dw' \\
&= dw \wedge dw + du \wedge (-f'(u)\, du - c^* dw - w\, dc) \\
&= -c^* du \wedge dw - w\, du \wedge dc.
\end{aligned} \tag{5.48}
$$

Here, the derivative refers to how the quantity $du \wedge dw$ changes when applied to two vectors evolving under (5.20) with $\epsilon = 0$. For notational convenience, we set

$$\alpha(z) := du \wedge dw(q_f'(z), \partial_c \Gamma(z,c)|_{c=c^*}). \tag{5.49}$$

Observing that $du \wedge dc(q_f'(z), \partial_c \Gamma(z,c)|_{c=c^*}) = u'(z) = w(z)$, we can apply an integrating factor to (5.48) to get

$$\partial_z \left(e^{c^* z} \alpha \right) = -w^2 e^{c^* z}. \tag{5.50}$$

Since $w \to 0$ faster than $e^{-c^* z}$ as $z \to -\infty$, this equation can be integrated to obtain

$$\alpha(z) = -e^{-c^* z} \int_{-\infty}^{z} e^{c^* s} w^2\, ds, \tag{5.51}$$

from which it is clear that

$$\lim_{z \to \infty} e^{c^* z} \alpha = L < 0. \tag{5.52}$$

The factor $e^{c^* z}$ ensures that the vectors $q_f'(z)$ and $\partial_c \Gamma(z,c)|_{c=c^*}$ stay bounded and nonzero in the limit. The Lemma then follows, since it is the direction of these vectors (and not the magnitude) that is of interest. $\square$

Notice that γ played no roll in the result of this subsection. Accordingly, the following is a byproduct of the proof of the Lemma.

Corollary 5.3.1. *Assume that* $\gamma > 0$ *is large enough so that* $u = \gamma f(u)$ *has three solutions* u_i *satisfying* $0 = u_1 < u_2 < u_3$. *Then for* $\epsilon > 0$ *sufficiently small, (5.19) possesses a heteroclinic orbit connecting the fixed points* $(0,0,0,0)$ *and* $Q := (u_3, u_3/\gamma, 0, 0)$ *for* $c = c^* + O(\epsilon)$. *The heteroclinic orbit corresponds to a traveling front solution for (5.1) and is locally unique.*

Proof. It is clear that (5.19) has three fixed points for the prescribed values of γ, and that $Q \in M_0^R$

for $\epsilon = 0$. A heteroclinic connection exists between the two points if the unstable manifold of 0 intersects the stable manifold of Q. From Fenichel theory [32], the limit as $\epsilon \to 0$ of $W^{cs}_{\epsilon}(Q)$ is exactly $W^s(M_0^R)$, which we just proved intersects $W^{cu}(0)$ transversely. The transverse intersection perturbs to the case $\epsilon > 0$, and the orbit in question is given by the intersection. The local uniqueness and dependence of the speed c on ϵ are a consequence of the Implicit Function Theorem. □

We remark that the framework of [21] and the calculation in Chapter 7 can be adapted to show that the traveling front just obtained is stable in the sense of Definition 1.1. However, we will not pursue that further here.

5.4 Exchange Lemma and completion of the construction

Armed with an understanding of $W^{cu}(0)$ as it moves along the front, we now turn our attention to the passage near the slow manifold M_{ϵ}^R and the back. As explained in §5.2, specifically (5.40), the Nagumo back is a heteroclinic connection between the jump-off point

$$q = (u^*, v^*, 0, (1/c^*)(\gamma v^* - u^*), c^*) \in M_0^R \tag{5.53}$$

and the point

$$\hat{q} = (u^* - 1, v^*, 0, (1/c^*)(\gamma v^* - u^* + 1), c^*) \tag{5.54}$$

on M_0^L. In $uw-$space, the equations take the form

$$u_b(z) = u^* - u_f(z), \quad w_b(z) = -w_f(z), \tag{5.55}$$

where u_f, w_f are the components of the front. The important facts for this section are that u_b is monotonically decreasing, and w_b decreases to a minimum and then increases from there. Now we focus on the first slow piece connecting the fast jumps, which involves an application of the Exchange Lemma.

To state and use the Exchange Lemma, we first rewrite the traveling wave equations (5.19) in Fenichel coordinates, see [46, 50] for more details. In a neighborhood B of the slow manifold M_{ϵ}^R, we can change coordinates so that (5.19) takes the form

$$\begin{aligned}\mathbf{a}' &= \Lambda(\mathbf{a}, \mathbf{b}, \mathbf{y}, \epsilon)\mathbf{a} \\ \mathbf{b}' &= \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{y}, \epsilon)\mathbf{b} \\ \mathbf{y}' &= \epsilon\left(U + G(\mathbf{a}, \mathbf{b}, \mathbf{y}, \epsilon)\right),\end{aligned} \tag{5.56}$$

with $\mathbf{a} \in \mathbb{R}^2, \mathbf{b} \in \mathbb{R}, \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, and $U = (1, 0)$. The region B can be taken to be of the form

$$B = \{(\mathbf{a}, \mathbf{b}, \mathbf{y}) : |\mathbf{a}| < \delta, |\mathbf{b}| < \delta, \mathbf{y} \in K\}, \tag{5.57}$$

where δ is small and K is a compact set containing the landing point p and jump-off point q in its interior. On account of normal hyperbolicity of M_0^R, we know that for sufficiently small δ the eigenvalues of $\Lambda(0, 0, \mathbf{y}, 0)$ are real, positive, and uniformly bounded away from 0. Likewise, $\Gamma(0, 0, \mathbf{y}, 0) < C_\delta < 0$. The function G in (5.56) is bilinear in $\mathbf{a}, \mathbf{b}$ due to the fact that the sets $\mathbf{a} = 0$ and $\mathbf{b} = 0$ are invariant. The special form of the $\mathbf{y}$ component is obtained by rectifying the flow on the slow manifold. It is clear that for this problem U is the "straightened out" graph of the cubic for fixed c–that is, the flow is in the direction $\partial_v H$–since there is no change in c in the trajectory through any point.

The (C^1) Exchange Lemma describes the configuration of a manifold of trajectories that spends a long time near M_ϵ^R before leaving the neighborhood B. The manifold of interest in our case is $W_\epsilon^{cu}(0)$, which the reader will recall is three-dimensional. (The subscript serves to emphasize the ϵ-dependence.) The following statement of the Exchange Lemma is specialized to the setting of (5.19). For the general statement and proof, the reader is directed to [50], or [56] for a sketch.

Theorem 5.4.1 ("Exchange Lemma" of [50])**.** *Assume that $W_0^{cu}(0) \pitchfork W^s(M_0^R)$. Let J be a compact segment of the trajectory through p for the limiting slow flow (5.42) that contains q. Then*

1. *For any $r_0 \in W_0^u(J) \cap \partial B$, there exists $q_\epsilon \in W_\epsilon^{cu}(0) \cap \partial B$ and a time $T_\epsilon > 0$ such that $q_\epsilon \cdot T_\epsilon \in \partial B$ and $|q_\epsilon \cdot T_\epsilon - r_0| = O(\epsilon)$. Furthermore, $T_\epsilon = O(\epsilon^{-1})$.*

2. *Let $\bar{q} \in W_\epsilon^{cu}(0) \cap \{|\mathbf{a}| = \delta\}$ be the exit point of a trajectory through $q \in W_\epsilon^{cu}(0) \cap \{|\mathbf{b}| = \delta\}$ that spends time $T = O(\epsilon^{-1})$ in B. Let $V \subset W_\epsilon^{cu}(0)$ be a neighborhood of q. Then the image*

of V under the time T map is $O(\epsilon)$-close in C^1 norm to $W^c(J)$ in a neighborhood of $\bar{q}$.

The first part of the theorem says that we can find points in $W^{cu}_\epsilon(0)$ near p that pass by the slow manifold and then exit the neighborhood B as close to the Nagumo back as we would like. The second part says that for such points, upon exiting the neighborhood B, the shooting manifold $W^{cu}_\epsilon(0)$ will be very close to the manifold $W^u(J)$. The fact that $W^{cu}(0)$ is crushed against an unstable manifold is to be expected, cf. [25, 72]. The strength of the Exchange Lemma lies in telling us which slow direction is picked out. (Recall that $W^u(M^R_\epsilon)$ is four-dimensional with the c equation appended, so there is only room for one of the two slow directions.) The result is that the dominant slow direction is that of the trajectory connecting the landing point and jump-off point. On the level of tangent planes, we have

$$T_{\bar{q}}W^{cu}_\epsilon(0) \approx T_q W^u(q) \oplus \mathrm{sp}\{\partial_v H(q)\}. \tag{5.58}$$

To complete the construction of the pulse, the final ingredient we need is that $W^u_0(J)$ intersects $W^s(M^L_0)$ transversely along the back. Indeed, the latter is the $\epsilon \to 0$ limit of $W^{cs}_\epsilon(0)$. If $W^s(M^L_0) \pitchfork W^u_0(J)$, then also $W^s(M^L_\epsilon) \pitchfork W^{cu}_\epsilon(0)$, since $W^{cu}_\epsilon(0)$ is $O(\epsilon)$ close to $W^u_0(J)$ by the Exchange Lemma. This is precisely what we need to show–that there is a (one-dimensional) transverse intersection between $W^{cu}_\epsilon(0)$ and $W^{cs}_\epsilon(0)$ for $\epsilon > 0$ small. Since $c' = 0$, the trajectory lying in the intersection therefore represents a homoclinic orbit to 0 for (5.19) with fixed ϵ. The required transversality along the back is recorded in the following lemma.

Lemma 5.4.1. *The invariant manifolds $W^u(J)$ and $W^s(M^R_0)$ intersect transversely along the second fast jump $q_b(z)$.*

The proof is identical to that of Lemma 5.3.1, so we omit the details. The reader is invited to check that it suffices to show that

$$\lim_{z\to\infty} e^{c^* z} du \wedge dw(q_b'(z), \partial_v q_b(z)) < 0, \tag{5.59}$$

where $\partial_v q_b(z)$–akin to $\partial_c \Gamma(z,c)|_{c=c^*}$ from the front–gives the change in the orbit $q_b(z)$ as v varies and is the unknown tangent direction to $W^u(M^R_0)$. The inequality (5.59) is confirmed using the

Melnikov integral

$$\lim_{z\to\infty} e^{c^* z} du \wedge dw(q_b'(z), \partial_v q_b(z)) = \int_{-\infty}^{\infty} e^{c^* z} w_b(z)\, dz. \tag{5.60}$$

As a remark, both transversality conditions are identical to those needed to construct the pulse for the 3D system (see [51] and [55]). The reason for this is that the extra (invariant) y direction is unstable, so it will not be duplicated in the tangent space to $W^s(M_0^{R/L})$ at the respective landing points. This is readily seen from the matrix in (5.47).

Putting together Lemmas 5.3.1 and 5.4.1 with Theorem 5.4.1, we can conclude the main result of this section.

Theorem 5.4.2. *For $\epsilon > 0$ sufficiently small, equation (5.19) possesses an orbit φ_ϵ homoclinic to 0 for a wave speed $c(\epsilon) = c^* + O(\epsilon)$. Furthermore, φ_ϵ is $O(\epsilon)$ close to the singular orbit consisting of two alternating fast and slow segments. Finally, the orbit is locally unique.*

Proof. We have already explained how the results of this section generate a transverse intersection of $W^{cs}_\epsilon(0)$ and $W^{cu}_\epsilon(0)$. The closeness to the singular orbit and the local uniqueness both follow from the Implicit Function Theorem, which is used to continue the transverse intersection to the $\epsilon \neq 0$ case. □

CHAPTER 6

The Maslov Box

We now return to the issue of using the Maslov index to understand the unstable spectrum of the operator L, defined by (2.3). We will work primarily with the eigenvalue equation written as a first-order system, as in (2.6). In this chapter, we will prove the main results of [21]. First, for general skew gradient systems (1.19), we prove that the Maslov index gives a lower bound on the number of unstable eigenvalues of L. The proof–given in §6.1–uses an adaptation of the "Maslov Box" argument of [41]. Next, in §6.2, we focus specifically on the FitzHugh-Nagumo equation (5.1) and prove that the Maslov index gives an exact count of all unstable eigenvalues in that case. In §6.3, we discuss an important connection with the Evans function, which is used to prove that the geometric and algebraic multiplicities of any unstable eigenvalues coincide.

6.1 The Maslov Box

We will see in this section that the set of positive, real eigenvalues of L is bounded above. Since the spectrum of L in $\mathbb{C} \setminus H$–recall (2.11)–consists of isolated eigenvalues of finite multiplicity (cf. [2, p. 172]), it follows that the quantity

$$\overline{\text{Mor}}(L) := \text{ the number of real, nonnegative eigenvalues of } L \text{ counting algebraic multiplicity}$$

is well defined. We use the notation $\overline{\text{Mor}}(L)$ to emphasize that $\lambda = 0$ is included in the count. Later we will use $\text{Mor}(L)$ for the count of strictly positive (i.e. unstable) eigenvalues. The rest of this section is dedicated to proving

Theorem 6.1.1.

$$|\text{Maslov}(\varphi)| \leq \overline{\text{Mor}}(L). \tag{6.1}$$

The strategy of the proof is to consider a contractible loop in $\Lambda(n) \times \Lambda(n)$ (the "Maslov box") consisting of four different curve segments. Since the total Maslov index must be zero, Proposition 3.1.1(i) guarantees that the sum of the constituent Maslov indices is zero. Two of these segments have Maslov index zero, one of them is Maslov(φ), and the final segment is bounded above by $\overline{\mathrm{Mor}}(L)$. This strategy has its roots in [41, 42, 47]. In particular, [41] coined the term "Maslov box," and that paper encounters many of the same difficulties that arise when considering homoclinic orbits (i.e. curves on infinite intervals). The difference between this paper and [41] is that the latter considered gradient reaction-diffusion equations. In that case, the linearized operator L is self-adjoint, and the Maslov index is computed using spectral flow of unitary matrices.

We remind the reader that $E^s(\lambda, z)$ and $E^u(\lambda, z)$–the stable and unstable bundles–each define two-parameter curves in $\Lambda(n)$ for real $\lambda > \beta$. We will also reuse the notation $S(\lambda), U(\lambda)$ for the stable and unstable subspaces of $A_\infty(\lambda)$. In the case $\lambda = 0$–whence (2.6) gives the variational equation for the traveling wave ODE (2.4)–we have $U(0) = V^u(0)$ and $S(0) = V^s(0)$. It follows from Lemma 3.2 of [2] that

$$\begin{aligned} \lim_{z\to-\infty} E^u(\lambda, z) &= U(\lambda) \\ \lim_{z\to\infty} E^s(\lambda, z) &= S(\lambda). \end{aligned} \tag{6.2}$$

In light of Theorem 2.5.2, this actually proves that

$$S(\lambda), U(\lambda) \in \Lambda(n), \tag{6.3}$$

since $\Lambda(n)$ is a closed submanifold of $\mathrm{Gr}_n(\mathbb{R}^{2n})$. In what follows, it will be important to know what happens to $E^u(\lambda, z)$ as $z \to \infty$. First, if $\lambda \in \sigma(L)$, then $E^u(\lambda, z) \cap E^s(\lambda, z) \neq \{0\}$, so it must be the case that

$$\lim_{z\to\infty} E^u(\lambda, z) \in \Sigma(S(\lambda)), \tag{6.4}$$

the train of $S(\lambda)$. On the other hand, if $\lambda \notin \sigma(L)$, then any solution of (2.6) is unbounded at $+\infty$, and it is proved in Lemma 3.7 of [2] that

$$\lim_{z\to\infty} E^u(\lambda, z) = U(\lambda). \tag{6.5}$$

There are a few facts to be gleaned from this observation. First, if $\lambda \notin \sigma(L)$, then $z \mapsto E^u(\lambda, z)$ (with domain $\mathbb{R}$) forms a loop in $\Lambda(n)$, in which case the Maslov index is independent of the choice of reference plane [4, §1.5]. This fact was used in [45], which is the first appearance of the Maslov index for solitary waves (known to the author). Also, it follows from (6.4) and (6.5) that

$$\lim_{z\to\infty} E^u(\lambda, z) \tag{6.6}$$

is discontinuous in λ at each eigenvalue of L. Indeed, $U(\lambda)$ is bounded away from $\Sigma(S(\lambda))$, since $\mathbb{R}^{2n} = S(\lambda) \oplus U(\lambda)$. This is the motivation for using the cutoff x_∞ (or τ in this paper) for the unstable bundle in [41], since the homotopy argument requires a continuous curve. We also have the additional motivation for the cutoff of using Maslov(φ) explicitly.

Proposition 2.2 of [2] guarantees that one can draw a simple, closed curve in $\mathbb{C}$ containing $\sigma(L) \cap (\mathbb{C} \setminus H)$ in its interior. An obvious consequence of this is that the real, unstable spectrum of L is bounded above by a constant M. We will make use of the following, slightly stronger fact.

Lemma 6.1.1. *There exists $\lambda_{\max} > M$ such that, for all $z \in \mathbb{R}$,*

$$E^u(\lambda_{\max}, z) \cap S(\lambda_{\max}) = \{0\}. \tag{6.7}$$

Proof. The intuition for this lemma is that (2.6) is essentially autonomous for large λ, and thus $U(\lambda)$ (which is an attracting fixed point for the autonomous system) is not able to drift far enough away to intersect the train of $S(\lambda)$. Similar arguments are used in [2, §5.B] and [35, §2.2]. We remark that the proof of the corresponding proposition in [41] relies on monotonicity that is not present here, so some adjustments must be made. To make the above rigorous, we introduce the scalings $y = \sqrt{\lambda} z$, $\tilde{p} = p$, and $\tilde{q} = (\lambda)^{-1/2} q$, which transform (2.6) to

$$\frac{d}{dy}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 0 & S \\ S^{-1} - Qf'(\hat{u}(z))/\lambda & -\frac{c}{\sqrt{\lambda}} I \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}. \tag{6.8}$$

(Note that the $'$ in $f'(\hat{u})$ came from the linearization of the traveling wave PDE and is unaffected by the current change of variables.) The scaled equation (6.8) has a corresponding asymptotic system

$$\frac{d}{dy}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 0 & S \\ S^{-1} - Qf'(\hat{u}(0))/\lambda & -\frac{c}{\sqrt{\lambda}} I \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}. \tag{6.9}$$

We call the stable and unstable subspaces of the matrix in (6.9) $\tilde{S}(\lambda)$ and $\tilde{U}(\lambda)$ respectively. Likewise, we use $\tilde{E}^u(\lambda, y)$ for the unstable bundle of (6.8). For fixed λ, it is clear that

$$E^u(\lambda, z^*) \cap S(\lambda) = \{0\} \iff \tilde{E}^u(\lambda, \sqrt{\lambda} z^*) \cap \tilde{S} = \{0\}, \tag{6.10}$$

since rescaling the independent variable does not change the eigenvectors of the asymptotic matrix, and the change in the dependent variables is linear. It therefore suffices to show that for large enough λ, $\tilde{E}^u(\lambda, y)$ is disjoint from $\tilde{S}(\lambda)$. Taking the limit $\lambda \to \infty$, we arrive at the system

$$\frac{d}{dy}\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 0 & S \\ S^{-1} & 0 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}, \tag{6.11}$$

which has stable and unstable subspaces

$$\tilde{S} = \mathrm{sp}\{(a, -S^{-1}a)^T : a \in \mathbb{R}^n\}, \quad \tilde{U} = \mathrm{sp}\{(a, S^{-1}a)^T : a \in \mathbb{R}^n\}. \tag{6.12}$$

Now, system (6.8) is a perturbation of (6.9), and both equations induce flows on $\mathrm{Gr}_n(\mathbb{R}^{2n})$. In the latter system, $\tilde{U}(\lambda)$ is an attracting fixed point, as is observed in the proof of Lemma 3.7 in [2]. We can therefore find a small ball B around $\tilde{U}(\lambda)$ in $\Lambda(n)$ on the boundary of which the vector field points inward. For large enough λ, the perturbation supplied by the non-autonomous part of (6.8) is negligible, so B will still be positively invariant for (6.8). Furthermore, since $\tilde{S}(\lambda)$ and $\tilde{U}(\lambda)$ are still transverse in the limit $\lambda \to \infty$, the radius of the ball can be taken small enough to be disjoint from $\Sigma(\tilde{S}(\lambda))$, *independently of* λ. Finally, since any $\lambda > M$ is not an eigenvalue of L, the curve $y \mapsto \tilde{E}^u(\lambda, y)$ will both emanate from and return to $U(\lambda)$. It will therefore be trapped in the ball B, and hence there will be no intersections with $\tilde{S}(\lambda)$. $\square$

Now fix the value $\lambda_{\max}$ guaranteed by the preceding lemma. The immediate goal is to set, once and for all, the value τ appearing in Definition 3.2. Since $E^s(\lambda_{\max}, z) \to S(\lambda_{\max})$ as $z \to \infty$, it follows from Lemma 6.1.1 that we can find a value $z = \tau_{\max}$ such that

$$E^u(\lambda_{\max}, z) \cap E^s(\lambda_{\max}, \zeta) = \{0\}, \quad \text{for all } z \in \mathbb{R} \text{ and for all } \zeta \geq \tau_{\max}. \tag{6.13}$$

Similarly, for each $\lambda \in [0, \lambda_{\max}]$ we can find τ_λ and an open interval I_λ containing λ such that

$$U(\lambda) \cap E^s(\lambda, z) = \{0\}, \quad \text{for all } z \geq \tau_\lambda, \lambda \in I_\lambda. \tag{6.14}$$

(For $\lambda = \lambda_{\max}$, the value $\tau_{\max}$ defined above works just fine.) Extracting a finite subcover $\cup_{k=1}^{N} I_{\lambda_k}$ of $[0, \lambda_{\max}]$, we set

$$\tau = \max\{\tau_{\lambda_1}, \dots, \tau_{\lambda_k}\}. \tag{6.15}$$

The preceding can be summarized in the following proposition.

Proposition 6.1.1. *With τ given by (6.15), the following are true.*

(i) $E^u(\lambda_{\max}, z) \cap E^s(\lambda_{\max}, \tau) = \{0\}$ for all $z \in (-\infty, \tau]$.

(ii) $U(\lambda) \cap E^s(\lambda, \tau) = \{0\}$ for all $\lambda \in [0, \lambda_{\max}]$.

Now consider the rectangle

$$Q = [0, \lambda_{\max}] \times [-\infty, \tau]. \tag{6.16}$$

Q is mapped into $\Lambda(n) \times \Lambda(n)$ by the function

$$G(\lambda, z) = (E^u(\lambda, z), E^s(\lambda, \tau)), \tag{6.17}$$

where $G(\lambda, -\infty)$ is defined to be $(U(\lambda), E^s(\lambda, \tau))$. Notice that G is continuous, see [2, §3]. Since Q is contractible, the image $G(Q) \subset \Lambda(n) \times \Lambda(n)$ is contractible as well. Let $F : Q \times [0, 1] \to Q$ be a deformation retract [64, p. 361] of Q onto the point $(0, -\infty)$. Composing F and G then gives a deformation retract of $G(Q)$ onto $G(0, -\infty) = (U(0), E^s(0, \tau))$. In particular, we see that the image of the boundary ∂Q (with a counterclockwise orientation) under G is homotopic with fixed endpoints to the constant path $(U(0), E^s(0, \tau))$. We will call this (closed) boundary curve α. Since $U(0) \cap E^s(0, \tau) = \{0\}$ by Proposition 6.1.1(ii), we see that Proposition 3.2.1 applies, so

$$\mu(\alpha) = \mu(U(0), E^s(0, \tau)) = 0. \tag{6.18}$$

The Maslov index in this case is for pairs of Lagrangian planes, since $\alpha \subset \Lambda(n) \times \Lambda(n)$. We can describe the loop α as the concatenation of four curve segments. (See Figure 6.1 below.) Define:

$$\begin{aligned}
\alpha_1 &= (E^u(0,z), E^s(0,\tau)), \quad z \in [-\infty, \tau] \\
\alpha_2 &= (E^u(\lambda,\tau), E^s(\lambda,\tau)), \quad \lambda \in [0, \lambda_{\max}] \\
\alpha_3 &= (E^u(\lambda_{\max}, -z), E^s(\lambda_{\max},\tau)), \quad z \in [-\tau, \infty] \\
\alpha_4 &= (U(\lambda_{\max} - \lambda), E^s(\lambda_{\max} - \lambda, \tau)), \quad \lambda \in [0, \lambda_{\max}].
\end{aligned} \tag{6.19}$$

Using the notation of [64], page 326, it is clear that $\alpha = \alpha_1 * \alpha_2 * \alpha_3 * \alpha_4$. As explained above, $\mu(\alpha) = 0$, since $G(Q)$ is contractible. Proposition 3.1.1(i) then asserts that

$$0 = \mu(\alpha) = \mu(\alpha_1) + \mu(\alpha_2) + \mu(\alpha_3) + \mu(\alpha_4). \tag{6.20}$$

It is a direct consequence of Proposition 6.1.1(i) that $\mu(\alpha_3) = 0$, since there are no conjugate points. Likewise, Proposition 6.1.1(ii) says that $\mu(\alpha_4) = 0$. Comparing (6.19) with Definition 3.2, we see that

$$\mu(\alpha_1) = \mathrm{Maslov}(\varphi). \tag{6.21}$$

Taken together with (6.20), these observations show that

$$|\mathrm{Maslov}(\varphi)| = |\mu(\alpha_2)|. \tag{6.22}$$

To prove Theorem 6.1.1, it therefore suffices to show that

$$|\mu(\alpha_2)| \leq \overline{\mathrm{Mor}}(L). \tag{6.23}$$

Remark 6.1. *Notice in Figure 6.1 the conjugate point in the upper left corner. This crossing corresponds to the translation invariance of (1.2) (i.e. $E^u(0,\tau) \cap E^s(0,\tau) \neq \{0\}$). The contributions of this crossing to $\mu(\alpha_1)$ and $\mu(\alpha_2)$ can be determined using (3.7).*

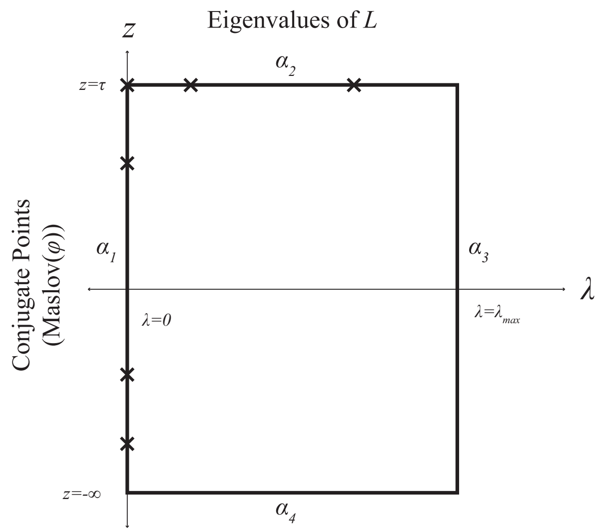


Figure 6.1: "Maslov Box": Domain in λz-plane

Suppose that λ^* is a conjugate point for α_2. By definition, this means that

$$E^u(\lambda^*, \tau) \cap E^s(\lambda^*, \tau) \neq \{0\}. \tag{6.24}$$

But this is precisely the condition that λ be an eigenvalue of L. Furthermore, the dimension of the intersection in (6.24) captures the geometric multiplicity of λ^* as an eigenvalue. By the triangle inequality, we therefore have

$$|\mu(\alpha_2)| \leq \sum_{\lambda^* \in [0, \lambda_{\max}]} \dim(E^u(\lambda^*, \tau) \cap E^s(\lambda^*, \tau)), \tag{6.25}$$

where the sum is taken over all conjugate points. Since $[0, \lambda_{\max}]$ contains all possible real, nonnegative eigenvalues of L, and the geometric multiplicity of an eigenvalue is no greater than its algebraic multiplicity, we see that $|\mu(\alpha_2)| \leq \overline{\mathrm{Mor}}(L)$, proving Theorem 6.1.1.

6.2 Counting unstable eigenvalues for the FitzHugh-Nagumo system

There are two reasons that the inequality in Theorem 6.1.1 cannot be improved to equality in general. First, the Maslov index counts *signed* intersections, so that two different eigenvalues of L might offset in the calculation of $\mu(\alpha_2)$ if the crossing forms have different signatures. Second, a given eigenvalue might be *deficient* (i.e. have lesser geometric than algebraic multiplicity). In the next few sections, we show that neither of these potential pitfalls occurs for the FitzHugh-Nagumo fast pulses. Additionally, we prove that any unstable spectrum must be real and that $\lambda = 0$ contributes 0 to $\mu(\alpha_2)$, so that the Maslov index actually counts the total number of unstable eigenvalues. For this section, the notation $f(u)$ refers to the cubic nonlinearity for the FitzHugh-Nagumo equation (5.1), and not the generic nonlinearity in (1.19).

Recall that the fast traveling pulses (whose existence was proved in Theorem 5.4.2) can be represented as homoclinic orbits $\varphi_\epsilon(z) = (\hat{u}(z), \hat{v}(z), \hat{u}'(z), v'(z)/\epsilon)$ of (5.19). The stability of the waves is determined by the spectrum of

$$L_\epsilon = \partial_z^2 + c\partial_z + \begin{pmatrix} f'(\hat{u}) & -1 \\ \epsilon & -\epsilon\gamma \end{pmatrix}, \tag{6.26}$$

acting on $BU(\mathbb{R}, \mathbb{R}^2)$. The subscript ϵ serves both to remind the reader that the operator is ϵ-dependent and to distinguish results that are general for (1.19) from those that are specific to (5.1). As before, we will study the eigenvalue problem as a first-order system

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \epsilon \\ \lambda - f'(\hat{u}) & 1 & -c & 0 \\ -1 & \frac{\lambda}{\epsilon} + \gamma & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}, \tag{6.27}$$

which we abbreviate

$$Y'(z) = A(\lambda, z)Y(z). \tag{6.28}$$

The results from Chapter 2 about the essential spectrum are still valid, but the spectral gap will depend on ϵ. This is not a problem, since ϵ will be fixed in the stability analysis. However, a few of the results to follow need ϵ to be "sufficiently small." For completeness, we record the following lemma on $\sigma_{\text{ess}}(L_\epsilon)$.

Lemma 6.2.1. *For each $\epsilon > 0$ sufficiently small, there exists $\beta_\epsilon < 0$ such that*

$$\sigma_{\text{ess}}(L_\epsilon) \subset H_\epsilon := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < \beta_\epsilon\}. \tag{6.29}$$

It then follows from (2.12) and (5.21) that for all real $\lambda \geq \beta_\epsilon$, the eigenvalues $\mu_i(\lambda)$ of $A_\infty(\lambda)$

are real and satisfy

$$
\begin{aligned}
\mu_1(\lambda) < \mu_2(\lambda) < 0 < -c < \mu_3(\lambda) < \mu_4(\lambda) \\
\mu_1(\lambda) + \mu_4(\lambda) = \mu_2(\lambda) + \mu_3(\lambda) = -c.
\end{aligned}
\tag{6.30}
$$

The benefit of having simple eigenvalues is that we can give analytically varying bases of $E^s(\lambda, z)$ and $E^u(\lambda, z)$ that separate solutions with different growth rates, cf. §4.2. This will be important in §6.3 when we revisit the symplectic Evans function.

We now proceed to show that any unstable eigenvalues of L_ϵ must be real. After that, we address the issue of direction of crossings by deriving the λ crossing form and showing that it is positive definite at all conjugate points. Finally, in §6.3 we show that the algebraic and geometric multiplicities of any unstable eigenvalues of L_ϵ are the same. This will prove:

Theorem 6.2.1.

$$
\operatorname{Maslov}(\varphi) = \operatorname{Mor}(L_\epsilon) = |\sigma(L_\epsilon) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}|. \tag{6.31}
$$

6.2.1 Realness of $\sigma(L_\epsilon)$

The analysis of L_ϵ is complicated by the presence of the ∂_z term in (6.26). We can sidestep this difficulty by considering instead the operator

$$
L_c := e^{cz/2} L_\epsilon e^{-cz/2}, \tag{6.32}
$$

as is done in [8, 41]. It is a routine calculation to see that for $(p, q)^T \in BU(\mathbb{R}, \mathbb{C}^2)$, we have

$$
L_c \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_{zz} + \left(f'(\hat{u}) - \frac{c^2}{4}\right) p - q \\ q_{zz} + \epsilon p - \left(\frac{c^2}{4} + \epsilon\gamma\right) q \end{pmatrix}. \tag{6.33}
$$

Furthermore, if $L_\epsilon P = \lambda P$, then $L_c(e^{cz/2}P) = \lambda e^{cz/2}P$. This proves that the eigenvalues of L_ϵ and L_c are the same, provided that $e^{cz/2}P$ is bounded for a given eigenvector P of L_ϵ. This is clearly the case as $z \to \infty$ since $c < 0$. For the other tail, let $\lambda \in \mathbb{C} \setminus H_\epsilon$ be an eigenvalue of L_ϵ with associated eigenvector P. Since $A_\infty(\lambda)$ is hyperbolic with simple eigenvalues, P must decay at least as fast as $e^{\mu_3(\lambda)z}$ as $z \to -\infty$. It follows that $e^{cz/2}P$ is bounded at $-\infty$ if

$$
\frac{c}{2} + \mu_3(\lambda) > 0. \tag{6.34}
$$

This is indeed the case, by (6.30). (For complex λ, the same inequality holds with $\mu_i(\lambda)$ replaced by $\operatorname{Re}\mu_i(\lambda)$.) We therefore consider the eigenvalue problem

$$L_c P = \lambda P. \tag{6.35}$$

Making the change of variables $\tilde{q} = \frac{1}{\sqrt{\epsilon}} q$, we can rewrite (6.35) as (dropping the tildes)

$$\begin{pmatrix} \partial_z^2 + \left(f'(\hat{u}) - \frac{c^2}{4}\right) & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & \partial_z^2 - \left(\frac{c^2}{4} + \epsilon\gamma\right) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \lambda \begin{pmatrix} p \\ q \end{pmatrix}. \tag{6.36}$$

L_c is now seen to be of the form

$$L_c = \begin{pmatrix} L_p & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & L_q \end{pmatrix}, \tag{6.37}$$

where $L_{p/q}$ are symmetric operators on $H^1(\mathbb{R})$. Since any eigenfunction of L_c in $BU(\mathbb{R}, \mathbb{C}^2)$ is exponentially decaying (provided $\lambda \in \mathbb{C} \setminus H_\epsilon$) and smooth (by elliptic regularity), we are free to consider the spectrum of L_c as an operator on the Hilbert space $H^1(\mathbb{R}, \mathbb{C}^2)$ instead. This also implies that $L_{p/q}$ are self-adjoint on some domain $\mathcal{D} \subset H^1(\mathbb{R})$ containing any relevant solutions. The payoff of studying L_c instead of L_ϵ is the following result, a general version of which was proved in Lemma 4.1 of [19]. We reproduce the proof here for convenience of the reader.

Lemma 6.2.2. *For $\epsilon > 0$ sufficiently small, if $\lambda \in \sigma_n(L_c) \cap (\mathbb{C} \setminus H_\epsilon)$ and $\operatorname{Re}\lambda \geq -\frac{c^2}{8}$, then $\lambda \in \mathbb{R}$. Consequently, the same is true for L_ϵ.*

Proof. Let $\lambda = a + bi$ be an eigenvalue for L_c with corresponding eigenvector $(p, q)^T$. Assume further that $a \geq -c^2/8$. Notice that the second equation in (6.36) can be solved for q, since $L_q + c^2/8$ is negative definite (and hence $\lambda \notin \sigma(L_q)$). Explicitly, we have

$$q = -\sqrt{\epsilon}\,(L_q - a - bi)^{-1}\, p. \tag{6.38}$$

Next, substitute this expression into the first equation of (6.36) to obtain

$$L_p p + \epsilon (L_q - a - bi)^{-1} p = (a + bi) p. \tag{6.39}$$

Taking the H^1 pairing $\langle \cdot, \cdot \rangle$ with p in (6.39) yields

$$\langle L_p p, p\rangle + \epsilon \langle (L_q - a - bi)^{-1} p, p\rangle = (a+bi)\langle p, p\rangle. \tag{6.40}$$

Recalling that L_p is self-adjoint on $\mathcal{D}$, we extract the imaginary parts of (6.40):

$$\epsilon \, \mathrm{Im}\langle (L_q - a - bi)^{-1} p, p\rangle = b\langle p, p\rangle. \tag{6.41}$$

The operator inverse in (6.41) can be decomposed into (self-adjoint) real and imaginary parts as follows:

$$(L_q - a - bi)^{-1} = \left((L_q - a)^2 + b^2\right)^{-1} (L_q - a) + ib\left((L_q - a)^2 + b^2\right)^{-1}. \tag{6.42}$$

Combining (6.41) and (6.42), we arrive at

$$b\langle \left[\epsilon \left((L_q - a)^2 + b^2\right)^{-1} - I\right] p, p\rangle = 0, \tag{6.43}$$

where I denotes the identity operator. For operators on $H^1(\mathbb{C}, \mathbb{C}^2)$ we write $A < B$ if $(B - A)$ is positive definite. Since $L_q - a < 0$ (independently of ϵ) it follows from the inequality

$$\left((L_q - a)^2 + b^2\right)^{-1} < (L_q - a)^{-2} \tag{6.44}$$

and the fact that $(L_q - a)^{-2}$ is bounded that

$$\epsilon \left((L_q - a)^2 + b^2\right)^{-1} - I < 0 \tag{6.45}$$

for ϵ small enough. In conjunction with (6.43), this implies that $b = 0$, as desired. □

6.2.2 The crossing form in λ

Recall that conjugate points along α_2 correspond to eigenvalues of L_ϵ. We will show below that the crossing form (in λ) is positive definite at all such crossings. This is the most significant difference between skew-gradient systems and the gradient systems considered in [41], since the crossing form is always positive definite in the latter case (cf. §4.1 and §5.5). Conversely, we rely on the smallness of ϵ to get monotonicity of the crossings for L_ϵ in (6.26). We stress that the λ

crossing form developed in this section would be the same for general systems (1.19). We focus on L_ϵ only because we are able to prove that the form is positive definite in this case.

To derive the λ crossing form, we first take a closer look at the z crossing form from Theorem 3.4.2. Suppose that z^* is a conjugate point for α_1, and that $\xi \in E^u(0,z^*) \cap E^s(0,\tau)$. By virtue of being in $E^u(0,z^*)$, we know that there exists a solution $u(z)$ of (6.27) such that $u(z) \in E^u(0,z)$ and $u(z^*) = \xi$. It follows that (3.18) can be rewritten

$$\Gamma(E^u(0,\cdot), E^s(0,\tau), z^*)(\xi) = \omega(\xi, A(0,z^*)\xi) = \omega(u(z), \partial_z u(z))|_{z=z^*}. \tag{6.46}$$

In other words, the crossing form simplifies when evaluated on a vector that is part of a solution to a differential equation.

Now suppose that $\lambda = \lambda^*$ is a conjugate point for α_2, with $\xi \in E^u(\lambda^*,\tau) \cap E^s(\lambda^*,\tau)$. From (3.6), we know that we must evaluate two crossing forms–one where the curve $E^s(\lambda,\tau)$ is frozen at $\lambda = \lambda^*$ and one where $E^u(\lambda,\tau)$ is frozen at $\lambda = \lambda^*$. To simplify the calculations, we will work with Ω instead of ω. Since one of these forms is just a scaled version of the other, it is clear that the signatures are the same, and hence the Maslov indices are as well. First consider the curve $\lambda \mapsto E^u(\lambda,\tau)$ and reference plane $E^s(\lambda^*,\tau)$. As in Chapter 3, we can write $E^u(\lambda,\tau)$ for $|\lambda - \lambda^*|$ small as the graph of an operator $B_\lambda : E^u(\lambda^*,\tau) \to J \cdot E^u(\lambda^*,\tau)$. This, in turn, generates a smooth curve $\gamma(\lambda) = (\xi + B_\lambda \xi) \in E^u(\lambda,\tau)$ with $\gamma(\lambda^*) = \xi$. By flowing backwards in z, we obtain a one-parameter family $u(\lambda,z)$ of solutions to (6.27) in $E^u(\lambda,\tau)$, with $u(\lambda^*,\tau) = \xi$. Applying the argument of Theorem 3.4.1–specifically (3.17)–with λ as the varying parameter, we see that

$$\Gamma(E^u(\cdot,\tau), E^s(\lambda^*,\tau), \lambda^*)(\xi) = \Omega(u(\lambda,z), \partial_\lambda u(\lambda,z))|_{\lambda=\lambda^*, z=\tau}. \tag{6.47}$$

The case where $E^s(\lambda,\tau)$ varies and $E^u(\lambda^*,\tau)$ is fixed is identical. We can generate a smooth family of solutions $v(\lambda,z) \in E^s(\lambda,z)$ with $v(\lambda^*,\tau) = \xi$. This half of the crossing form is then given by

$$\Gamma(E^s(\cdot,\tau), E^u(\lambda^*,\tau), \lambda^*)(\xi) = \Omega(v(\lambda,z), \partial_\lambda v(\lambda,z))|_{\lambda=\lambda^*, z=\tau}. \tag{6.48}$$

By uniqueness of solutions, we importantly have

$$u(\lambda^*, z) = v(\lambda^*, z) := P(z), \tag{6.49}$$

which is a λ^*-eigenvector of L_ϵ. Putting together (3.6), (6.47), and (6.48), it follows that

$$\begin{aligned}\Gamma(E^u(\cdot,\tau), E^s(\cdot,\tau), \lambda^*)(\xi) &= \{\Omega(u(\lambda,z), \partial_\lambda u(\lambda,z)) - \Omega(v(\lambda,z), \partial_\lambda v(\lambda,z))\}|_{\lambda=\lambda^*, z=\tau} \\ &= \{\Omega(u(\lambda,z), \partial_\lambda u(\lambda,z)) + \Omega(\partial_\lambda v(\lambda,z), v(\lambda,z))\}|_{\lambda=\lambda^*, z=\tau} \\ &= \partial_\lambda \Omega(v(\lambda,z), u(\lambda,z))|_{\lambda=\lambda^*, z=\tau},\end{aligned} \tag{6.50}$$

where the last equality follows from (6.49).

The expression obtained in (6.50) will be useful in the next section when we relate the crossing form to the Evans function. For now, we compute (6.47) and (6.48) directly. For (6.47), we use the equality of mixed partials and the fact that u solves (6.27) to obtain

$$(\partial_\lambda u(\lambda,z))_z = A(\lambda,z)\partial_\lambda u(\lambda,z) + A_\lambda u(\lambda,z), \tag{6.51}$$

where

$$A_\lambda := \partial_\lambda A(\lambda,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \end{pmatrix}. \tag{6.52}$$

Next, apply $\omega(u(\lambda,z),\cdot)$ to (6.51) to see that

$$\begin{aligned}\omega(u(\lambda,z), A_\lambda u(\lambda,z)) &= \omega(u(\lambda,z), (\partial_\lambda u(\lambda,z))_z) - \omega(u(\lambda,z), A(\lambda,z)\partial_\lambda u(\lambda,z)) \\ &= \omega(u(\lambda,z), (\partial_\lambda u(\lambda,z))_z) + \omega(A(\lambda,z)u(\lambda,z), \partial_\lambda u(\lambda,z)) + c\omega(u(\lambda,z), \partial_\lambda u(\lambda,z)) \\ &= \partial_z \omega(u(\lambda,z), \partial_\lambda u(\lambda,z)) + c\omega(u(\lambda,z), \partial_\lambda u(\lambda,z)).\end{aligned} \tag{6.53}$$

The second equality follows from the proof of Theorem 2.5.1, specifically (2.38).

Applying an integrating factor and using (2.36) and (2.19) then shows that

$$\begin{aligned}\Gamma(E^u(\cdot,\tau),E^s(\lambda^*,\tau),\lambda^*)(\xi) &= \Omega(u,\partial_\lambda u)(\lambda^*,\tau) \\ &= \int_{-\infty}^{\tau} \partial_z \Omega(u(\lambda,z),\partial_\lambda u(\lambda,z))|_{\lambda=\lambda^*}\,dz = \int_{-\infty}^{\tau} e^{cz}\omega(P,A_\lambda P)\,dz.\end{aligned} \tag{6.54}$$

The preceding calculation makes use of the fact that $u(\lambda,z)\in E^u(\lambda,z)$, and hence it decays faster than e^{cz} as $z\to-\infty$, by (6.30). The calculation of the crossing form for the stable bundle using the solutions $v(\lambda,z)$ is identical until the last step. Indeed, those solutions decay at $+\infty$, so an application of the Fundamental Theorem gives

$$\begin{aligned}\Gamma(E^s(\cdot,\tau),E^u(\lambda^*,\tau),\lambda^*)(\xi) &= \Omega(v,\partial_\lambda v)(\lambda^*,\tau) \\ &= -\int_{\tau}^{\infty} \partial_z \Omega(v,\partial_\lambda v)(\lambda^*,z)\,dz = -\int_{\tau}^{\infty} e^{cz}\omega(P,A_\lambda P)\,dz.\end{aligned} \tag{6.55}$$

Combining (3.6), (6.54), and (6.55), we obtain the following.

Lemma 6.2.3. *The relative crossing form (in λ) for the curves $\lambda\mapsto(E^u(\lambda,\tau),E^s(\lambda,\tau))$ at a conjugate point $\lambda=\lambda^*$ is given by*

$$\Gamma(E^u(\cdot,\tau),E^s(\cdot,\tau),\lambda^*)(\xi) = \int_{-\infty}^{\infty} e^{cz}\,\omega(P,A_\lambda P)\,dz, \tag{6.56}$$

where $P\in E^u(\lambda^*,z)\cap E^s(\lambda^*,z)$ is the λ^*-eigenfunction of L_ϵ satisfying $P(\tau)=\xi$.

6.2.3 Monotonicity of λ-crossings

Writing $P:=(p,q,p_z,q_z/\epsilon)$, it is straightforward to calculate from (2.34) that

$$\omega(P,A_\lambda P) = p^2 - \frac{q^2}{\epsilon}. \tag{6.57}$$

The following theorem shows that Γ is positive definite for each conjugate point of α_2, which proves that Maslov(φ) equals the sum of the geometric multiplicities of all unstable eigenvalues of L_ϵ.

Theorem 6.2.2. *Let $\lambda\in\sigma(L_\epsilon)\cap(\mathbb{R}^+\cup\{0\})$ with corresponding eigenvector $P=(p,q)^T$. Suppose* *further that* $0 < \epsilon < \frac{c^4}{16}$. *Then*

$$\int_{-\infty}^{\infty} e^{cz}\left(p^2 - \frac{q^2}{\epsilon}\right) dz > 0. \tag{6.58}$$

In other words, the crossing form (6.50) is positive definite for all $\lambda \in [0, \lambda_{\max}]$.

The proof of this theorem uses the following Poincaré-type inequality.

Lemma 6.2.4. *Suppose* $h \in H^1(\mathbb{R})$ *satisfies*

$$\int_{-\infty}^{\infty} e^{cz}\left(h^2 + (h_z)^2\right) dz < \infty. \tag{6.59}$$

Then for all $R \in \mathbb{R}$ *(including* $R = \infty$*), we have*

$$\frac{c^2}{4}\int_{-\infty}^{R} e^{cz} h^2 \, dz \le \int_{-\infty}^{R} e^{cz} (h_z)^2 \, dz. \tag{6.60}$$

The proof of this inequality is a simple estimate using the fact that (6.59) defines a norm on an exponentially weighted Sobolev space. For more details, we refer the reader to Lemma 4.1 of [59], the source of this result.

Proof of Theorem 6.2.2. Written as a system, the eigenvalue equation $L_\epsilon P = \lambda P$ is

$$\begin{aligned} p_{zz} + cp_z + (f'(\hat{u}) - \lambda)p - q &= 0 \\ q_{zz} + cq_z + \epsilon p - (\epsilon\gamma + \lambda)q &= 0. \end{aligned} \tag{6.61}$$

Now, multiply the second equation in (6.61) by $e^{cz}q$ to obtain

$$\left(e^{cz} q_z\right)_z q - (\epsilon\gamma + \lambda)e^{cz} q^2 = -e^{cz}\epsilon pq. \tag{6.62}$$

Since p, q and their derivatives all decay exponentially in both tails, we can integrate (6.62) to obtain (after an integration by parts)

$$\int_{-\infty}^{\infty} e^{cz}(q_z)^2 \, dz + (\epsilon\gamma + \lambda)\int_{-\infty}^{\infty} e^{cz} q^2 \, dz = \epsilon \int_{-\infty}^{\infty} e^{cz} pq \, dz. \tag{6.63}$$

It then follows from (6.60), (6.63), and the Cauchy-Schwarz inequality that

$$\begin{aligned}\frac{c^2}{4\epsilon}\int_{-\infty}^{\infty} e^{cz}q^2\,dz &\le \frac{1}{\epsilon}\int_{-\infty}^{\infty} e^{cz}(q_z)^2\,dz \\ &< \int_{-\infty}^{\infty} pq\,dz \le \left(\int_{-\infty}^{\infty} e^{cz}p^2\,dz\right)^{1/2}\left(\int_{-\infty}^{\infty} e^{cz}q^2\,dz\right)^{1/2}.\end{aligned} \tag{6.64}$$

Dividing the first and last terms in the inequality by $||q||_{1,c}$ (the e^{cz}-weighted L^2 norm) and squaring yields

$$\frac{c^4}{16\epsilon}\int_{-\infty}^{\infty} e^{cz}\frac{q^2}{\epsilon}\,dz < \int_{-\infty}^{\infty} e^{cz}p^2\,dz, \tag{6.65}$$

and the result now follows. □

Remark 6.2. *The proof of the preceding theorem uses estimates that are very similar to calculations in [17]. However, the objectives of the calculations are very different. In [17], the goal is to establish the existence of a traveling wave using variational techniques. By contrast, we are considering the stability issue, in particular the monotonicity of the crossing form as λ varies.*

To close this section, observe that Theorem 6.2.2 applies to $\lambda = 0$. It then follows from (3.7) that this conjugate point contributes 0 to $\mu(\alpha_2)$, as it is a left endpoint crossing. This proves that Maslov(φ) only counts unstable eigenvalues for L_ϵ, which is implied in the statement of Theorem 6.2.1.

6.3 Multiplicity of eigenvalues: return of the Evans function

There is one remaining loose end to tie up if we want the Maslov index to give a complete picture of the unstable spectrum of L_ϵ, namely, the multiplicity of eigenvalues. In general, for $\lambda \in \sigma_n(L)$, the *geometric multiplicity* of λ is given by $\dim\ker(L-\lambda I)$. For $\lambda \in \mathbb{C}\setminus H_\epsilon \cap \sigma_n(L_\epsilon)$, this number is bounded above by two (or n, in the general setting of (1.19)), since $E^u(\lambda,z)$ and $E^s(\lambda,z)$ are only two-dimensional. Since $\dim\ker(L-\lambda I) = \dim(E^u(\lambda,z)\cap E^s(\lambda,z))$, it is clear from (6.19) that the dimension of a crossing for α_2 gives the geometric multiplicity of λ.

The algebraic multiplicity of λ, on the other hand, is trickier. It is given by $\dim\ker(L-\lambda I)^\alpha$, where α is the ascent of λ, i.e. the smallest α for which $\dim\ker(L-\lambda I)^\alpha = \dim\ker(L-\lambda I)^{a+1}$. See [2, §6.D] for more details. There is nothing obvious about the Maslov index that addresses the algebraic multiplicity of an eigenvalue. In [41], self-adjoint operators are studied, and this issue is moot, since the two multiplicities coincide. However, for our purposes it is not obvious that the two multiplicities are the same.

One tool that demonstrably gives information about the algebraic multiplicity of eigenvalues is the Evans function. In [2, §6] it is proved that the order of $\lambda \in \mathbb{C}$ as a root of $D(\lambda)$ is equal to its algebraic multiplicity as an eigenvalue of L_ϵ. We will use this fact to give a new, geometric interpretation of algebraic multiplicity in terms of the Maslov index. We remind the reader that we have spanning solutions $u_i(\lambda, z)$, $i = 1 \ldots 4$ satisfying (4.11), on account of the simplicity of the eigenvalues of $A_\infty(\lambda)$. Furthermore, the Evans function is given by one of the two equivalent formulations (4.15) or (4.17). The following theorem is the main result of this section.

Theorem 6.3.1. *Let $\lambda^* \in \sigma_n(L_\epsilon) \cap (\mathbb{C} \setminus H_\epsilon)$. Then the geometric and algebraic multiplicities of λ^* are equal. This is equivalent to $\lambda = \lambda^*$ being a regular conjugate point of α_2.*

Proof. We prove this separately for λ with geometric multiplicity one and two. Recall that a crossing is regular if the associated crossing form (6.50) is nondegenerate. First, suppose that λ^* is an eigenvalue of L_ϵ with geometric multiplicity one. The goal is to show that $D'(\lambda^*) \neq 0$. Let $P(z)$ be a corresponding eigenfunction. We can perform a change of basis near $\lambda = \lambda^*$ so that

$$\begin{aligned} E^s(\lambda, z) &= \mathrm{sp}\{U(\lambda, z), a_s(\lambda, z)\} \\ E^u(\lambda, z) &= \mathrm{sp}\{V(\lambda, z), a_u(\lambda, z)\} \end{aligned}, \tag{6.66}$$

with $U(\lambda^*, z) = V(\lambda^*, z) = P(z)$. Doing so changes $D(\lambda)$ by multiplication with a nonzero analytic function $C(\lambda)$ (§4.1 of [71]). Since $D(\lambda^*) = 0$, we have

$$\frac{d}{d\lambda}\left[D(\lambda)C(\lambda)\right]|_{\lambda=\lambda^*} = D'(\lambda^*)C(\lambda^*), \tag{6.67}$$

so making this change of basis does not affect whether or not the derivative of D at λ^* vanishes. Comparing with (4.16), it therefore suffices to consider $\tilde{D}'(\lambda^*)$, with

$$\tilde{D}(\lambda) = -\det \begin{bmatrix} \Omega(U(\lambda, z), V(\lambda, z)) & \Omega(U(\lambda, z), a_u(\lambda, z)) \\ \Omega(a_s(\lambda, z), V(\lambda, z)) & \Omega(a_s(\lambda, z), a_u(\lambda, z)) \end{bmatrix}. \tag{6.68}$$

We compute $\tilde{D}'(\lambda^*)$ as in (4.19) to obtain

$$\tilde{D}'(\lambda^*) = \Omega(a_s(\lambda, z), a_u(\lambda, z))\, \partial_\lambda \Omega(U(\lambda, z), V(\lambda, z))|_{\lambda=\lambda^*, z=\tau}. \tag{6.69}$$

Define $\xi = P(\tau)$. Comparing with (6.50), we see that

$$\partial_\lambda \Omega(U(\lambda, z), V(\lambda, z))|_{\lambda=\lambda^*, z=\tau} = -\Gamma(E^u(\cdot, \tau), E^s(\cdot, \tau), \lambda^*)(\xi), \tag{6.70}$$

which is nonzero, since λ^* is a regular crossing by Theorem 6.2.2. It would follow that $D'(\lambda^*) \neq 0$, and hence that λ^* is a simple eigenvalue of L_ϵ, if $\Omega(a_s(\lambda^*, z), a_u(\lambda^*, z)) \neq 0$. It turns out that this is equivalent to λ^* having geometric multiplicity one. Indeed, if $\Omega(a_s(\lambda^*, z), a_u(\lambda^*, z)) = 0$, then $\mathrm{sp}\{a_s(\lambda^*, z), a_u(\lambda^*, z)\}$ is a Lagrangian plane. A simple dimension-counting argument (cf. [15, p. 85]) then implies that

$$E^s(\lambda^*, z) = \mathrm{sp}\{U, a_s\} = \mathrm{sp}\{V, a_u\} = E^u(\lambda^*, z). \tag{6.71}$$

We now turn to the case where λ^* is a two-dimensional crossing, meaning that

$$\dim(E^u(\lambda^*, z) \cap E^s(\lambda^*, z)) = 2. \tag{6.72}$$

By making another change of basis if necessary, we are free to assume that

$$\begin{aligned} u_1(\lambda^*, z) &= u_4(\lambda^*, z) \\ u_2(\lambda^*, z) &= u_3(\lambda^*, z) \end{aligned}. \tag{6.73}$$

We then set $\xi_1 = u_1(\lambda^*, \tau)$ and $\xi_2 = u_2(\lambda^*, \tau)$. Since the algebraic multiplicity of λ^* is no less than its geometric multiplicity, we know *a priori* that $D'(\lambda^*) = 0$. This is easily verified by applying the product rule to (2.1), which is the zero matrix for $\lambda = \lambda^*$. What we need to verify is that $D''(\lambda^*) \neq 0$, and that this is equivalent to the regularity of the crossing form. To see this, we use (2.1) to write out

$$D(\lambda) = \Omega(u_1(\lambda, z), u_4(\lambda, z))\Omega(u_2(\lambda, z), u_3(\lambda, z)) - \Omega(u_1(\lambda, z), u_3(\lambda, z))\Omega(u_2(\lambda, z), u_4(\lambda, z)). \tag{6.74}$$

Evaluating at $\lambda = \lambda^*$, each of the four terms in (6.74) is zero, using (6.73) and the fact that $E^{u/s}(\lambda, z)$ are Lagrangian planes. As mentioned above, we can see from (6.74) that $D'(\lambda^*) = 0$, since the derivative produces a series of four terms, each of which is a product with a factor of zero. Computing $D''(\lambda^*)$ from the general Leibniz rule, we see that the only surviving terms are those for which each factor in (6.74) is differentiated once. Explicitly, we compute that

$$
\begin{aligned}
D''(\lambda^*) =& 2\,\{\partial_\lambda \Omega(u_1(\lambda,z), u_4(\lambda,z)) \partial_\lambda \Omega(u_2(\lambda,z), u_3(\lambda,z)) \\
& - \partial_\lambda \Omega(u_1(\lambda,z), u_3(\lambda,z)) \partial_\lambda \Omega(u_2(\lambda,z), u_4(\lambda,z))\}\,|_{\lambda=\lambda^*, z=\tau} \\
&= -2 \det \begin{bmatrix} \partial_\lambda \Omega(u_1(\lambda,z), u_3(\lambda,z)) & \partial_\lambda \Omega(u_1(\lambda,z), u_4(\lambda,z)) \\ \partial_\lambda \Omega(u_2(\lambda,z), u_3(\lambda,z)) & \partial_\lambda \Omega(u_2(\lambda,z), u_4(\lambda,z)) \end{bmatrix} |_{\lambda=\lambda^*, z=\tau} \\
&= 2 \det \begin{bmatrix} \partial_\lambda \Omega(u_1(\lambda,z), u_4(\lambda,z)) & \partial_\lambda \Omega(u_1(\lambda,z), u_3(\lambda,z)) \\ \partial_\lambda \Omega(u_2(\lambda,z), u_4(\lambda,z)) & \partial_\lambda \Omega(u_2(\lambda,z), u_3(\lambda,z)) \end{bmatrix} |_{\lambda=\lambda^*, z=\tau}.
\end{aligned}
\tag{6.75}
$$

We see from (6.73) that the last matrix (obtained by switching columns and taking a transpose in the previous line) is exactly the matrix of the crossing form Γ in (6.50). To say that Γ is nondegenerate means that the determinant in (6.75) is nonzero, hence $D''(\lambda^*) \neq 0$, as desired.

□

Remark 6.3. *Although we phrased the preceding theorem for the operator L_ϵ, it is clear that the proof generalizes to L in (2.3). At an n-dimensional crossing λ^*, the first $(n-1)$ derivatives of $D(\lambda)$ are forced to vanish. The n^{th} derivative will then contain a factor corresponding to the λ-crossing form Γ. The number of zeros of Γ in the normal form [79, p. 186] then gives the discrepancy between the algebraic and geometric multiplicities of λ^* as an eigenvalue of L.*

Although algebraic versus geometric multiplicity seems like a picayune detail, it is actually critical in the case $\lambda = 0$, as we see from Theorem 1.1.1. It is possible that $E^u(0,z) \cap E^s(0,z) = \mathrm{sp}\{\varphi'(z)\}$ is one-dimensional, but that $\lambda = 0$ is still not a simple eigenvalue. In [3, pp. 57-60], it is shown that $\lambda = 0$ is simple if and only if the wave is *transversely constructed*, in the sense of (A3), cf. Chapter 4. Thus the geometric interpretation of simplicity is that two manifolds intersect transversely in augmented phase space.

By contrast, the understanding of simplicity afforded by the symplectic structure requires no variation in c. Instead, we see that $\lambda = 0$ (or any other eigenvalue) is simple if the curve $\lambda \mapsto E^u(\lambda, \tau)$ transversely intersects the train of $E^s(0, \tau)$ for all sufficiently large τ. To see this, notice that Theorem 6.3.1 proves that the eigenvalue is simple if and only if the relative crossing form (6.50) is regular. But if the integral (6.56) is nonzero, then so will be the integral in (6.47) for τ large enough. Alternatively, for an eigenvalue λ^* with geometric multiplicity one, being simple is equivalent to the curves $\lambda \mapsto E^u(\lambda, \tau), E^s(\lambda, \tau)$ intersecting non-tangentially at $\lambda = \lambda^*$.

CHAPTER 7

Calculation of the Maslov index

In this chapter, we carry out the calculation of the Maslov index for the FitzHugh-Nagumo fast pulses. This calculation is the centerpiece of [22]; in conjunction with Theorem 6.2.1 proved in the preceding chapter, it allows us to conclude that the fast traveling pulses obtained in Theorem 5.4.2 are stable:

Theorem 7.0.1. *For $0 < \epsilon < c^4/16$, the traveling waves $\varphi_\epsilon(z)$ guaranteed to exist by Theorem 5.4.2 are stable in the sense of Definition 1.1.*

Recall that Maslov(φ) is calculated by following the curve $E^u(0,z)$ from $z = -\infty$ to $z = \tau$. Solving (6.27) directly (thus determining the curve of interest) is a tall order, since that equation is nonautonomous and dependent on ϵ. Instead, we will use the fact that, for $\lambda = 0$, (6.27) is the variational equation for (5.19) along the traveling wave $\varphi(z)$. Consequently, we have the following (general) theorem.

Theorem 7.0.2. *For all $z \in \mathbb{R}$, the unstable bundle $E^u(0,z)$ is the tangent space to $W^u(0)$ at $\varphi(z)$. Likewise, the stable bundle $E^s(0,z)$ is everywhere along $\varphi(z)$ tangent to $W^s(0)$.*

This fact is well-known (e.g. [3, p. 73] and [84, p. 196]) and true for traveling waves in general systems (1.1), as long as the end state of the wave is a hyperbolic fixed point of the traveling wave ODE. Finding a proof in the literature is difficult, so we offer one here.

Proof. We will prove this statement for the unstable bundle, as the other case is identical. Since $E^u(0,z)$ and $T_{\varphi(z)}W^u(0)$ are both two-dimensional vector spaces, it suffices to show that

$$T_{\varphi(z_0)}W^u(0) \subseteq E^u(0,z_0) \tag{7.1}$$

for any $z_0 \in \mathbb{R}$. To that end, take $v \in T_{\varphi(z_0)}W^u(0)$. By definition, this means we can find a smooth curve $\alpha : (-\delta,\delta) \to W^u(0)$ such that $\alpha(0) = \varphi(z_0)$ and $\alpha'(0) = v$. We already know that

$\varphi'(z) \in E^u(0,z)$, so assume that α is transverse to φ. For each $s \in (-\delta, \delta)$, we can consider the trajectory $\alpha(s) \cdot (-\infty, 0]$ through $\alpha(s)$. This generates a one-parameter family of orbits $\varphi_s(z)$, with $\varphi_0(z) = \varphi(z)$ for all $z \leq z_0$. (Thus the family is parametrized so that $\varphi_s(z_0) = \alpha(s)$.) It is clear that

$$v = \partial_s \varphi_s(z_0)|_{s=0}. \tag{7.2}$$

We claim that the solution to the variational equation (6.27), $\lambda = 0$, along $\varphi(z)$ with initial condition v at $z = z_0$ is given by $\partial_s \varphi_s(z)|_{s=0}$. It then follows that $v \in E^u(0, z_0)$, since $\partial_s \varphi_s(z)|_{s=0}$ converges to 0 exponentially as $z \to -\infty$. (This is because $\varphi_s(z) \to 0$ uniformly in s by virtue of being in $W^u(0)$.) To prove the claim, we have

$$\partial_z \partial_s(\varphi_s(z)) = \partial_s F(\varphi_s(z)) = DF(\varphi_s(z)) \partial_s \varphi_s(z). \tag{7.3}$$

Evaluating at $s = 0$ shows that the derivative with respect to the parameter s satisfies the variational equation. Evaluating at $z = z_0$ shows that the initial condition is v. $\square$

The preceding theorem says that Maslov(φ) is equal to the number of twists that $W^u(0)$ makes as $\varphi(z)$ moves through phase space. On the one hand, this is our motivation for studying the Maslov index, as it relates spectral information to structural properties of the wave (or, more precisely, an invariant manifold containing the wave), à la Sturm-Liouville theory. On the other hand, this provides a means to the end of *calculating* the Maslov index if, for some reason, more information is known about the nonlinear problem (5.19) than the linear problem (6.27). Such is the case for singularly perturbed systems, for which Fenichel theory and subsequent developments provide a means for tracking invariant manifolds throughout phase space. By contrast, the timescale separation makes the eigenvalue problem itself no more or less tractable.

7.1 Plücker coordinates revisited

The calculation in this chapter will use the detection form of §4.3 to identify conjugate points. Most of the calculations will be done in Plücker coordinates, so it will be helpful to see how the two concepts are related. Let $V = \mathrm{sp}\{v_1, v_2\}, W = \mathrm{sp}\{w_1, w_2\} \in \Lambda(2)$ be Lagrangian planes with

Plücker coordinates (p_{ij}) and (q_{ij}) respectively. Then

$$\begin{aligned} W \cap V \neq \{0\} \iff \det[v_1, v_2, w_1, w_2] = 0 \\ = p_{12}q_{34} - p_{13}q_{24} + p_{14}q_{23} + p_{23}q_{14} - p_{24}q_{13} + p_{34}q_{12}, \end{aligned} \tag{7.4}$$

using (2.25) and cofactor expansion to compute the determinant. If instead $W = W(z)$ is a curve of subspaces, then the right-hand side of (7.4) becomes a function of z, which is exactly $\beta(z)$ from (4.32).

In general, using the function β directly is difficult, because it is only defined up to nonzero multiples. Thus, arguments asserting the existence of a conjugate point based on a change in sign are difficult. However, we will see later that such arguments can often be applied if the underlying equation determining the curve $W(z)$ is relatively simple.

7.2 Phase portrait of flow induced on $\Lambda(2)$ by a constant coefficient system

In §7.5 and §7.7, we will rely on properties of the dynamics induced on $\Lambda(2)$ by a linear, constant coefficient system. The phase portrait of such systems is described completely in [73] and is of interest in control theory. In this section we catalog the relevant results for this work, tailored to the linearization of the traveling wave ODE (5.19) at any point on $M_0^{R/L}$. The reader should be aware that the presentation in [73] assumes that the flow on $\Lambda(n)$ is given by the action of a $2n \times 2n$ symplectic matrix on Lagrangian subspaces. This is not the case here, since the solution operator for (5.20) is not symplectic. However, this does not change the geometry of the flow on $\Lambda(2)$, which is an invariant manifold of the system on $\mathrm{Gr}_2(\mathbb{R}^4)$. It is therefore clear that the following facts remain true, although the assumptions of the corresponding theorems in [73] sometimes require modification.

To fix some notation, first recall that there are three "corners" at which transitions from fast-to-slow dynamics (or vice-versa) occur: $p = (1, 0, 0, -1/c^*)$, $q = (u^*, v^*, 0, (1/c^*)(\gamma v^* - u^*))$, and $\hat{q} = (u^* - 1, v^*, 0, (1/c^*)(\gamma v^* - u^* + 1))$. With a calculation similar to (5.23), one finds that there are four eigenvalues for the linearization at each point, which satisfy (when $\epsilon = 0$)

$$\mu_1 < \mu_2 = 0 < \mu_3 = -c < \mu_4. \tag{7.5}$$

μ_1 and μ_4 depend on u, but their sum is always equal to $-c$. For fixed u, the linearized system

(5.20) is of the form

$$Y'(z) = BY(z), \tag{7.6}$$

which has solution operator $\exp(Bz) \in \mathrm{GL}_4(\mathbb{R})$. For each eigenvalue μ_i, assign a nonzero eigenvector η_i. Since eigenspaces of B are invariant under the action of $\exp(Bz)$, we see that the equation induced on $\mathrm{Gr}_2(\mathbb{R}^4)$ by (7.6) has six fixed points

$$X_{ij} = \mathrm{sp}\{\eta_i, \eta_j\}, \quad \{i, j\} \in \binom{\{1,2,3,4\}}{2}. \tag{7.7}$$

Of these, X_{12}, X_{13}, X_{24} and X_{34} are Lagrangian planes, making them the points of interest. The following theorem holds for the flow on $\Lambda(2)$ induced by the constant coefficient system (7.6) based at each corner point mentioned above. Recall that $\dim \Lambda(2) = 3$. We refer the reader to [73] for proofs.

Theorem 7.2.1 (Shayman [73])**.** *For the equation induced by (7.6) on $\Lambda(2)$, the following are true:*

1. *Each fixed point for (7.6) is hyperbolic. We have*

$$\begin{aligned} \dim W^u(X_{12}) &= \dim W^s(X_{34}) = 3 \\ \dim W^u(X_{13}) &= \dim W^s(X_{24}) = 2 \\ \dim W^u(X_{24}) &= \dim W^s(X_{13}) = 1. \end{aligned} \tag{7.8}$$

   Furthermore, each of $W^u(X_{12})$ and $W^s(X_{34})$ is open and dense in $\Lambda(2)$.

2. $\Lambda(n) = \bigcup W^u(X_{ij}) = \bigcup W^s(X_{ij})$*, due to the fact that* $\Lambda(2)$ *is compact.*

3. *Each $W^{u/s}(X_{ij})$ is a Schubert cell. In particular, it is diffeomorphic to $\mathbb{R}^d$, where d is the dimension of the invariant manifold.*

4. *For any i, j, i', j', either $W^u(X_{ij}) \cap W^s(X_{i'j'}) = \phi$ or $W^u(X_{ij}) \pitchfork W^s(X_{i'j'})$.*

Remark 7.1. *It is also true that the vector field on $\Lambda(2)$ induced by (7.6) is Morse-Smale.*

7.3 Calculation using geometric singular perturbation theory

Our strategy will be to use elements from the construction of $\varphi(z)$ to follow $W^u(0)$ along the different fast and slow segments. Notice that there are seven–not four–pieces that we must consider, since the singular orbit in $\Lambda(2)$ corresponding to $T_{\varphi(z)}W^u(0)$ has jump discontinuities at the corners. This is because the orientation of the unstable manifold is different upon entering a neighborhood of $M_\epsilon^{L/R}$ than it is along the slow pieces. In fact, the analysis of the corners is the most challenging aspect of the calculation. We will then use the additivity property from Proposition 3.1.1 to calculate the index.

By Theorem 7.0.2, $E^s(0,z)$ is tangent to $W^s(0)$ everywhere along $\varphi(z)$. It follows that the reference plane $E^s(0,\tau)$ is given by $T_{\varphi(\tau)}W^s(0)$, where $\varphi(\tau)$ is as close as we like to returning to 0. By Fenichel theory, this subspace is spanned (to leading order) by the tangent vector to M_ϵ^L and the stable eigenvector of the same point on the critical manifold. We label the components of this point $\varphi(\tau) = (u_\tau, v_\tau, w_\tau, y_\tau)$. Throughout the calculation, we will make heavy use of the robustness of transverse intersections. More precisely, the train of $E^s(0,\tau)$ is a codimension one subset of $\Lambda(2)$. If the curve $E^u(0,z)$ crosses it transversely for some value $z = z^*$, then the crossing would persist for sufficiently small perturbations of either the curve or the reference plane. We are therefore justified in taking the leading order approximations of both $E^u(0,z)$ and $E^s(0,\tau)$. This allows us to search for intersections on the fast and slow timescales with $\epsilon = 0$, which is significantly easier. In particular, we can use the singular value $c = c^*$ throughout. We drop the * for the rest of the section.

As mentioned above, the most difficult part of the analysis is determining what happens to $\Lambda(2)$ near the corners. To figure out how the gaps between the entry and exit points in $\Lambda(2)$ are bridged, we will analyze the flow induced on $\Lambda(2)$ by the constant coefficient systems obtained by linearizing (5.19) at the corners. We will rely on the results of §7.2 to understand this flow. We will also make heavy use of the Plücker coordinates in this chapter, so it will be helpful for the reader to refer back to §2.3 and §7.1. For convenience of the reader, we separate the rest of this chapter into sections wherein each piece of the wave is considered separately.

7.4 First fast jump

As explained in §5.2, we have a very clear picture of the unstable bundle (i.e. of $T_{\varphi(z)}W^u(0)$) along the fast front; at each point along the orbit, it is $O(\epsilon)$ close to $W^u(0_f)$, the unstable manifold for 0 in (5.24). The latter is just a cylinder over the Nagumo front, so its tangent space at any point along the jump is known. In anticipation of computing the crossing form, we include the δv component, even though it will be 0 for both basis vectors. We can differentiate (5.30) with respect to u to determine one vector tangent to $W^u(0)$, and the other is given by the invariant y direction. We therefore have

$$T_{\varphi(z)}W^u(0) \approx \text{sp}\left\{ \begin{bmatrix} 1 \\ 0 \\ \sqrt{2}/2 - u\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{7.9}$$

To detect conjugate points, we need a working basis for $E^s(0,\tau)$. In light of the discussion in §7.1, one basis vector is found by differentiating the equation defining M_0 in (5.25) with respect to v. The other is computed by finding the stable eigenvector for the linearization of (5.24) around $\varphi(\tau)$ with $\epsilon = 0$. It is then a calculation to see that

$$E^s(0,\tau) \approx \text{sp}\left\{ \begin{bmatrix} 1 \\ f'(u_\tau) \\ 0 \\ \frac{1}{c}(\gamma f'(u_\tau) - 1) \end{bmatrix}, \begin{bmatrix} f'(u_\tau) \\ 0 \\ f'(u_\tau)\mu_1(u_\tau) \\ \mu_1(u_\tau) \end{bmatrix} \right\}, \tag{7.10}$$

where $\mu_1(u_\tau)$ is the stable eigenvalue for the linearization of (5.24) about $\varphi(\tau)$. (Notice that the eigenvalues and eigenvectors for points in M_0^L can be written as a function of u.) We use $\approx$ to remind the reader that this the leading order (in ϵ) approximation to $E^s(0,\tau)$. Although we could use (7.10) directly to find conjugate points, the calculation would be tedious due to the way $\mu_1(u_\tau)$ depends on u. Instead, we claim that the Maslov index contribution is the same if we instead look for intersections with the train of

$$V^s(0) = \text{sp}\left\{ \begin{bmatrix} 1 \\ -a \\ 0 \\ \frac{1}{c}(1 + \gamma a) \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -a\sqrt{2} \\ \sqrt{2} \end{bmatrix} \right\}, \tag{7.11}$$

which one obtains by substituting $u = 0$ in (7.10) and using (5.28) and (5.32). Indeed, we know that $u_\tau \to 0$ as $\tau \to \infty$, so that $\Sigma(E^s(0,\tau))$ will be very close to $\Sigma(V^s(0))$ as long as τ is large enough. Thus, as long any crossings of $E^u(0,z)$ with $V^s(0)$ are one-dimensional and transverse, then $E^u(0,z)$

would have to cross $E^s(0,\tau)$ nearby and in the same direction.

Using this new reference plane, we see from (7.9) and (7.11) that an intersection occurs if and only if

$$u = a + \frac{1}{2}. \tag{7.12}$$

Since u increases monotonically along the fast jump from 0 to 1, it follows that there is a unique conjugate point, and the intersection is spanned by $\xi := [1, 0, -a\sqrt{2}, \sqrt{2}]^T$. To determine the direction of the crossing, we evaluate Γ from (3.13) on this vector. We call the conjugate point z^* and use (2.34) to compute:

$$\begin{aligned} \omega(\xi, A(0, z^*)\xi) = \langle \xi, JA(0, z^*)\xi\rangle &= -f'(u) + ca\sqrt{2} - 2a^2 \\ &= -f'\left(\frac{1}{2} + a\right) + ca\sqrt{2} - 2a^2 \\ &= a^2 - \frac{1}{4} < 0. \end{aligned} \tag{7.13}$$

This shows that the crossing is negative, and we conclude that the contribution to Maslov(φ) is -1 along the fast front.

7.5 First corner

Near the first landing point $p = (1, 0, 0, -1/c)$, the shooting manifold will undergo an abrupt reorientation. As the front approaches p, the tangent space to the shooting manifold will be spanned (approximately) by the stable eigenvector of the fixed point $(1, 0, -1/c)$ for the fast subsystem and the invariant y direction, as in the previous section. Combining (5.34) with the observation $f'(1) = a - 1$ and a calculation analogous to (7.10), we see that

$$T_{p_{\text{in}}} W^u(0) \approx \text{sp}\{\eta_1(p), \eta_3(p)\} = \text{sp}\left\{ \begin{bmatrix} a-1 \\ 0 \\ (1-a)\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{7.14}$$

The subscript "in" on p refers to the fact that this is the tangent space to $W^u(0)$ upon entrance into a neighborhood of p, as opposed to the trip away from p, up the slow manifold. The notation $\eta_i(p)$ indicates that the corresponding vector is an eigenvector for (5.33) with eigenvalue $\mu_i(p)$.

The next task is to determine the configuration of $W^u(0)$ as it moves up the slow manifold M_ϵ^R. For this part of the journey, the derivative of the wave is given (to leading order) by the tangent vector to M_ϵ^R. At p, this corresponds to the 0-eigenvector

$$\eta_2(p) = \begin{bmatrix} 1 \\ a-1 \\ 0 \\ \frac{1}{c}\left(\gamma(a-1)-1\right) \end{bmatrix}. \tag{7.15}$$

It is less obvious which is the second direction picked out. Deng's Lemma [25, 72] asserts that $W^u(0)$ will be crushed against $W^u(M_\epsilon^R)$, the unstable manifold of the right slow manifold. However, there are two unstable directions for each point on the critical manifold, and it is unclear which of these is picked out. (Since the approach to p was in the weak unstable direction, it is not unreasonable to think that this direction would persist.) Thankfully, the symplectic structure is able to break the tie. We know from Theorem 2.5.2 that $T_{\varphi(z)}W^u(0)$ is a Lagrangian subspace of $\mathbb{R}^4$. Since $\Lambda(2)$ is closed in $\mathrm{Gr}_2(\mathbb{R}^4)$, the symplectic form ω must vanish on the leading order approximation to $T_{\varphi(z)}W^u(0)$ as well. A direct computation shows that

$$\omega(\eta_2(p), \eta_3(p)) = 1 - a \neq 0, \tag{7.16}$$

so it must be that

$$T_{p_{\mathrm{out}}}W^u(0) \approx \mathrm{sp}\{\eta_2(p), \eta_4(p)\} = \mathrm{sp}\left\{ \begin{bmatrix} 1 \\ a-1 \\ 0 \\ \frac{1}{c}\left(\gamma(a-1)-1\right) \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \sqrt{2}(1-a) \\ -\sqrt{2} \end{bmatrix} \right\}. \tag{7.17}$$

Again, the vector $\eta_4(p)$ is obtained by using the formula for a generic eigenvector from (7.10) in conjunction with (5.34). For the rest of this section, we will write η_i instead of $\eta_i(p)$. The goal is to show that there are no conjugate points during the transition from p_{in} to p_{out}. Although this appears to be an $\epsilon \neq 0$ consideration, it is actually understood by analyzing the constant coefficient linear system obtained by setting $u \equiv 1$ in (5.20) with $\epsilon = 0$. Indeed, by taking ϵ very small, we can ensure that $W^u(0)$ is as close to p as desired while still maintaining (approximately) the shape of the cylinder. Similarly, since the traveling wave is C^1 $O(\epsilon)$-close to the singular object, the slow (i.e. tangent) direction is picked up arbitrarily close to p on M_ϵ^R. The second tangent vector is a solution to the linearized equation (6.27) with initial condition close to η_3, which is already an unstable direction. It follows that this direction must remain close to the unstable subspace of p, since the wave itself stays arbitrarily close to p during the transition, and the unstable subspace of (5.24) at p is invariant. The above discussion of the symplectic structure then implies that this solution must be bumped to η_4, the strong unstable direction, during this transition.

Setting $X_{ij} = \text{sp}\{\eta_1, \eta_j\}$ (cf. §7.2), we therefore must solve a pseudo-boundary value problem to connect the points $T_{p_{\text{in}}} W^u(0) = X_{13}$ and $T_{p_{\text{out}}} W^u(0) = X_{24}$ in $\Lambda(2)$ for the equation induced by

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1-a & 1 & -c & 0 \\ -1 & \gamma & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}, \tag{7.18}$$

which is (5.20) evaluated at $\hat{u} = 1$. These two points are both equilibria for said equation, since the eigenspaces of the matrix in (7.18) are invariant. It follows that the desired connection must be a heteroclinic orbit. It is explained in §7.2 that $W^u(X_{13}) \cap W^s(X_{24})$ is one-dimensional, so it suffices to find a single point in each distinct orbit to describe the intersection completely. The Schubert cell description of $W^{s/u}(X_{ij})$ makes it easy to see that there are two distinct heteroclinic connections from X_{13} to X_{24}. These orbits–call them $\gamma_\pm$–pass through the points

$$W_\pm = \text{sp}\{\eta_1 \pm \eta_2, \eta_3 \pm k\eta_4\}. \tag{7.19}$$

The constant

$$k = -\frac{\omega(\eta_2, \eta_3)}{\omega(\eta_1, \eta_4)} = \frac{\sqrt{2}}{3-2a} > 0 \tag{7.20}$$

is needed to ensure that the planes $W_\pm$ are Lagrangian. This restriction is very beneficial; were we looking for the same connections in the full Grassmannian, then there would be a two-dimensional set of orbits indexed by $k (\neq 0)$. Now, to prove that there is no contribution to the Maslov index near the corner, it suffices to show that the trajectory through $W_\pm$ is disjoint from $\Sigma(V^s(0))$. Since the "boundary data" for this equation are given in terms of the basis of eigenvectors at $u=1$, the easiest way to describe the solution is to use this basis for the Plücker coordinates as well. The drawback is that the reference plane $V^s(0)$ must be rewritten in terms of this new basis, which can be done with the help of Maple:

$$\begin{aligned}
&V^s(0) = \mathrm{sp}\,\{\nu_1, \nu_2\}\,, \\
&\quad \nu_1 = -2\eta_1 + 2c(2a-3)\eta_3 + (2a-1)(a-1)\eta_4 \\
&\quad \nu_2 = 2(2a-1)\eta_1 + a(3-2a)\eta_2 + \frac{(2a-1)(2a-3)}{-c}\eta_3 + (1-2a)\eta_4.
\end{aligned} \tag{7.21}$$

The heteroclinic orbit in $\Lambda(2)$ through $W_\pm$ is the projectivized version of the solution to the equation induced by (7.18) on $\bigwedge^2(\mathbb{R}^4)$ with initial condition

$$\tilde{W}_\pm = (\eta_1 \pm \eta_2) \wedge (\eta_3 \pm k\eta_4) = (0, 1, \pm k, \pm 1, k, 0). \tag{7.22}$$

The ordered 6-tuple in (7.22) gives the Plücker coordinates of $W_\pm$ in the new basis. Using (2.21), we can now give the explicit solution through this point, since the η_i are eigenvectors for the matrix in (7.18):

$$\gamma_\pm(z) = (0, e^{(\mu_1+\mu_3)z}, \pm k e^{(\mu_1+\mu_4)z}, \pm e^{(\mu_2+\mu_3)z}, k e^{(\mu_2+\mu_4)z}, 0). \tag{7.23}$$

Since these coordinates are projective, we can divide by $e^{-cz} = e^{(\mu_1+\mu_4)z} = e^{(\mu_2+\mu_3)z}$ to obtain a more tractable representation of the same planes,

$$\tilde{\gamma}_\pm(z) = (0, e^{-\frac{\sqrt{2}}{2}z}, \pm k, \pm 1, k e^{\frac{\sqrt{2}}{2}z}, 0). \tag{7.24}$$

We claim that it suffices to show that $\tilde{\gamma}_+$ does not cross $\Sigma(V^s(0))$. Indeed, consider the concatenated curve $\gamma_0 := \tilde{\gamma}_+ * -\tilde{\gamma}_-$, which is a loop in $\Lambda(2)$. It is known that the Maslov index of a loop does not depend on the choice of reference plane, since the Maslov index can be interpreted as an element in the cohomology group $H^1(\Lambda(2), \mathbb{Z})$ [4, 5, 26]. Taking the reference plane to be $V = \mathrm{sp}\{\eta_3, \eta_4\} = (0,0,0,0,0,1)$, it follows from (7.4) that crossings are given by the equation $p_{12} = 0$. For $\gamma_0(z)$, $p_{12} \equiv 0$, so $\gamma_0(z)$ is entirely contained in $\Sigma(V)$. However, the plane V itself is not in the image of γ_0, which means that $\dim(\gamma_0(z) \cap V) \equiv 1$. It then follows from Proposition 3.1.1(iii) that the Maslov index of γ_0 is 0. Since the Maslov index is additive by concatenation, it follows that the Maslov indices of γ_+ and γ_- with respect to any reference plane are opposite of each other. We now show that γ_+ has no crossings with $\Sigma(V^s(0))$, which proves that there is no contribution to the Maslov index at this corner, regardless of which path is taken.

Let (p_{ij}) be the Plücker coordinates of $V^s(0)$. From (7.4), we see that z^* is a conjugate time if and only if

$$0 = -e^{-z^*\sqrt{2}/2} p_{24} + k p_{23} + p_{14} - k e^{z^*\sqrt{2}/2} p_{13}. \tag{7.25}$$

To prove that the expression in (7.25) never vanishes, we first calculate using (7.21) and (2.25) that

$$\begin{aligned} -p_{24} &= a(1-2a)(3-2a)(1-a) > 0 \\ k p_{23} = p_{14} &= -2a(1-2a)(3-2a) < 0 \\ -k p_{13} &= 16a(1-a) > 0. \end{aligned} \tag{7.26}$$

As a function of z, the right-hand side of (7.25) can therefore be written as

$$h(z) := A e^{-z\sqrt{2}/2} - B + C e^{z\sqrt{2}/2}, \tag{7.27}$$

with $A, B, C > 0$. It is clear that $h(z) > 0$ for $|z|$ sufficiently large. Furthermore, h has a single local minimum at $z = \ln(A/C)/\sqrt{2}$, at which point $h(z) = 2\sqrt{AC} - B$. To show that there are no conjugate points for γ_+, it therefore suffices to show that $2\sqrt{AC} - B > 0$. We compute

$$\begin{aligned} 2\sqrt{AC} - B &= 8a(1-a)\sqrt{(1-2a)(3-2a)} - 4a(1-2a)(3-2a) \\ &= 4a\Big(2(1-a)\sqrt{(1-2a)(3-2a)} - (1-2a)(3-2a)\Big) \\ &= 4a\sqrt{(1-2a)(3-2a)}\left(\sqrt{4(1-a)^2} - \sqrt{(1-2a)(3-2a)}\right) \\ &= 4a\sqrt{(1-2a)(3-2a)}\left(\frac{1}{\sqrt{4(1-a)^2} + \sqrt{(1-2a)(3-2a)}}\right) > 0, \end{aligned} \tag{7.28}$$

as desired. This proves that the connecting orbit γ_+ from X_{13} to X_{24} has no conjugate points, and by the argument above the same is true of γ_-. We thus see that there is no contribution to the Maslov index in the corner near p.

7.6 Passage near M_ϵ^R

We now consider the tangent space to $W^u(0)$ as it moves by M_ϵ^R. Since M_0^R is one-dimensional, it will be helpful to think of the curve $T_\varphi W^u(0)$ as being parametrized by v (and sometimes u). As noted in previous sections, $\varphi'(z) \in E^u(0,z)$ is tangent to leading order to $T_\varphi(z)M_\epsilon^R$ for this part of the journey. Due to its being crushed against $W^u(M_\epsilon^R)$, the other vector spanning $T_\varphi W^u(0)$ will be in an unstable direction, which must be $\eta_4 = \eta_4(v)$ by the symplectic considerations. As was the case for the fast jumps, we are free to take $\epsilon = 0$ due to the robustness of transverse crossings. This time, the limit $\epsilon \to 0$ is the singular limit on the slow timescale. At any point $P = (u, v, 0, y)$ on M_0^R, the shooting manifold therefore has tangent space

$$T_P W^u(0) = \mathrm{sp}\left\{ \begin{bmatrix} 1 \\ f'(u) \\ 0 \\ \frac{1}{c}(\gamma f'(u) - 1) \end{bmatrix}, \begin{bmatrix} f'(u) \\ 0 \\ f'(u)\mu_4(u) \\ \mu_4(u) \end{bmatrix} \right\}. \tag{7.29}$$

In this section, we must be more careful about the reference plane. The cubic is symmetric about its inflection point, meaning that

$$f'(1/3(a+1) + u) = f'(1/3(a+1) - u). \tag{7.30}$$

In particular, this implies that $f'(0) = f'(u^*)$, and hence the linearization of (5.24) at the two jump-off points 0 and q has the same set of eigenvectors and eigenvalues. This is problematic, since $\varphi'(z)$ approaches q in the direction $\eta_2(q)$, which we now see is in the subspace $V^s(0)$. Moreover, we cannot use any perturbation arguments at this point, since there is another non-smooth (in the limit) reorientation at q to prepare for the jump back to M_0^L. To sidestep this issue, we simply use the reference plane (7.10), with τ chosen so that $\varphi(\tau)$ is on the slow manifold M_0^L, but not at 0 or the landing point $\hat{q}$. We will see that this slides the conjugate point down M_ϵ^R to a point safely away from either corner. Now, it is clear that a crossing occurs at a point (u, v, w, y) if and only if

$$\det \begin{bmatrix} f'(u_\tau) & 1 & 1 & f'(u) \\ 0 & f'(u_\tau) & f'(u) & 0 \\ f'(u_\tau)\mu_1(u_\tau) & 0 & 0 & f'(u)\mu_4(u) \\ \mu_1(u_\tau) & \frac{1}{c}(\gamma f'(u_\tau) - 1) & \frac{1}{c}(\gamma f'(u) - 1) & \mu_4(u) \end{bmatrix} = 0. \tag{7.31}$$

For sure, the expression in (7.31) vanishes at least once. Indeed, u ranges from 1 to $u^* = 2/3(a+1)$ on M_0^R, so since $2/3(a - 1/2) < u_\tau < 0$, it follows from (7.30) that u must attain the unique value u_* such that $f'(u_*) = f'(u_\tau)$. At this point (call it $\varphi(z_*)$), $\varphi'(z_*) = T_{\varphi(z_*)}M_0^R$ is parallel to $\eta_2(u_\tau) \in E^u(0, \tau)$, which means that z_* is a conjugate point. At any other point on M_0^R, a tedious (but routine) calculation of the determinant in (7.31) reveals that it does not vanish, hence there are no other other conjugate points on this segment.

To calculate the contribution to the Maslov index, we need the dimension and direction of the single crossing, which occurs at the point $P_* := \varphi(z_*) = (u_*, v_*, w_*, y_*)$ and time $z = z_*$. Since $f'(u_\tau) = f'(u_*)$ but $\mu_1(u_\tau) \neq \mu_4(u_*)$, it is clear from (7.31) that the intersection $E^s(0, \tau) \cap T_{P_*}W^u(0)$ is one-dimensional, spanned by $\eta_2(u_\tau) = \eta_2(u_*)$, the velocity of φ. For the direction of the crossing, observe that (3.13) evaluated on the velocity φ' at a conjugate time z^* can be rewritten

$$\omega(\varphi', \frac{d}{dz}\varphi')|_{z=z_*} = \epsilon\omega(\varphi', \frac{d}{d\zeta}\varphi')|_{\zeta=\epsilon z_*}. \tag{7.32}$$

Since we only care about the sign of this expression, we can ignore the ϵ in front. Furthermore, $\varphi' \approx \eta_2(v)$ along M_ϵ^R, and v increases as ζ increases for the reduced flow, so it follows that

$$\operatorname{sign}\Gamma(E^u, E^s(0, \tau); z_*)(\varphi'(z_*)) = \operatorname{sign}\omega(\eta_2(v), \partial_v\eta_2(v))|_{v=v_*} = \frac{g''(v_*)}{c} > 0. \tag{7.33}$$

In the above calculation, $g = f^{-1}$, so $g''(v_*) = -f''(u_*)/(f'(u_*))^3 < 0$. This shows that the crossing near the slow manifold contributes $+1$ to the Maslov index, so it offsets the crossing in the opposite direction along the fast jump.

7.7 Second corner

As the slow flow carries $\varphi(z)$ up M_ϵ^R, it approaches the jump-off point q, which is the scene of another abrupt reorientation of $W^u(0)$. At the bottom right corner, we saw that there was no contribution to the Maslov index, irrespective of which of the two possible paths $T_{\varphi(z)}W^u(0)$ took to get to its starting position for the slow flow. Unfortunately, we will not be so lucky at the right jump-off point.

First, let us determine the correct "boundary conditions" for the corner problem. From the previous section, we know that $T_{\varphi(z)}W^u(0)$ will be $O(\epsilon)$ close to X_{24} as $\varphi(z)$ approaches q. In this section, the notation X_{ij} refers to the plane $\mathrm{sp}\{\eta_i(q), \eta_j(q)\}$ spanned by eigenvectors of the linearization (5.20) evaluated at q, for which $u = u^*$ and $f'(u^*) = -a$. The exit position of φ along the back can be determined by using the singular solution; as was the case for the front, the wave will be launched from q in the weak unstable direction, $\eta_3(u^*)$. We argue that the second direction present is the most unstable eigenvector $\eta_4(u^*)$. Indeed, the tangent vectors to $W^u(0)$ solve (6.27), which is essentially autonomous in the neighborhood of q. We know that the initial condition will be $O(\epsilon)$ close to η_4, and therefore this direction must dominate near the corner, since it is the direction of most rapid growth for the autonomous system. It follows that we are searching for a heteroclinic connection from X_{24} to X_{34}.

From §7.2, we know that $W^u(X_{24})$ is one-dimensional and X_{34} is a global attractor, so the only way to move from one point to the other in $\Lambda(2)$ is to exchange η_2 for η_3 in the basis for $T_{\varphi(z)}W^u(0)$. There are again two orbits that make this connection, γ_+ through $\mathrm{sp}\{\eta_2 + \eta_3, \eta_4\}$ and γ_- through $\mathrm{sp}\{\eta_2 - \eta_3, \eta_4\}$. As before, it is easy to find the solutions for the equation induced on $\Lambda(2)$ by the constant coefficient system

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ a & 1 & -c & 0 \\ -1 & \gamma & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{7.34}$$

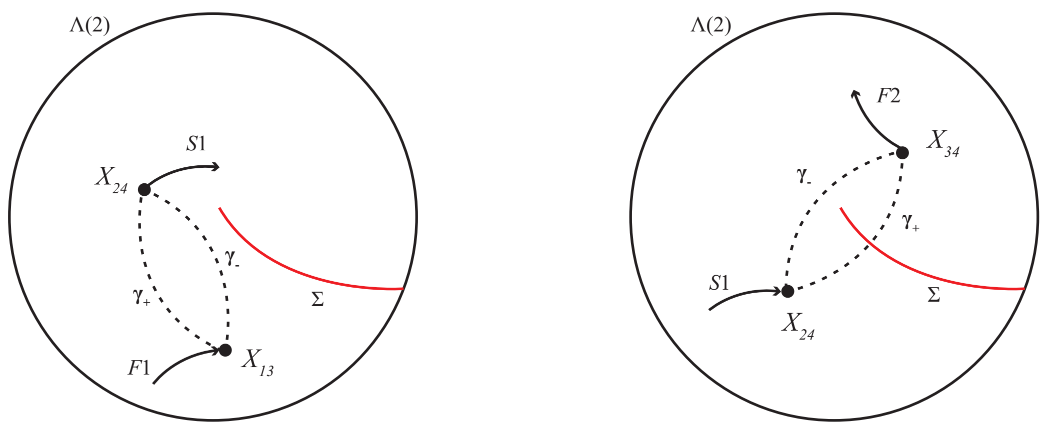
pinned at the corner q. In Plücker coordinates, these two paths are

$$\gamma_\pm(z) = (0, 0, 0, 0, 1, \pm e^{(\mu_3 - \mu_2)z}). \tag{7.35}$$

This time the concatenated path $\gamma_0 = \gamma_+ * -\gamma_-$ has Maslov index 1. Indeed, one can reparametrize γ_0 to see that it has the same homotopy class as

$$\tilde{\gamma}_0 = \begin{cases} (0,0,0,0,1-t,t) & t \in [0,1] \\ (0,0,0,0,t-1,2-t) & t \in [1,2]. \end{cases} \tag{7.36}$$

(This is a loop since the coordinates are homogeneous.) We are now free to use any reference plane to compute the Maslov index, so we choose the convenient subspace $V = \mathrm{sp}\{\eta_1, \eta_3\}$. By (7.4), the train of V is given by $p_{24} = 0$, so one sees that there is a unique conjugate point for $\tilde{\gamma}_0$ at $t = 1$, with η_3 spanning the intersection. It is not difficult to see that this crossing is regular, so it contributes ± 1 to the Maslov index. (The sign is not important.) This is the only crossing, so by homotopy invariance of the Maslov index, Proposition 3.1.1(ii), it follows that the Maslov index of γ_0 is ± 1. Thus the Maslov indicies of γ_+ and γ_-–which are integers summing to ± 1–must be different.



(a) Corner near p (b) Corner near q

Figure 7.1: Schematic of corners where transitions occur between fast and slow dynamics.

We actually make the stronger claim that one of the indices is 0 and the other is ± 1. To see this, recall from (4.32) that we detect crossings by evaluating a fixed one-form on (7.35). Doing so yields a monotone function in z, which can have only 0 or 1 zeros. It therefore suffices to check the sign of this one-form at the endpoints (i.e. $|z| \gg 1$) of the correct curve. Before doing so, we must determine which of γ_+ and γ_- is traversed to connect the two states.

The scalings of η_2 and η_3 are important for distinguishing the paths $\gamma_+(z)$ and $\gamma_-(z)$, so we fix the basis vectors

$$\eta_2 = \begin{bmatrix} -1/a \\ 1 \\ 0 \\ \frac{1}{c}\left(\gamma + \frac{1}{a}\right) \end{bmatrix}, \ \eta_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \eta_4 = \begin{bmatrix} -a \\ 0 \\ -a\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{7.37}$$

As explained above, one tangent direction to the shooting manifold is η_4, which will not move in the limit, since it is an eigenvector of (7.34). It is therefore evident that the trajectory in $\Lambda(2)$ is driven by the change in the velocity φ'. To see which of the paths $\gamma_\pm$ is taken, we must know the sign of the multiple of η_2 (resp. η_3) that φ' is upon entrance to (resp. exit from) a neighborhood of q. The entrance is clear from the slow flow (5.26); to leading order, v is increasing, u is decreasing, $w \approx 0$ and y is decreasing, hence φ' is a positive multiple of η_2, comparing with (7.37).

To see the orientation at exit, set $\tilde{y} = y - \frac{1}{c}(\gamma v^* - u^*)$. Along the back, y goes from $\frac{1}{c}(\gamma v^* - u^*)$ at $q \in M_0^R$ to $\frac{1}{c}(\gamma v^* - (u^* - 1))$ at $\hat{q} \in M_0^L$, so $\tilde{y}$ goes from 0 to $-1/c$. Furthermore, we compute that $\tilde{y}$ satisfies

$$\tilde{y}' = -y' = cy - \gamma v^* + u = -c\tilde{y} - (u^* - u) = -c\tilde{y} - u_f, \tag{7.38}$$

where u_f is the equation for u on the front, as in (5.29). This is the same equation and boundary conditions satisfied by y along the front, so we have

$$\tilde{y}(z) = Ke^{-cz} - e^{-cz} \int_{-\infty}^{z} e^{cs} u_f(s)\, ds, \tag{7.39}$$

where K is given by (5.37). Notice that K is positive, so

$$\lim_{z \to -\infty} e^{cz} y'(z) = -\lim_{z \to -\infty} e^{cz} \tilde{y}'(z) = cK < 0, \tag{7.40}$$

using (7.39). As along the front, u and w still decay faster than y at $-\infty$, so it follows that φ' leaves q along the back in the direction $cK\eta_3$. This proves that the connecting orbit in $\Lambda(2)$ from X_{24} to X_{34} is γ_-.

To determine the contribution to the Maslov index, it therefore suffices to compare the signs of $\det\left[E^s(0,\tau), \eta_2, \eta_4\right]$ and $\det\left[E^s(0,\tau), -\eta_3, \eta_4\right]$. Representing $E^s(0,\tau)$ in the basis (7.10), a calculation gives that

$$\det\left[E^s(0,\tau),\eta_2,\eta_4\right] = \frac{\delta(\delta a\sqrt{2} + 2Qa^2 - \delta\mu_1(u_\tau))}{-2ac}, \tag{7.41}$$

where $\delta = -f'(u_\tau) - a > 0$ and $Q = \sqrt{2}/2 - \mu_1(u_\tau) > 0$. The introduction of these variables simplifies the calculation because $f'(u_\tau)$ approaches $-a$ from above as $u_\tau \to 0$. It is thus clear that the determinant in (7.41) is positive. Similarly, we compute that

$$\det\left[E^s(0,\tau),-\eta_3,\eta_4\right] = \frac{(f'(u_\tau))^2 a(\sqrt{2} - 2\mu_1(u_\tau))}{2} > 0. \tag{7.42}$$

Since the detection form is monotone in z on $\gamma_-(z)$, the fact that it has no changes in sign implies that it has no zeros, and therefore there are no conjugate points near q. To recap, the cumulative Maslov index as we enter the back is 0: -1 from the front $+1$ near the right slow manifold.

7.8 Second fast jump

The analysis of the back is nearly identical to that of the front, so we will skip many of the details. Along the back, $W^u(0)$ is $O(\epsilon)$ close to $W^u(q)$, the cylinder over the Nagumo back. As a remark, the full power of the Exchange Lemma is not needed to see this; we are not carrying any extra center/slow directions in the Maslov index calculation. We are once again free to consider intersections of $T_{q_b(z)}W^u(q)$ with the train of $V^s(0)$, since $V^s(0)$ is transverse to the tangent space to the cylinder near q and $\hat{q}$. Recycling the notation $u_f(z)$ from the front, we have $u_b = u^* - u_f$, hence $w_b = -w_f$. We can again solve for w as a function of u to obtain

$$w(u) = -\frac{\sqrt{2}}{2}(u^* - u)(1 - (u^* - u)), \tag{7.43}$$

where now u ranges from u^* to $u^* - 1$. This yields the basis

$$\operatorname{sp}\left\{\begin{bmatrix} 1 \\ 0 \\ \frac{\sqrt{2}}{2} - \sqrt{2}(u^* - u) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} \tag{7.44}$$

of $T_{q_b(z)}W^u(q)$. Comparing with (7.11), we see that there is a unique conjugate point, which is the value z^* such that $(u^* - u) = \dfrac{1}{2} + a$. The intersection $V^s(0) \cap T_{q_b(z)}W^u(q)$ is again spanned by

$\xi = \{1, 0, -a\sqrt{2}, \sqrt{2}\}$. Since $f'(u^* - 1/2 - a) = f'(1/2 + a)$ by (7.30), the crossing form calculation is identical to (7.13). Explicitly, we have

$$\begin{aligned} \omega(\xi, A(0, z^*)\xi) &= -f'\left(u^* - \left(\frac{1}{2} + a\right)\right) + ca\sqrt{2} - 2a^2 \\ &= -f'\left(\frac{1}{2} + a\right) + ca\sqrt{2} - 2a^2 \\ &= a^2 - \frac{1}{4} < 0. \end{aligned} \tag{7.45}$$

Thus the Maslov index of the second fast jump is -1.

7.9 Final corner, passage near M_ϵ^L, and return to equilibrium

The analysis of the corner $\hat{q}$ is identical to that of p. First, the symmetry of f ensures that the set of eigenvectors and eigenvalues for the system (5.20) evaluated at p and at $\hat{q}$ are the same when $\epsilon = 0$. Also, the tangent space of $W^u(0)$ is $O(\epsilon)$ close to X_{13} upon entrance into a neighborhood of both points. Finally, Deng's Lemma and the already-proved existence of the wave necessitate that $T_{p_{\text{out}}} W^u(0)$ and $T_{\hat{q}_{\text{out}}} W^u(0)$ are both $O(\epsilon)$ close to X_{24}. Since there are only two possible paths of (Lagrangian) planes connecting X_{13} and X_{24}–neither of which has any conjugate points–there is no need to investigate the corner $\hat{q}$ further. We therefore turn our attention to the slow return to equilibrium.

As for M_ϵ^R, we expect one conjugate point for the final slow piece. This one is actually easier to find; by definition of Maslov(φ), there is a conjugate point at $z = \tau$, for which value of z we have

$$\mathrm{sp}\{\varphi'(\tau)\} = E^u(0, \tau) \cap E^s(0, \tau). \tag{7.46}$$

The fact that φ is transversely constructed implies that the intersection is only one-dimensional. In terms of the singular orbit, we see that the intersection is spanned by the tangent vector to $T_{\varphi(\tau)} M_\epsilon^L$. The non-existence of any other conjugate points is identical to §7.5; one simply shows that the determinant which detects conjugate points does not vanish unless $u = u_\tau$.

The signature of the crossing form is also computed as in §7.5. This time, we have

$$\operatorname{sign} \Gamma(E^u(0, \cdot), E^s(0, \tau); \tau)(\varphi'(\tau)) = -\operatorname{sign} \omega\left(\eta_2(v), \partial_v \eta_2(v)\right)|_{v = v_\tau}, \tag{7.47}$$

since v decreases as $\zeta = \epsilon z$ increases on M_0^L. Once again defining $g(v) = f^{-1}(v)$–this time on the left branch of M_0–one computes from (2.34) that

$$\omega\left(\eta_2(v), \partial_v \eta_2(v)\right)|_{v=v_\tau} = \frac{g''(v)}{c} < 0, \tag{7.48}$$

where $g''(v) = -f''(u_\tau)/(f'(u_\tau))^3 > 0$. Hence the crossing is positive, as it was for the conjugate point on M_ϵ^R. Since this crossing occurs at the right endpoint of the curve $E^u(0,z)$, the contribution to the Maslov index is $+1$, by Definition 3.1.

7.10 Concluding remarks

For those keeping score at home, it follows from Proposition 3.1.1(i) that:

Theorem 7.10.1.

$$\text{Maslov}(\varphi) = -1 + 1 - 1 + 1 = 0. \tag{7.49}$$

In conjunction with Theorem 6.2.1, this proves Theorem 7.0.1, i.e. that the fast traveling pulses for the FitzHugh-Nagumo system are (nonlinearly) stable. Although the profiles and speeds of the waves in (5.1) and those for the same equation without diffusion on v are very similar, we point out that the stability proofs are entirely different and independent of each other. In [44, 81], the stability result is obtained by showing that the eigenvalues of the linearized operator are close to those for the reduced systems corresponding to the fast front and back. Conversely, the eigenvalue problem for L in (6.26) is analyzed entirely as an operator on $BU(\mathbb{R}, \mathbb{R}^2)$. Thus the smallness of ϵ in each setting appears in different ways. In Chapter 6, it is used to achieve monotonicity for the Maslov index in the spectral parameter, as well as to prove that the unstable spectrum of L must be real. Most notably, the small parameter allowed us to calculate Maslov(φ) in this chapter using geometric singular perturbation theory.

CHAPTER 8

Summary of results and future directions

In this thesis, we obtained several results related to the existence and stability of traveling pulses for three variations of reaction-diffusion equations. The broadest equations studied were skew-gradient systems (1.19) with arbitrarily many components. We also studied a subset of (1.19) consisting of two-component activator-inhibitor-systems (4.1). Finally, we studied the doubly-diffusive FitzHugh-Nagumo equation (5.1), which is a singularly-perturbed example of (4.1). The overarching goal of the thesis was to develop the Maslov index as a powerful tool in the stability analysis of traveling pulses in skew-gradient systems.

The main results are as follows. For two-component activator inhibitor systems, we proved an analog of a result of Chardard & Bridges [15] that the parity of the Maslov index determines the sign of the derivative of the Evans function at $\lambda = 0$ (§4.3). Our proof is quite different from (and more elementary than) that of [15]. For general skew-gradient systems, we proved that the Maslov index gives a lower bound on the number of unstable eigenvalues for the linearization about a traveling pulse (§6.1). Additionally, we provide a blueprint for the obstacles that arise in trying to make the inequality into equality when dealing with non-self-adjoint operators (§6.2).

For the FitzHugh-Nagumo system, we proved the existence of traveling fronts (§5.3) and pulses (§5.4) using geometric singular perturbation theory. The existence of the fronts was not previously known. The existence of the pulses was known [17], but our proof provides a more detailed description of the pulses, as well as the means for proving stability. We also proved that the Maslov index gives an exact count of all unstable eigenvalues by addressing the issues identified in §6.2. This involved a careful consideration of multiplicity of eigenvalues, which brought in the Evans function again and led to a new understanding of the relationship between the Evans function (particularly its derivatives) and the Maslov index (§6.3). Finally, we proved that the traveling pulses are stable (§7.10) by developing a novel way of calculating the Maslov index using geometric singular perturbation to tack invariant manifolds through phase space. To our knowledge, this is the first instance in which a complete calculation of the Maslov index has been used to prove the stability of a soliton.

8.1 Future directions

Regarding the use of the Maslov index in stability problems, this thesis addressed two different types of problems. First, it broadened the applicability of the Maslov index by using it to analyze the stability of nonlinear waves in a larger class of systems. Second, it addressed the critical issue of calculating the index. Below we describe several possible new directions under both of these headings.

8.1.1 (In)stability of patterns via the Maslov index

In §1.4, we explained that much of the motivation for studying skew-gradient systems is that gradient systems do not support Turing bifurcations, whose role we want to understand in pattern formation. Armed with an understanding of the challenges of skew-gradient systems from this work, a natural next step is to use the Maslov index to study spatially periodic structures directly. The idea is that any patterns that exist when $D \approx I$ should be unstable. My approach to this problem will begin with the work of Gardner [34], who showed how the Evans function can be used to locate unstable spectrum for the linearization about a periodic wave train. The idea is to recast the problem as an eigenvalue problem involving the Floquet spectrum. It is then shown in [47, 48] how the Maslov index can be used to count unstable eigenvalues for self-adjoint Schrödinger operators with θ-periodic boundary conditions, which can easily be related to Gardner's formulation. My plan is to generalize the result of [47] to skew-gradient systems (for which L is not self-adjoint) and give a lower bound on the Morse index in the same way that [21] adapts the Maslov Box of [41] for traveling pulses. Since the objective is to use the Maslov index as an *instability* index for the patterns of interest, having an inequality involving the Maslov index is of tremendous value.

8.1.2 Generalized Maslov index

To elucidate the relationship between diffusivities and stability, it is desirable to consider systems that *do* exhibit patterns emerging from Turing bifurcations, such as the Gray-Scott and Gierer-Meinhardt models. These are not skew-gradient systems due to their respective nonlinearities. As such, there is not a symplectic structure preserved by the eigenvalue equations. In ongoing work with Graham Cox, Chris Jones, Robert Marangell, and Yuri Latushkin, we are developing an integer index that can be applied to analyze these models. We have identified a very large (i.e. open and dense) subset $M \subset \mathrm{Gr}_n(\mathbb{R}^{2n})$ which deformation retracts onto the Lagrangian Grassmannian $\Lambda(n)$. Since M and $\Lambda(n)$ consequently have the same topology, we will be able to associate an integer index (the "generalized Maslov index") to any loop in M. The next step is to understand how this integer is related to the dynamics of the eigenvalue equation obtained by linearizing about a pattern. This will involve developing an intersection-based theory of the generalized Maslov index (cf. [4, 70]), as it is intersections of curves in the Grassmannian which encode unstable eigenvalues.

8.1.3 Understanding instability in skew-gradient systems

Generic standing pulses in gradient reaction-diffusion equations were recently shown to be unstable using the Maslov index [7]. The idea of the proof is to use the reversibility of the standing wave ODE to assert the existence of a conjugate point. Since the crossing form is monotone in z for gradient systems, this proves that the Maslov index is greater than zero, hence so is the Morse index. It would be interesting to pursue similar results for skew-gradient systems. This is a highly non-trivial problem for traveling waves in general, since the steady state equation does not possess the same symmetry. However, there is a more insidious issue for skew-gradient systems, which is that the Maslov index is generally *not* monotone in z. For example, the calculation in Chapter 7–see Theorem 7.10.1–shows that there are four conjugate points, but there are two each in the positive and negative directions, which offset to give an index of 0. Thus the instability of the wave cannot be inferred (a priori) from the mere existence of a conjugate point. I will investigate how the reversibility symmetry for standing waves in skew-gradient systems can be utilized to say something about the existence of unstable eigenvalues. I expect that the crossing form will reflect the symmetry and that it will be key to obtaining instability results.

8.1.4 Calculations in singularly perturbed systems with fold points

Chapter 7 offers a framework for calculating the Maslov index for singularly perturbed PDE. For the fast waves in (5.1), the slow manifold is normally hyperbolic, so Fenichel theory applies. However, it is important for many applications (especially to climate modeling) to understand regimes in which normal hyperbolicity breaks down at so-called *fold points*. The state-of-the-art tool for analyzing dynamics near fold points is called the "blow-up" method [28]. The corner analysis of §7.5 and §7.7 in this work is actually reminiscent of the blow-up technique, in that the focus is on a single point on M_0. It would be worthwhile to see if similar calculations work for waves that pass through fold points. One example could be provided by multi-pulses in the FitzHugh-Nagumo equation, which are known to exist in the 3D system [13, 12]. The existence of these pulses is proved using the blow-up method, which should be applicable when v is allowed to diffuse as well. Another possible example is slow pulses (i.e. speed $c = O(\epsilon)$) for (5.1).

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