Cohomology of Flag Varieties and the BK-Filtration

by
Charles Hague

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2007

Approved by

Advisor: Professor Shrawan Kumar
Reader: Professor Prakash Belkale
Reader: Professor Robert Proctor
Reader: Professor Richard Rimanyi
Reader: Professor Jonathan Wahl

ABSTRACT

CHARLES HAGUE: Cohomology of Flag Varieties and the BK-Filtration

(Under the direction of Professor Shrawan Kumar)

Let G be a complex algebraic group and let P be a parabolic subgroup of G. Let $T^*(G/P)$ denote the cotangent bundle of the flag variety G/P. In this thesis we describe results connecting cohomology of bundles on $T^*(G/P)$ to purely combinatorial objects such as filtrations on irreducible G-modules and Lusztig's q-analog of weight multiplicity.

To my family, for their continued and loving support.

CONTENTS

CHAPTER 1

Introduction

Let G be complex semisimple algebraic group. For a dominant weight μ let $V(\mu)$ denote the irreducible representation of G with highest weight μ and for any weight λ of $V(\mu)$ let $V_\lambda(\mu)$ denote the λ-weight space of $V(\mu)$. In [**6**], R. Brylinski constructed a filtration on weight spaces of irreducible G-modules. This filtration, often referred to as the Brylinski-Kostant (or BK) filtration, studies the action of certain special nilpotent elements of the Lie algebra of G (the **principal** nilpotents). Explicitly, let e be a principal nilpotent and set

$$\mathcal{F}^n\big(V_\lambda(\mu)\big) := \{v \in V_\lambda(\mu) : e^{n+1}.v = 0\},$$

where μ, λ are as above. This filtration was motivated by fundamental work of Kostant [**21**], [**22**] on adjoint orbits and actions of sl_2-triples.

In the presence of a particular higher cohomology vanishing result for line bundles on the cotangent bundle $T^*(G/B)$ of G/B, Brylinski showed that certain polynomials, Luzstig's **q-analogs of weight multiplicity**, compute the dimensions of the various degrees of this filtration. These polynomials were initially introduced in [**25**] and were proven in [**17**] to be equal to certain **Kazhdan-Lusztig polynomials** for affine Weyl groups. These Kazhdan-Lusztig polynomials are deep objects in combinatorial representation theory, cf [**18**], [**19**], and [**25**].

In chapter 3 we generalize Brylinski's results to the filtration defined in a similar way by arbitrary even nilpotents (see section 2.3 for the definition of an even nilpotent). The main theorem in this chapter, Theorem 3.2.1, states that certain "parabolic analogs" of Kazhdan-Lusztig polynomials compute the dimensions of the various degrees of these filtrations. This requires us to describe the correct generalization of the cohomology vanishing condition required for Brylinski's result. This generalization involves cohomology vanishing results for bundles on cotangent bundles of partial flag varieties of G.

The sheaf cohomology groups of G-equivariant bundles on flag varieties of G (often called **homogeneous bundles**) are G-representations, and a central question is to determine which G-representations appear in the cohomology of these bundles. In particular, the study of G-equivariant bundles on cotangent bundles of flag varieties has been extensively studied in [**1**], [**3**], [**4**], and [**12**]; additionally, [**30**] extends some of these results to algebraic groups in positive characteristic. Broer ([**4**]) in particular has extensive results in this direction and we will generalize some of his results.

Explicitly, for any standard parabolic subgroup P of G, let $p_P : T^*(G/P) \to G/P$ denote the cotangent bundle of the flag variety G/P. For any P-dominant weight λ there is a corresponding homogeneous bundle $\mathcal{L}_P(\lambda)$ on G/P (see section 2.2 below for the precise definition of these bundles). For $i \geq 0$ set

$$H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) := H^i\big(T^*(G/P),\, p_P^*\,\mathcal{L}_P(\lambda)\big)$$

and for any weight λ set

$$H^{i,\mathfrak{n}}(\lambda) := H^i\big(T^*(G/B),\, p_B^*\,\mathcal{L}_B(\lambda)\big)\,.$$

The higher cohomology vanishing required for Brylinski's result is that $H^{i,\mathfrak{n}}(\lambda) = 0$ for all $i > 0$ and dominant λ; in [**4**] Broer proved this result (Broer's result is actually stronger than this, cf Theorem 4.2.4 below). Previously this vanishing had been known for many (although not all) dominant λ, cf [**1**], [**6**], and [**12**]. Broer also proved that $H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) = 0$ for all $i > 0$ when (1) P is a minimal parabolic corresponding to a short simple root and λ is dominant; and (2) $\mathcal{L}_P(\lambda)$ is a line bundle on G/P and λ is dominant (in general, the bundles $\mathcal{L}_P(\lambda)$ will not be line bundles).

In chapter 4 we prove the following extensions of these results, cf Theorems 4.2.5, 4.4.4, and 4.5.5. We show that $H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) = 0$ for all $i > 0$ when: (1) P is an even parabolic and both λ and $\lambda + 2\rho_P$ are dominant (see Section 2.2 for the definition of ρ_P and Definition 2.3.7 for the definition of even parabolics); (2) P is any minimal parabolic and λ is dominant; and (3) G is of type A and λ is regular dominant. (3) was already known to S. Kumar as an easy application of Frobenius vanishing results; we include it here for completeness.

These vanishing results provide a basic technical tool in analyzing filtrations on weight spaces but are interesting in their own right; another application of these cohomology vanishing results is in the geometry of $\mathrm{Ad}(G)$-orbits of nilpotent elements in $\mathrm{Lie}(G)$. These orbits and their closures are subvarieties of $\mathrm{Lie}(G)$ with rich geometric structure. Using cohomology vanishing results on cotangent bundles of flag varieties it has been shown that some of these orbits have normal closures and that, in general, the normalizations of these orbit closures have rational singularities, cf [**4**], [**11**], [**13**], [**27**], [**28**], and [**29**].

In chapter 5 we conclude by explicitly giving examples of these filtrations on various irreducible representations.

CHAPTER 2

Background

2.1. Notation

Let G denote a complex semisimple simply-connected algebraic group over $\mathbb{C}$ and T a maximal torus in G (we assume G simply-connected only for notational convenience; all results in my thesis generalize easily to G of arbitrary isogeny type). Set $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{h} = \text{Lie } T$. Let W be the Weyl group of G. Fix a Borel subgroup $B \subseteq G$ containing T. Let $\Lambda \subseteq \mathfrak{h}^*$ denote the weight lattice of G. Let $\Lambda_R \subseteq \Lambda$ denote the root lattice. Let ρ be the half sum of all elements of Δ^+; ρ has the property that $\rho(\alpha^\vee) = 1$ for all $\alpha \in \pi$. Note that $\rho \in D$. There is a shifted action of W on $\mathfrak{h}^*$ defined by $w * \lambda := w(\lambda + \rho) - \rho$. This action clearly keeps Λ and Λ_R stable.

Let Δ (resp. Δ^-, Δ^+), denote the roots (resp. positive and negative roots) with respect to T and B and let $\pi \subseteq \Delta^+$ denote the simple roots. For A, $B \subseteq \Lambda$ and $\mu \in \Lambda$ set $A + B := \{\lambda_1 + \lambda_2 : \lambda_1 \in A,\ \lambda_2 \in B\}$ and set $\mu + A := \{\mu + \lambda_1 : \lambda_1 \in A\}$. For a subset $C \subseteq \Lambda$ set $\langle C \rangle := \sum_{\lambda \in C} \lambda$. Set $\langle \emptyset \rangle = 0$. For an element $w \in W$ set

$$R(w) := \{\beta \in \Delta^+ : w\beta \in \Delta^-\}\,.$$

One can check that for $w \in W$ and $\lambda \in \Lambda$ we have $w * \lambda = w\lambda - \langle R(w^{-1}) \rangle$.

Set

$$D := \{\lambda \in \Lambda : \lambda(\alpha^\vee) \geq 0 \text{ for all } \alpha \in \pi\},$$

the collection of **dominant weights** in Λ. We say that a weight $\mu \in D$ is **regular** if $\mu(\alpha^\vee) > 0$ for all $\alpha \in \pi$. Also set

$$D' := \{H \in \mathfrak{h} : \alpha(H) \in \mathbb{R} \text{ and } \alpha(H) \geq 0 \text{ for all } \alpha \in \pi\},$$

the **dominant chamber** in H. Irreducible representations of G are parametrized by elements of D; let $V(\mu)$ denote the irreducible representation of G with highest weight $\mu \in D$.

For any $\alpha \in \pi$ let χ_α denote the fundamental weight corresponding to α. That is, $\chi_\alpha(\alpha^\vee) = 1$ and $\chi_\alpha(\beta^\vee) = 0$ for all $\beta \in \pi \setminus \{\alpha\}$. For any nonempty $S \subsetneq \pi$ set $\chi_S := \sum_{\alpha \in S} \chi_\alpha$. If $S = \emptyset$ set $\chi_S = 0$. Hence we can write $\rho = \chi_\pi$.

For any T-module M and any weight μ of M, let M_μ denote the μ-weight subspace of M. In particular, for dominant λ,

$$V_\mu(\lambda) := V(\mu)_\lambda$$

is the μ-weight space of the irreducible G-module $V(\lambda)$.

2.2. Flag Varieties and Homogeneous Bundles

Remark 2.2.1. Although we are assuming that G is a semisimple group, most results in this section are vaild for arbitrary reductive groups.

Parabolic subgroups and G-homogeneous spaces.

Definition 2.2.2. A subgroup $P \subseteq G$ is called **parabolic** subgroup if it contains a Borel subgroup of G. We say that P is a **standard** parabolic subgroup if $B \subseteq P$. A subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is a **parabolic** subalgebra if $\mathfrak{p} = \text{Lie } P$ for a parabolic subgroup P of

G, and $\mathfrak{p}$ is a **standard** parabolic subalgebra if P is a standard parabolic subgroup (note of course that $\mathfrak{p}$ is a standard parabolic subalgebra iff $\mathfrak{p} \supseteq \mathfrak{b}$, where $\mathfrak{b} = \text{Lie } B$).

Remark 2.2.3. For the rest of this paper, we assume that all parabolics are standard parabolics, both on the group level and on the Lie algebra level. We may do this without loss of generality, since if P is a parabolic not containing B some conjugate of P will contain B (and similarly on the Lie algebra level).

Every parabolic subgroup can be written as a semidirect product $L \rtimes U_P$ where U_P is the unipotent radical of P and L is a reductive subgroup containing T. Since T is a maximal torus of L as well, the root system Δ_L of L is naturally a sub-root system of Δ, and Δ_L is generated by a collection of simple roots $\pi_L \subseteq \pi$. In particular, $\Delta_L = \Delta \cap \mathbb{Z}\pi_L$. We also have $\Delta_L^+ = \Delta^+ \cap \Delta^L$ and $\Delta_L^- = \Delta^- \cap \Delta^L$. In a slight abuse of notation we may refer to Δ_L as Δ_P, and similarly for $\Delta_L^{\pm}$ and π_L. In particular, $\pi_B = \Delta_B^+ = \emptyset$. Let ρ_L (also denoted by ρ_P) be the half-sum of all roots in Δ_L^+. We obviously have $\rho_B = 0$ and $\rho_G = \rho$.

Let W_L be the subgroup of W generated by the simple reflections corresponding to the elements of π_L, and let W^L denote the set of minimal-length coset representatives of W/W_L. In particular, W_L has a longest element that we will denote w_0^L. As above, we may sometimes write W_P, W^P, etc.

A parabolic subgroup P of G is uniquely determined by π_L (recall that we are assuming P to be a standard parabolic); and conversely, every subset of π gives rise to a (standard) parabolic subgroup. Explicitly, we can write $P = L \rtimes U_P$ as above, where L is generated by T and the U_α with $\alpha \in \Delta_P$.

Similarly, $\mathrm{Lie}(P) =: \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{p}}$ where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{n}_{\mathfrak{p}} = \mathrm{Lie}(U_P)$. As U_P is normal in P we have that $\mathfrak{n}_{\mathfrak{p}}$ is an ideal of $\mathfrak{p}$. Moreover, we have

$$\mathfrak{l} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_P} \mathfrak{g}_\alpha \right)$$

and

$$\mathfrak{n}_{\mathfrak{p}} = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_P^+} \mathfrak{g}_\beta \,.$$

For any algebraic group G over $\mathbb{C}$ and any algebraic subgroup $H \subseteq G$ we have a scheme-theoretic quotient G/H. Topologically, we give G/H the natural quotient topology under the map $h : G \to G/H$. To define the structure sheaf of G/H we set (for any open $U \subseteq G/H$)

$$\mathcal{O}_{G/H}(U) = \{ f \in \mathcal{O}_G\big(h^{-1}(U)\big) : f(ga) = f(g) \text{ for all } g \in G \text{ and } a \in H \}\,.$$

This gives G/H the structure of a locally-ringed space, and with this structure G/H is a scheme over $\mathbb{C}$. Every G-homogeneous space is of this form: if X is a variety with transitive G-action then $X \cong G/H_x$, where H_x is the stabilizer of any point $x \in X$.

We have that P is a parabolic subgroup if and only if G/P is a projective variety. When $P \neq B$ these varieties are called **partial flag varieties**, and G/B is called a **full flag variety**.

Representations of parabolics.

Every irreducible representation of a parabolic subgroup P arises from an irreducible representation of L with trivial U_P-action. Indeed, if V is an irreducible L-module, then since U_P is normal in P we may extend trivially over U_P to obtain an irreducible

P-module. Conversely, an irreducible P-module M must be a completely reducible U_P-module: since U_P is normal in P, we have that $\mathrm{soc}_{U_P} M$ is a P-submodule of M (where $\mathrm{soc}_{U_P} M$ is the direct sum of all completely reducible U_P-submodules of M). But, as U_P is unipotent, a completely reducible U_P-module must be trivial.

The highest weights of irreducible L-modules are parametrized by the P**-dominant** weights in Λ, where we say that $\lambda \in \Lambda$ is P-dominant if $\lambda(\alpha^\vee) \geq 0$ for all $\alpha \in \pi_L$. We say that λ is P**-regular dominant** if $\lambda(\alpha^\vee) > 0$ for all $\alpha \in \pi_L$. For P-dominant λ let $V^P(\lambda)$ denote the irreducible L-module with highest weight λ.

In the special case $P = B$ we have $L = T$ and hence all weights of G are B-dominant. In this case we simply have $V^B(\lambda) \cong \mathbb{C}_\lambda$. On the other extreme, the set of G-dominant weights is the set D of all dominant weights. We have $V^G(\lambda) \cong V(\lambda)$, where $V(\lambda)$ denotes the irreducible G-module with highest weight $\lambda \in D$.

Homogenous bundles on flag varieties.

We define a functor

$$\{P\text{-modules}\} \mapsto \{G\text{-equivariant bundles on } G/P\}$$

as follows. Let M be any P-module. Define a right action of P on $G \times M$ by

$$(g, v).p := (gp, p^{-1}v)\,.$$

Set

$$G \times^P M := (G \times M)/P\,.$$

For $g \in G$ and $m \in M$ let $g * m$ denote the image of (g, m) in $G \times^P M$. One checks easily that the map $\phi : G \times^P M \to G/P$ given by $g * m \mapsto gP$ is a vector bundle map with fibers isomorphic to M.

Furthermore, there is a natural G-action on $G \times^P M$ given by $g_1.(g_2 * m) = g_1 g_2 * m$. With this action, the map ϕ is G-equivariant. We call $G \times^P M$ a **homogeneous vector bundle** on G/P with fiber M. For a fixed parabolic P the association $M \mapsto G \times^P M$ is functorial and hence we obtain the functor mentioned above. We will denote this functor by $\mathcal{L}_P$. This functor is exact and commutes with dualizing, tensoring, and the direct sum operation. When no confusion results we will make no distinction between the bundle $\mathcal{L}_P(M)$ and its sheaf of sections, which is a G-equivariant locally free sheaf of $\mathcal{O}_{G/P}$-modules. We obtain a G-action on sections of $\mathcal{L}_P(M)$ via

$$(g_1.s)(g_2 P) := g_1.\big(s(g_1^{-1} g_2 P)\big)\,.$$

Let $\psi : E \to G/P$ be any G-equivariant bundle on G/P; then $\psi^{-1}(eP)$ is a P-module. Let F denote the exact functor

$$\{G\text{-equivariant bundles on } G/P\} \mapsto \{P\text{-modules}\}$$

that takes a bundle to its fiber over eP. It is straightforward to check that this functor gives an inverse of the functor $\mathcal{L}_P$ and hence we obtain an equivalence of categories

$$\{G\text{-equivariant bundles on } G/P\} \cong \{P\text{-modules}\}\,.$$

Two particular homogeneous bundles on G/P will be of special interest to us. Let $\mathfrak{n}_\mathfrak{p}$ be the Lie algebra of U_P. Then $T(G/P) \cong G \times^P \mathfrak{n}_\mathfrak{p}^*$ and hence $T^*(G/P) \cong G \times^P \mathfrak{n}_\mathfrak{p}$.

We will also be concerned with $G \times^P M$ in the case where the representation M of P is irreducible. In this case we are considering homogeneous bundles of the form $\mathcal{L}_P\big(V^P(\lambda)\big)$ for P-dominant $\lambda \in \Lambda$. For notational convenience, set

$$\mathcal{L}_P\,(\lambda) := \mathcal{L}_P\big(V^P(\lambda)^*\big)\,, \tag{2.1}$$

the dual bundle to $\mathcal{L}_P\big(V^P(\lambda)\big)$.

It is also worth mentioning that the bundles $G \times^P V$ are trivial for all G-modules V. Indeed, the map $G \times^P V \to G/P \times V$ given by $g * v \mapsto (g, gv)$ is an isomorphism of G-equivariant bundles over G/P.

The Borel-Weil-Bott Theorem.

Note that the bundles $\mathcal{L}_B\,(\gamma)$ for $\gamma \in \Lambda$ are line bundles on G/B, since $V^B(\gamma) = \mathbb{C}_\gamma$. In fact, every line bundle on G/B is of this form. The next result gives the G-module structure of the sheaf cohomology groups of these line bundles. For the statement of the next theorem, assume that G is an arbitrary reductive group.

THEOREM 2.2.4. *(The Borel-Weil-Bott Theorem) Choose $\lambda \in \Lambda$.*

(i) If there is $w \in W$ such that $w * \lambda \in D$ then

$$H^{l(w)}\left(G/B,\ \mathcal{L}_B\big(\lambda\big)\right)\ \cong\ V(\lambda)^*$$

and $H^i\left(G/B,\ \mathcal{L}_B\big(\lambda\big)\right) = 0$ for all $i \neq l(w)$. Note that such w is unique if it exists.

(ii) If there is no such $w \in W$ (equivalently, if there is an $\alpha \in \pi$ and $\nu \in W$ such that $(\nu * \lambda)\,(\alpha^\vee) = -1$), then $H^i\left(G/B,\ \mathcal{L}_B\big(\lambda\big)\right) = 0$ for all i.

(iii) Let P be a standard parabolic. Then for all $i \geq 0$,

$$H^i\,(G/P,\ \mathcal{L}_P\,(\lambda))\ \cong\ H^i\,(G/B,\ \mathcal{L}_B\,(\lambda))\,.$$

In general, if $P \neq B$ and $\lambda \in \Lambda$ is P-dominant, $\mathcal{L}_P(\lambda)$ need not be a line bundle, as $V^P(\lambda)$ will in general not be 1-dimensional.

Remark 2.2.5. Let $P \subseteq Q$ be two parabolics and let L_Q denote the Levi factor of Q. Then $L_Q \cap P$ is a parabolic subgroup of L_Q and we have $Q/P \cong L_Q/(L_Q \cap P)$. Hence Theorem 2.2.4 gives the sheaf cohomology of all L_Q-equivariant bundles on Q/P.

Positivity properties of homogeneous bundles.

As noted above, $\mathcal{L}_P(\lambda)$ will in general not be a line bundle for $P \neq B$. However, if λ satisfies $\lambda(\alpha^\vee) = 0$ for all $\alpha \in \pi_P$, then $V^P(\lambda)$ is 1-dimensional and $\mathcal{L}_P(\lambda)$ is a line bundle on G/P. The converse is also true, as it is easy to check that if λ is P-dominant and $\lambda(\alpha^\vee) > 0$ for some $\alpha \in \Delta_L^+$ then $\dim\,(V^P(\lambda)) > 1$.

Remark 2.2.6. We briefly review the notion of intersection product on an irreducible complex projective variety X (for a brief introduction to this subject see section 1.1.C of [**23**]). For any line bundle L of X we have the first Chern class

$$c_1(L) \in H^2(X;\mathbb{Z})$$

of L. Let $L_1, \ldots, L_k$ be line bundles on X and let $Y \subseteq X$ be a closed subspace of dimension k. The cup product

$$c_1(L_1) \cdot c_1(L_2) \cdot \ldots \cdot c_1(L_k)$$

is an element of $H^{2k}(X;\mathbb{Z})$ and hence

$$\big(c_1(L_1) \cdot c_1(L_2) \cdot \ldots \cdot c_1(L_k)\big) \cap [Y] \in H_0(X;\mathbb{Z}) \cong \mathbb{Z}\,,$$

where $\cap$ is the cap product and $[Y] \in H_{2k}(X;\mathbb{Z})$ is the fundamental class of Y. We denote this integer, called the **intersection product**, by $\int_Y c_1(L_1) \cdot c_1(L_2) \cdot \ldots \cdot c_1(L_k)$. One also has an intersection product for Cartier divisors on X through the line bundles associated to the divisors.

DEFINITION 2.2.7. Let L be a line bundle on a projective variety X. Then L is **nef** if $\int_C c_1(L) \geq 0$ for every irreducible curve $C \subseteq X$.

REMARK 2.2.8. By the Nakai-Moishezon-Kleiman criterion ([**23**], Theorem 1.2.23), a line bundle L on a projective variety X is ample iff

$$\int_Y c_1(L)^{dimY} > 0$$

for all irreducible subvarieties $Y \subseteq X$. Thus ample implies nef.

One has the following standard results for ample and nef line bundles.

THEOREM 2.2.9. *(cf [**23**], [**24**]) Let C and E be line bundles on a variety X over $\mathbb{C}$.*

(i) Assume that X is projective. If C is a quotient of a trivial bundle, then C is nef.

(ii) Assume that X is projective. If C is nef and E is ample then $C \otimes E$ is ample.

(iii) Assume that there is a proper morphism from X to an affine scheme. Then C is ample iff for all coherent sheaves $\mathcal{F}$ on X there is an integer m_0 such that $H^i(X, \mathcal{F} \otimes C^m) = 0$ for all $i > 0$ and $m \geq m_0$. In particular, note that the map $X \to Spec\,\mathbb{C}$ is a proper morphism when X is projective.

(iv) C is ample iff for all coherent sheaves $\mathcal{F}$ on X there is an integer n_0 such that $\mathcal{F} \otimes C^n$ is generated by global sections for all $n \geq n_0$.

(v) *Assume that X is projective. Let $f : X \to Y$ be a finite map. If A is an ample (resp. nef) bundle on Y then f^*A is ample (resp. nef) on X.*

We have the following standard positivity result.

Proposition 2.2.10. *Let P be a parabolic and let $\lambda \in \Lambda$ be such that $\mathcal{L}_P(\lambda)$ is a line bundle on G/P. Then:*

(i) *$\mathcal{L}_P(\lambda)$ is nef iff $\lambda \in D$.*

(ii) *$\mathcal{L}_P(\lambda)$ is ample iff $\lambda(\alpha^\vee) > 0$ for all $\alpha \in \pi \setminus \pi_P$.*

Bundle-theoretic results from algebraic geometry.

In this section we collect various results from algebraic geometry.

Proposition 2.2.11. *(Projection formula) Let $f : X \to Y$ be a morphism of varieties, let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{G}$ be a locally free sheaf on Y. Then*

$$R^i f_* (f^*\mathcal{G} \otimes \mathcal{F}) \cong \mathcal{G} \otimes R^i f_* \mathcal{F}$$

for all $i \geq 0$.

Proposition 2.2.12. *(Leray spectral sequence) Let $f : X \to Y$ be a morphism of varieties and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a first-quadrant spectral sequence in cohomology*

$$H^i \left(Y,\, R^j f_* \mathcal{F}\right) \implies H^{i+j} (X,\, \mathcal{F}).$$

The following corollary of the Leray spectral sequence is immediate.

Corollary 2.2.13. *Let $f : X \to Y$ be an affine morphism. Then for every coherent sheaf $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F})$$

for all $i \geq 0$.

In particular, this implies that if $h : E \to X$ is a vector bundle then

$$H^i(E, \mathcal{F}) \cong H^i(X, f_*\mathcal{F})$$

for all $i \geq 0$ and coherent sheaves $\mathcal{F}$ on E.

We have the following formulation of the Grauert-Riemenschneider vanishing theorem due to Kempf [**20**].

Theorem 2.2.14. *(Grauert-Riemenschneider) Let X be a smooth complex variety and let $f : X \to Y$ be a proper morphism of complex varieties. Let ω_X denote the canonical bundle of X. If n is the dimension of a generic fiber of f then $R^i f_*(\omega_X) = 0$ for all $i > n$.*

Remark 2.2.15. For a morphism $f : X \to Y$ of varieties, the function $y \mapsto \dim f^{-1}(y)$ is upper semicontinuous on Y. Thus, if Y is irreducible, the dimension of a generic fiber of f is the minimum of the dimensions of the fibers of f.

For a standard parabolic P of G, let $p : T^*(G/P) \to G/P$ denote the cotangent bundle of P. By a slight abuse of notation, we will use the letter p to denote this map regardless of which parabolic P we have chosen. Using the results above, we get the following proposition.

Proposition 2.2.16.

(i) Let $f : X \to Y$ *be a proper morphism of smooth complex varieties, where* Y *is an affine variety. Let* n *be the dimension of a generic fiber of* f*. Then* $H^i(X, \omega_X) = 0$ *for all* $i > n$*.*

(ii) Let $P \subseteq Q$ *be standard parabolics of* G *and let* M *be a* P*-module. Let* $g : G/P \to G/Q$ *be the projection map. For any* $i \geq 0$, $H^i(Q/P, \mathcal{L}_P(M))$ *is a* Q*-module and we have*

$$R^i g_*\big(\mathcal{L}_P(M)\big) \cong \mathcal{L}_Q\Big(H^i(Q/P, \mathcal{L}_P(M))\Big). \tag{2.2}$$

(iii) Let P *be any parabolic and* M *any* P*-module. Recall the bundle map* $p : T^*(G/P) \to G/P$ *from above. We have*

$$H^i\big(T^*(G/P), p^*\mathcal{L}_P(M)\big) \cong H^i\big(G/P, \mathcal{L}_P(S(\mathfrak{n}_{\mathfrak{p}}{}^*) \otimes M)\big) \tag{2.3}$$

for all i*.*

(iv) Let $P \subseteq Q$ *be parabolics and let* $\mu \in \Lambda$ *be* Q*-dominant. Then*

$$H^i\big(T^*(G/Q), p^*\mathcal{L}_Q(\mu)\big) \cong H^i\Big(G/P, \mathcal{L}_P\big(\mu\big) \otimes \mathcal{L}_P\big(S(\mathfrak{n}_{\mathfrak{q}}{}^*)\big)\Big) \tag{2.4}$$

for all i*.*

Proof. (i) Consider the Leray spectral sequence

$$H^i\left(Y, R^j f_*(\omega_X)\right) \Rightarrow H^{i+j}(X, \omega_X).$$

Now, $R^j f_*(\omega_X)$ is a coherent sheaf on Y for all j. Thus $H^i\big(Y, R^j f_*(\omega_X)\big) = 0$ for all $i > 0$. Hence the spectral sequence collapses and the statement follows immediately from Theorem 2.2.14.

(ii) The fact that $H^i\left(Q/P,\, \mathcal{L}_P\left(M\right)\right)$ is a Q-module follows from the fact that $\mathcal{L}_P\left(M\right)$ is a Q-equivarant bundle on Q/P. For any point $aQ \in G/Q$ we have

$$H^i\left(g^{-1}(aQ),\, \mathcal{L}_P\left(M\right)\big|_{g^{-1}(aQ)}\right) \;\cong\; H^i\left(Q/P,\, \mathcal{L}_P\left(M\right)\right)$$

for all i, since g is a fiber bundle with fibers Q/P. Thus $R^i g_*(\mathcal{L}_P\left(M\right))$ is a locally free sheaf on G/Q which is also G-equivariant since $\mathcal{L}_P\left(M\right)$ is a G-equivariant sheaf on G/P and g is a G-equivariant morphism. As a Q-equivariant bundle on Q/P is completely determined by its fiber at eQ, the result now follows by considering the (geometric) fiber of $R^i g_*(\mathcal{L}_P\left(M\right))$ at eQ.

(iii) For any vector bundle map $b : E \to Z$ it is easy to check that

$$b_* \mathcal{O}_E \;\cong\; S(\mathcal{E}^\vee),$$

where $\mathcal{E}$ is the sheaf of sections of the bundle E, $\mathcal{E}^\vee$ is the dual sheaf, and $S(\mathcal{E}^\vee)$ is the symmetric algebra sheaf of $\mathcal{E}^\vee$.

We have

$$T^*(G/P) \;\cong\; G \times^P \mathfrak{n}_\mathfrak{p}.$$

Thus

$$p_*\, \mathcal{O}_{T^*(G/P)} \;\cong\; \mathcal{L}_P\big(S({\mathfrak{n}_\mathfrak{p}}^*)\big).$$

By the projection formula,

$$p_*\, p^* \mathcal{L}_P\left(M\right) \;\cong\; \mathcal{L}_P\big(S({\mathfrak{n}_\mathfrak{p}}^*)\big) \otimes \mathcal{L}_P\left(M\right) \;\cong\; \mathcal{L}_P\big(S({\mathfrak{n}_\mathfrak{p}}^*) \otimes M\big).$$

Now, as p is an affine map, using the Leray spectral sequence we have that

$$H^i\big(T^*(G/P),\, p^* \mathcal{L}_P(M)\big) \;\cong\; H^i\left(G/P,\, p_*\, p^* \mathcal{L}_P\left(M\right)\right) \;\cong\; H^i\big(G/P,\, \mathcal{L}_P(S({\mathfrak{n}_\mathfrak{p}}^*) \otimes M)\big)$$

for all i, as desired.

(iv) Let $g : G/P \to G/Q$ be the projection map as above. Consider the Leray spectral sequence

$$H^i\big(G/Q,\ R^j g_* \left(\mathcal{L}_P\big(\mu\big) \otimes \mathcal{L}_P\left(S(\mathfrak{n_q}^*)\right)\right)\big) \ \Rightarrow\ H^{i+j}\big(G/P,\ \mathcal{L}_P\big(\mu\big) \otimes \mathcal{L}_P\left(S(\mathfrak{n_q}^*)\right)\big)\,.$$

By Borel-Weil we have

$$H^j\left(Q/P,\ \mathcal{L}_P\left(\mu\right)\right) = 0$$

for all $j > 0$, and

$$H^0\big(Q/P,\ \mathcal{L}_P\left(\mu\right)\big) = \mathcal{L}_Q\left(\mu\right).$$

By (ii) we have

$$\begin{aligned} R^j g_*\Big(\mathcal{L}_P\left(\mu\right) \otimes \mathcal{L}_P\big(S(\mathfrak{n_q}^*)\big)\Big) \ &\cong\ \mathcal{L}_Q\Big(H^j\big(Q/P,\ \mathcal{L}_P\left(\mu\right) \otimes \mathcal{L}_P\big(S(\mathfrak{n_q}^*)\big)\big)\Big) \\ &\cong\ \mathcal{L}_Q\Big(H^j\big(Q/P,\ \mathcal{L}_P\left(\mu\right)\big)\Big) \otimes S(\mathfrak{n_q}^*) \\ &=\ 0 \end{aligned}$$

for all $j > 0$. Hence the spectral sequence collapses and the result follows from (iii).

$\square$

2.3. Nilpotent Orbits

All of the material in this section is drawn from [**8**]. In this section we assume that all algebras and varieties are over $\mathbb{C}$.

Semisimple and nilpotent elements.

Definition 2.3.1. Let $\mathfrak{g}$ denote the Lie algebra of G. An element $X \in \mathfrak{g}$ is said to be **nilpotent** if $\mathrm{ad}(X)$ is a nilpotent operator on $\mathfrak{g}$. An element $H \in \mathfrak{g}$ is said to be **semisimple** if $\mathrm{ad}(H)$ is a semisimple operator on $\mathfrak{g}$, ie the action of $\mathrm{ad}(H)$ on $\mathfrak{g}$ is diagonalizable.

For any $g \in G$ and $A \in \mathfrak{g}$, we will denote $\mathrm{Ad}(g).A$ by $g.A$. From the definition one sees immediately that $A \in \mathfrak{g}$ is nilpotent (resp. semisimple) iff $g.A$ is nilpotent (resp. semisimple) for some (equivalently, all) $g \in G$. Hence it makes sense to talk about a nilpotent or semisimple G-orbit in $\mathfrak{g}$.

For any $A \in \mathfrak{g}$ we can write A uniquely as $A = X + H$, where X is nilpotent, H is semisimple, and $[X, H] = 0$. The following is a useful characterization of nilpotent and semisimple elements of $\mathfrak{g}$.

Proposition 2.3.2. *An element $A \in \mathfrak{g}$ is nilpotent (resp. semisimple) iff the action of A on any representation of $\mathfrak{g}$ is nilpotent (resp. semisimple).*

From here on, an **adjoint orbit** in $\mathfrak{g}$ will mean an orbit under the adjoint action of the group G. We will denote the adjoint orbit $G.X$ by $\mathfrak{O}_X$. Every semisimple adjoint orbit is closed in $\mathfrak{g}$ and the union of all semisimple orbits forms a dense subset of $\mathfrak{g}$. On the other hand, the only closed nilpotent orbit is $\{0\}$ and the union $\mathcal{N}$ of all nilpotent elements of $\mathfrak{g}$ is a closed subvariety of $\mathfrak{g}$ called the **nullcone**. $\mathcal{N}$ is the union of a finite number of G-orbits. An adjoint orbit in $\mathfrak{g}$ is called **regular** if the dimension of the orbit is maximal. Regular adjoint orbits have codimension $\mathrm{rank}(G)$ in $\mathfrak{g}$, and regular semisimple and nilpotent orbits exist.

There is a unique regular nilpotent orbit $\mathfrak{O}_{reg}$ which is dense in $\mathcal{N}$; elements of this orbit are called **principal** nilpotents. See section 2.3 below for an explicit representative of this orbit. There is also a unique nilpotent orbit dense in $\mathcal{N} \setminus \mathfrak{O}_{reg}$; this orbit is called the **subregular** nilpotent orbit and is denoted by $\mathfrak{O}_{subreg}$. There is a unique nonzero nilpotent orbit of minimal dimension in $\mathcal{N}$, denoted by $\mathfrak{O}_{min}$; this orbit is also the unique nonzero orbit that is contained in the closure of every other nilpotent orbit.

sl_2-triples, associated parabolics, and even elements.

DEFINITION 2.3.3. A triple $\{X, Y, H\} \subseteq \mathfrak{g}$ is called an sl_2-triple if there is a Lie algebra isomorphism $sl_2 \xrightarrow{\sim} \operatorname{span}\{X, Y, H\}$ such that the standard basis $\{X', Y', H'\}$ of sl_2 maps to $\{X, Y, H\}$.

In particular, since an sl_2-triple gives $\mathfrak{g}$ the structure of an sl_2-representation, we see that if $\{X, Y, H\}$ is an sl_2-triple then X and Y are nilpotent and H is semisimple. X is called the **nilpositive** element of the triple and H is the **semisimple** element of the triple. We have the following important theorem.

THEOREM 2.3.4. *(Jacobson-Morozov) Let $X \in \mathcal{N}$. Then there is an sl_2-triple through X, ie an sl_2-triple $\{X, Y, H\}$.*

The sl_2-triple guaranteed by the theorem will not necessarily be unique. It is also important to point out that while every nilpotent element lies in an sl_2-triple, the same is *not* true of semisimple elements. If a semisimple element H is the semisimple element in some sl_2-triple then we say that H is **distinguished**. This also gives a notion of distinguished semisimple orbits.

Definition 2.3.5. We say that a nilpotent element X is in **good position** if (1) $X \in \mathfrak{n}$ and (2) there is an sl_2-triple $\{X, Y, H\}$ with $H \in D'$ (recall that $D' \subseteq \mathfrak{h}$ is the dominant chamber).

Let $\{X, Y, H\}$ be an sl_2-triple. Then, viewing $\mathfrak{g}$ as an sl_2-representation, we can write the eigenspace decomposition of $\mathfrak{g}$ under the action of H as

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n \,.$$

This decomposition depends only on the nilpositive element X and not on the rest of the elements of the triple, cf [**8**], Remark 3.8.5. Thus, given any nilpotent element X, we obtain such a decomposition of $\mathfrak{g}$ into eigenspaces. We say that X is **even** if $\mathfrak{g}_n = 0$ for all odd n.

One may check that $\bigoplus_{n \geq 0} \mathfrak{g}_{\mathfrak{n}}$ is a parabolic subalgebra of $\mathfrak{g}$; we call this the parabolic subalgebra **associated to** X. It has Levi factor $\mathfrak{g}_0 = \mathfrak{g}^H$ and nilradical $\bigoplus_{n>0} \mathfrak{g}_n$ (where $\mathfrak{g}^H$ denotes the centralizer of H in $\mathfrak{g}$). This associated parabolic is a standard parabolic iff X is in good position. Furthermore, X is regular iff the associated parabolic is a Borel subalgebra; in this case, H will be regular as well.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. We say that X is a **Richardson** element for $\mathfrak{p}$ if $\mathfrak{O}_X \cap \mathfrak{n}_{\mathfrak{p}}$ is dense in $\mathfrak{n}_{\mathfrak{p}}$. The Richardson elements for $\mathfrak{p}$ form a single orbit in $\mathcal{N}$.

Lemma 2.3.6. *(Kostant) Let $X \in \mathcal{N}$ and let $\mathfrak{p}$ be the associated parabolic. Let $P \subseteq G$ be the parabolic subgroup with Lie algebra $\mathfrak{p}$. Set*

$$\mathfrak{p}_2 := \bigoplus_{n \geq 2} \mathfrak{g}_n \subseteq \mathfrak{n}_{\mathfrak{p}} \,.$$

Then

$$\mathfrak{O}_X \cap \mathfrak{p}_2 = \mathfrak{O}_X \cap \mathfrak{n}_{\mathfrak{p}} = P.X$$

and $\mathfrak{O}_X \cap \mathfrak{p}_2$ *is open dense in* $\mathfrak{p}_2$.

This implies that X is even nilpotent iff X is a Richardson element for its associated parabolic, since one may check via sl_2-theory that X is even iff $\mathfrak{p}_2 = \mathfrak{n}_{\mathfrak{p}}$.

DEFINITION 2.3.7. Assume we have chosen a Borel subgroup $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$. A standard parabolic subalgebra $\mathfrak{p}$ is a **standard even parabolic** if there is an even nilpotent X such that $\mathfrak{p}$ is the parabolic associated to X. A subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ is a **standard even Levi** if it is the Levi factor of a standard even parabolic. When no confusion results we shall also use these definitions to refer to the appropriate subgroups of G.

REMARK 2.3.8. It can happen that there are $X, Y \in \mathcal{N}$ such that X and Y are not in the same orbit but the parabolics associated to X and Y are the same; this happens when the weighted Dynkin diagrams (defined below) associated to X and Y have 0's on the exact same nodes. In particular, in some types there are non-even $X \in \mathcal{N}$ whose associated parabolics are standard even parabolic; see the table on p. 49 of [**8**] for an example in type A_3.

Weighted Dynkin diagrams.

We would like to find a way of describing nilpotent orbits in $\mathcal{N}$. The coarsest level of information would simply be to know how many nilpotent orbits there are; we would then hopefully want to find a way of explicitly constructing them. The first method of doing this involves Dynkin diagrams. The essential result here is the following theorem.

Theorem 2.3.9. *([8], Theorems 3.2.10 and 3.2.14) Let Or_N denote the set of nonzero nilpotent orbits in $\mathfrak{g}$ and let Or_S denote the set of standard semisimple orbits.*

(i) If $\{X, H, Y\}$ and $\{X, H', Y'\}$ are two sl_2-triples with the same nilpositive element X then the two triples are conjugate by an element of G. In particular, this gives a well-defined map $\varphi : Or_N \to Or_S$ via $\mathfrak{O}_X \mapsto \mathfrak{O}_H$.

(ii) The map φ above is a bijection.

Thus, to describe nilpotent orbits, we will first describe standard semisimple orbits. Fix a Borel subgroup B and a torus $T \subseteq B$. Let $\mathfrak{b}$ and $\mathfrak{h}$ be the corresponding Lie algebras, and set $\mathfrak{n} = \mathrm{Lie}(U)$ (recall that U is the unipotent radical of B). Every semisimple orbit has nonempty intersection with D' and in fact each semisimple orbit has a unique element in D' up to the action of the Weyl group. Further, if $H \in D'$ is distinguished semisimple then $\alpha(H) \in \{0, 1, 2\}$ for all $\alpha \in \pi$.

For each nonzero nilpotent orbit $\mathfrak{O}_X$ we now obtain a unique weighted Dynkin diagram with weights 0, 1, or 2 as follows. By Theorem 2.3.9 there is a unique distinguished semisimple orbit $\mathfrak{O}_H$ corresponding to $\mathfrak{O}_X$ and we may choose a unique element $H \in \mathfrak{O}_H \cap D'$. To each node of the Dynkin diagram of G we now associate the integer $\alpha(H)$, where α is the simple root corresponding to the chosen node. In particular, for the 0 orbit, the associated weighted Dynkin diagram has every weight 0.

This shows that the number of nilpotent orbits in $\mathfrak{g}$ is at most 3^l, where l is the rank of G. In general, though, the correspondence won't be bijective; there may be some weighted Dynkin diagrams that do not correspond to nilpotent orbits. Note that $\mathfrak{O}_X$ is an even nilpotent orbit iff the corresponding weighted Dynkin diagram has no 1's. The regular nilpotent orbit (also called the **principal** nilpotent orbit) corresponds to the weighted Dynkin diagram where every weight is 2. An orbit representative for the principal nilpotent orbit is given by

$$X_{reg} = \sum_{\alpha \in \pi} X_\alpha \, ,$$

where for each $\alpha \in \pi$, $X_\alpha \in \mathfrak{g}_\alpha$ is a nonzero root vector.

There is also another method of classifying nilpotent orbits, called Bala-Carter theory, which works for any simple group. For more details see [**8**], Chapter 8.

Explicit orbit representatives in type A and even nilpotent orbits.

In the case of classical groups, one can explicitly construct representatives for each nilpotent orbit, cf [**8**], Chapter 5. We will do this in the case of sl_n, and also indicate which orbits are even. For the other classical types the combinatorics are rather ugly and non-enlightening.

In type sl_n, there is a bijective correspondence between partitions $\underline{\mathbf{p}} = [p_1, \ldots, p_k]$ of n where the p_i are nonzero and nonincreasing, and nilpotent orbits in $\mathcal{N}$. Orbit representatives for these partitions are constructed as follows.

If the partition is such that $p_i = 1$ for all $1 \leq i \leq k$ then the associated orbit is the 0 orbit, so we assume that $\underline{\mathbf{p}} \neq [1, \ldots, 1]$. For any $\beta \in \Delta^+$ let $X_\beta \in \mathfrak{g}_\beta$ be a Chevalley basis element. Let $\{\alpha_1, \ldots, \alpha_{n-1}\}$ denote the simple roots from left to right in the Dynkin diagram of type A_{n-1}.

Given a partition $\underline{\mathbf{p}}$ as above, set $m = \max\{j : p_j > 1\}$. For each $1 \leq i \leq m$ set $N_i := \sum_{l=1}^{i} p_{l-1}$ (where we set $p_{-1} = 0$) and set

$$X_i := \sum_{j=N_i+1}^{N_{i+1}-1} X_{\alpha_j} \, .$$

We now obtain an orbit representative corresponding to $\underline{\mathbf{p}}$ by setting

$$X_{\underline{\mathbf{p}}} := \sum_{i=1}^{m} X_i \,.$$

We may explicitly describe the matrix corresponding to $X_{\underline{\mathbf{p}}}$ as follows. For any $q > 0$ let $J(q)$ be the $q \times q$ matrix with $1's$ on the superdiagonal and $0's$ everywhere else. Then $X_{\underline{\mathbf{p}}}$ is the matrix with Jordan blocks of size $p_1, \ldots, p_k$, i.e. the block diagonal matrix

$$\operatorname{diag}\big(J(1), \ldots, J(q)\big) := \begin{pmatrix} J(p_1) & & & \\ & J(p_2) & & \\ & & \ddots & \\ & & & J(p_k) \end{pmatrix} .$$

The bijection between partitions and orbits is given by $\underline{\mathbf{p}} \mapsto \mathfrak{O}_{X_{\underline{\mathbf{p}}}}$. Even nilpotent orbits correspond to partitions consisting of only even or only odd parts.

Remark 2.3.10. Given a partition $\underline{\mathbf{p}}$ of n, let X be the orbit representative of this partition in type A_{n-1} constructed above. One can also explicitly construct $H \in \mathfrak{h}$ such that H is the semisimple element of an sl_2-triple containing X. Generally H itself will *not* be in D', though, and often no G^X-conjugate of H will be in D' either (where G^X is the stabilizer of X in G). This is a subtlety worth noting: if no G^X-conjugate of H is in D' then X is not be a standard nilpotent element and hence, if we want to obtain a standard triple (or a weighted Dynkin diagram), we must conjugate the sl_2-triple containing X and H. This can always be done by an element of W, as described below.

Given a partition $\underline{\mathbf{p}} = [p_1, \ldots, p_k]$, we now describe how (cf [**8**], section 3.6) to (a) construct a semisimple element H in an sl_2-triple containing $X_{\underline{\mathbf{p}}}$; (b) how to conjugate the resulting triple to a standard triple; and (c) how to obtain a weighted Dynkin diagram corresponding to $\mathfrak{O}_{X_{\underline{\mathbf{p}}}}$ (all of this in type A only, of course).

Retain the notation from the construction of $X = X_{\underline{\mathbf{p}}}$ above. Let

$$\{\chi_1^\vee, \ldots, \chi_{n-1}^\vee\} \subseteq \mathfrak{h}$$

denote the fundamental coweights; that is, $\chi_i^\vee$ is defined by $\alpha_j(\chi_i^\vee) = \delta_{i,j}$ for $1 \leq i, j \leq n-1$. For $1 \leq i \leq \min\{m, k-1\}$ set

$$H_i := \left(\sum_{j=N_i+1}^{N_{i+1}-1} \chi_j^\vee \right) - (p_i + p_{i+1} - 2)\chi_{N_{i+1}}^\vee .$$

If $m = k$ set

$$H_k := \sum_{j=N_k+1}^{n-1} \chi_j^\vee .$$

We now set $H := \sum_{i=1}^{m} H_i$, a semisimple element in an sl_2-triple containing X; this is always an element of the coroot lattice.

Explicitly, we can write a matrix for H as follows. For each $q > 0$ let $D(q)$ be the $q \times q$ diagonal matrix $\mathrm{diag}(q-1, q-3, \ldots, -(q-1))$. We now set

$$H = \mathrm{diag}(D(p_1), \ldots, D(p_k))$$

(recall the notation diag(?) from above).

To conjugate H to a standard semisimple element it is straightforward to find $w \in W$ such that $wH \in D'$. Explicitly, wH will be the diagonal matrix where the entries of H have been rearranged into nonincreasing order. Furthermore, wX will be a standard nilpotent element (cf [**8**], section 3.6) and the weighted Dynkin diagram associated to $\mathfrak{O}_X$ is the Dynkin diagram whose nodes are labelled by $\alpha(wH)$ for $\alpha \in \pi$. See chapter 5 below for an explicit example.

CHAPTER 3

The BK-filtration and generalizations

3.1. The BK-filtration

Definitions.

Choose a Borel B and a torus T of G. The **BGG category** $\mathcal{O}$ consists of $\mathfrak{g}$ representations V that have finite-dimensional weight spaces and are **$\mathfrak{n}$-locally finite**; i.e., for any $v \in V$, v lies in a finite-dimensional $\mathfrak{n}$-submodule of V.

Let V be an object in the BGG category $\mathcal{O}$ and let $U \subseteq V$ be any vector subspace. Let e be a principal nilpotent in good position (recall the definition of principal nilpotents and nilpotents in good position from section 2.3). In [**6**], Ranee Brylinski defined a filtration on U inside of V, called the **Brylinski-Kostant filtration** (or **BK-filtration**), as follows: set

$$\mathcal{F}^n_V(U) := \{v \in U : e^{n+1} v = 0\}\,.$$

In particular, we would like to consider this filtration in the case where V is an irreducible representation $V(\mu)$ of G and U is a weight subspace $V_\lambda(\mu)$ of V, for $\mu \in D$ and $\lambda \in \Lambda$. In this case we shall suppress the subscript and write

$$\mathcal{F}^n\big(V_\lambda(\mu)\big) := \mathcal{F}^n_{V(\mu)}\big(V_\lambda(\mu)\big)\,. \tag{3.1}$$

Define

$$r_\mu^\lambda(q) := \sum_{n\geq 0} \dim\left(\frac{\mathcal{F}^n\big(V_\lambda(\mu)\big)}{\mathcal{F}^{n-1}\big(V_\lambda(\mu)\big)}\right) q^n \in \mathbb{Z}[q]$$

(where $\mathcal{F}^{-1}\big(V_\lambda(\mu)\big) = \{0\}$). This **jump polynomial** counts the dimensions of the degrees of the filtration, and clearly $\dim\left(V_\lambda(\mu)\right) = r_\mu^\lambda(1)$.

Kazhdan-Lusztig polynomials and Brylinski's theorem.

For $\gamma \in \Lambda_R$ let $p_q(\gamma) \in \mathbb{Z}[q]$ be the coefficient of e^γ in $\prod_{\beta\in\Delta^+}(1-qe^\beta)^{-1}$. This is a q-analog of Kostant's partition function $p(\gamma) = p_q(\gamma)\big|_{q=1}$, which counts the number of ways of writing γ as a sum of positive roots. The degree-n coefficient of $p_q(\gamma)$ counts the number of ways of writing γ as a sum of precisely n (not necessarily distinct) positive roots.

For μ, $\lambda \in D$ such that $\lambda - \mu \in \Lambda_R$ set

$$m_\mu^\lambda(q) := \sum_{w\in W} (-1)^{l(w)} p_q(w * \mu - \lambda)\,,$$

Lusztig's **q-analog of weight multiplicity** [**25**]. It is called an analog of weight multiplicity because, by the Weyl character formula, $m_\mu^\lambda(1) = \dim\left(V_\lambda(\mu)\right)$. The polynomials $m_\mu^\lambda(q)$ for λ, $\mu \in D$ are equal to certain **Kazhdan-Lusztig polynomials** for affine Weyl groups, which are important objects in combinatorial representation theory (see [**17**], [**19**], [**18**], and [**25**]).

Using this setup, we have the following theorem of Brylinski which relies on a cohomology vanishing condition. Recall the bundle $\mathcal{L}_B(\lambda)$ on G/B from equation (2.1), and let $p : T^*(G/B) \to G/B$ be the cotangent bundle of G/B.

Theorem 3.1.1. *(Brylinski, [**6**]) Let e be a principal nilpotent in good position. Let $\mu,\,\lambda \in D$. If*

$$H^i\big(T^*(G/B),\, p^*\mathcal{L}_B(\lambda)\big) = 0$$

for all $i > 0$ then $m_\mu^\lambda(q) = r_\mu^\lambda(q)$.

We also have the following result of Broer which is a special case of Theorem 4.2.4 below.

Theorem 3.1.2. *(Broer, [**3**]) $H^i\big(T^*(G/B),\, p^*\mathcal{L}_B(\lambda)\big) = 0$ for all $\lambda \in D$ and $i > 0$.*

Thus we have

Corollary 3.1.3. *For $\mu,\,\lambda \in D$, $m_\mu^\lambda(q) = r_\mu^\lambda(q)$.*

3.2. Generalization of the BK-filtration

Definitions and main result.

We now generalize the BK-filtration to the case of an arbitrary nilpotent element. Choose $X \in \mathcal{N}$ and V in the category $\mathcal{O}$. Let $U \subseteq V$ be any vector subspace. Define a filtration $\mathcal{F}_{X,V}$ on U by

$$\mathcal{F}^n_{X,V}(U) := \left\{v \in U : X^{n+1}v = 0\right\}.$$

If e is a principal nilpotent then clearly $\mathcal{F}^n_{e,V}(U) = \mathcal{F}^n_V(U)$, the original BK-filtration.

As before, we want to consider subspaces of an irreducible G-module V. However, we will not consider the filtration on an entire weight subspace of V, but rather on special subspaces of weight spaces. We will also restrict our attention to the case where $X \in \mathfrak{n}$ is even.

Let $X \in \mathfrak{n}$ be an even nilpotent element in good position and let P be its associated standard even parabolic. Let L be the Levi factor of P containing T. For any $\mu \in D$ and weight λ of $V(\mu)$ let $\mathcal{W}^P_\lambda(\mu) \subseteq V_\lambda(\mu)$ denote the subspace consisting of L-highest weight vectors. Set

$$\mathcal{F}^n_X\big(\mathcal{W}^P_\lambda(\mu)\big) := \mathcal{F}^n_{X,V(\mu)}\big(\mathcal{W}^P_\lambda(\mu)\big)\,.$$

This generalizes the filtration $\mathcal{F}\big(V_\lambda(\mu)\big)$ from (3.1) above. Note that $\mathcal{W}^B_\lambda(\mu) = V_\lambda(\mu)$, so that

$$\mathcal{F}^n_e\big(\mathcal{W}^B_\lambda(\mu)\big) = \mathcal{F}^n\big(V_\lambda(\mu)\big)\,.$$

We obtain a jump polynomial as before: set

$$r^{X,\lambda}_\mu(q) := \sum_{n\geq 0} \dim\left(\frac{\mathcal{F}^n_X\big(\mathcal{W}^P_\lambda(\mu)\big)}{\mathcal{F}^{n-1}_X\big(\mathcal{W}^P_\lambda(\mu)\big)}\right) q^n\,.$$

We also need to generalize the polynomials p and m from above. This is straightforward, as we will simply restrict our attention to weights of $\mathfrak{n}_\mathfrak{p}$ instead of $\mathfrak{n}$. Let $p^P_q(\gamma) \in \mathbb{Z}[q]$ be the coefficient of e^γ in $\displaystyle\prod_{\beta\in\Delta^+\setminus\Delta^+_P} (1-qe^\beta)^{-1}$ and set

$$m^{P,\lambda}_\mu(q) := \sum_{w\in W} (-1)^{l(w)} p^P_q(w*\mu-\lambda)$$

. It is clear that when $P = B$ we obtain the earlier setting due to Brylinski.

Let $p : T^*(G/B) \to G/P$ be the cotangent bundle of G/P. The following theorem is the main theorem in this section; it generalizes Theorem 3.1.1 above.

Theorem 3.2.1. *Let $X \in \mathfrak{n}$ be a standard even nilpotent. Let μ, $\lambda \in D$. If*

$$H^i\big(T^*(G/P),\, p^*\mathcal{L}_P(\lambda)\big) = 0$$

for all $i > 0$, then $r_\mu^{X,\lambda}(q) = m_\mu^{P,\lambda}(q)$, where P is the parabolic associated to X.

In chapter 4 below, we will prove this cohomology vanishing for various λ.

Remark 3.2.2. Note that this implies, in the presence of the cohomology vanishing condition, that the coefficients of $m_\mu^{P,\lambda}(q)$ are positive when μ, $\lambda \in D$. This is by no means clear from the definition and, in fact, can fail if $\lambda \notin D$.

3.3. Proof of Theorem 3.2.1

Outline of proof.

Remark 3.3.1. Our proof is an adaptation of Brylinski's proof of her Theorem 3.4 in [**6**] (which is stated as Theorem 3.1.1 above).

Let us first give a sketch of the proof. Fix $\mu, \lambda \in D$, and choose an even nilpotent element X in good position. Let P be the parabolic subgroup corresponding to X. We first construct two G-equivariant locally-free sheaves $q_* \mathcal{A}_P(\lambda)$ and $p_* p^* \mathcal{L}_P(\lambda)$ of infinite rank on G/P such that (a) $q_* \mathcal{A}_P(\lambda)$ is filtered by G-equivariant locally-free sheaves $\left(q_* \mathcal{A}_P(\lambda)\right)^{\leq n}$ of finite rank and (b) $p_* p^* \mathcal{L}_P(\lambda)$ is graded by G-equivariant locally-free sheaves $\left(p_* p^* \mathcal{L}_P(\lambda)\right)^n$ of finite rank.

We then show in Theorem 3.3.18 below that the multiplicity of $V(\mu)^*$ in

$$\frac{H^0\left(G/P,\ \left(q_* \mathcal{A}_P(\lambda)\right)^{\leq n}\right)}{H^0\left(G/P,\ \left(q_* \mathcal{A}_P(\lambda)\right)^{\leq n-1}\right)}$$

is precisely the degree-n coefficient of the jump polynomial $r_\mu^{X,\lambda}(q)$. In Proposition 3.3.19 we show that this quotient of global sections is isomorphic, in the presence of the desired cohomology vanishing condition, to $H^0\left(G/P,\ \left(p_*\, p^*\mathcal{L}_P(\lambda)\right)^n\right)$ as G-modules. To complete the proof of Theorem 3.2.1 we show in the same proposition that the multiplicity of $V(\mu)^*$ in $H^0\left(G/P,\ \left(p_*\, p^*\mathcal{L}_P(\lambda)\right)^n\right)$ is the degree-n coefficient of $m_\mu^{P,\lambda}(q)$.

The proof.

Fix μ, λ, P and X as above. Let L be the Levi factor of P. Recall that $D' \subseteq \mathfrak{h}$ is the dominant chamber. Let $H \in D'$ be the unique distinguished semisimple element in D' occuring in an sl_2-triple with nilpositive element X. Recall that $G^H = L$ and $\mathfrak{g}^H = \mathfrak{l}$, where G^H is the stabilizer of H in G and $\mathfrak{g}^H$ is the stabilizer of H in $\mathfrak{g}$.

LEMMA 3.3.2. *The natural projection $G/L \twoheadrightarrow G/P$ induces an isomorphism*

$$G/L \;\cong\; G \times^P (H + \mathfrak{n}_\mathfrak{p})$$

of G-equivariant fiber bundles on G/P, where we consider G/L as an equivariant fiber bundle on G/P through the isomorphism $G/L \;\cong\; G \times^P (P/L)$ given by $g * pL \mapsto gpL$ and where P acts on $H + \mathfrak{n}_\mathfrak{p}$ through the adjoint action.

PROOF. It suffices to show that $P/L \;\cong\; H + \mathfrak{n}_\mathfrak{p}$ as P-varieties. As a variety, $P \cong U_P \times L$. Since $U_P^H = \{e\}$ by the construction of P, we have $P^H = L$ and hence $P/L \;\cong\; P.H$. Since $L.H = H$ we have $P.H = U_P.H$ and hence a variety isomorphism $U_P \xrightarrow{\sim} U_P.H$.

Note that $\mathfrak{n}_\mathfrak{p}.H \subseteq \mathfrak{n}_\mathfrak{p}$, so that $U_P.H \subseteq H + \mathfrak{n}_\mathfrak{p}$ (cf [**7**], Lemma 1.4.12(i)). As U_P and $H + \mathfrak{n}_\mathfrak{p}$ are both isomorphic to $\mathbb{A}^n$ for $n = \dim(\mathfrak{n}_\mathfrak{p})$, the variety injection $U_P.H \hookrightarrow H + \mathfrak{n}_\mathfrak{p}$ must be an isomorphism. Thus we have

$$P/L \;\cong\; P.H = U_P.H \;\cong\; H + \mathfrak{n}_\mathfrak{p}$$

as desired. $\square$

Remark 3.3.3. From here on, we will use the G-equivariant isomorphism $G/L \cong G \times^P (H + \mathfrak{n}_\mathfrak{p})$ to write elements of G/L in the form $g * (H + Z)$, for $g \in G$ and $Z \in \mathfrak{n}_\mathfrak{p}$.

Recall that the cotangent bundle of G/P is a homogeneous bundle: we have

$$T^*(G/P) \cong G \times^P \mathfrak{n}_\mathfrak{p}.$$

There is a natural fiberwise action of $T^*(G/P)$ on G/L via

$$g * Z_1 \, + \, g * (H + Z_2) \, = \, g * (H + Z_1 + Z_2),$$

for $g \in G$, $\; g * Z_1 \in T^*(G/P)$, and $g * (H + Z_2) \in G/L$.

Notation 3.3.4. Let $p : T^*(G/P) \to G/P$ and $q : G/L \to G/P$ be the bundle maps. For $\lambda \in D$ we will denote by $\underline{L}_P(\lambda)$ the bundle $G \times^P \big(V^P(\lambda)^*\big)$ on G/P. Recall that $\mathcal{L}_P\,(\lambda)$ is the notation for the sheaf of sections of this bundle.

Let $\underline{A}_P(\lambda)$ denote the bundle $q^*\underline{L}_P(\lambda)$ on G/L and let $\mathcal{A}_P(\lambda)$ denote the sheaf of sections of $\underline{A}_P(\lambda)$ on G/L.

For $gP \in G/P$ let $\underline{L}_P(\lambda)_{gP}$ (resp. $T^*(G/P)_{gP}$, resp. $\big(G/L\big)_{gP}$) denote the fiber of $\underline{L}_P(\lambda)$ (resp. $T^*(G/P)$, resp. G/L) over gP. For $H_0 \in G/L$ let $\underline{A}^P(\lambda)_{H_0}$ denote the fiber of $\underline{A}_P(\lambda)$ over H_0.

Remark 3.3.5. Given a variety Y and a sheaf $\mathcal{G}$ on Y, by an abuse of notation we shall write $g \in \mathcal{G}$ to mean $g \in \mathcal{G}(U)$ for some open subset U of Y. If g is meant to be a gloabl section of $\mathcal{G}$ this will be specified.

Remark 3.3.6. We have that $p_*\,\mathcal{O}_{T^*(G/P)} \,\cong\, \mathcal{L}_P\,(S({\mathfrak{n}_\mathfrak{p}}^*))$ and $q_*\,\mathcal{O}_{G/L} \,\cong\, \mathcal{L}_P\,(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}])$. Hence

$$p_*\,p^*\mathcal{L}_P(\lambda) \,\cong\, \mathcal{L}_P\big(S({\mathfrak{n}_\mathfrak{p}}^*) \otimes V^P(\lambda)^*\big)$$

and

$$q_*\,\mathcal{A}_P(\lambda)\;\cong\;\mathcal{L}_P\big(\mathbb{C}[H+\mathfrak{n}_\mathfrak{p}]\otimes V^P(\lambda)^*\big),$$

for P-dominant λ, by the projection formula (Proposition 2.2.11). In particular, these sheaves are locally-free and G-equivariant on G/P.

For any two varieties X and Y over $\mathbb{C}$ let $\mathrm{Hom}_{var}(X,Y)$ denote the set of variety morphisms from X to Y over $\mathbb{C}$. Note that

$$S(\mathfrak{n}_\mathfrak{p}{}^*)\otimes V^P(\lambda)^*\;\cong\;\mathbb{C}[\mathfrak{n}_\mathfrak{p}]\otimes V^P(\lambda)^*\;\cong\;\mathrm{Hom}_{var}\big(\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)$$

as P-modules, where the action of P on $\mathrm{Hom}_{var}\big(\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)$ is given by

$$(p.f)(X)=p.\big(f(p^{-1}.X)\big)\,.$$

Similarly, we see that

$$\mathbb{C}[H+\mathfrak{n}_\mathfrak{p}]\otimes V^P(\lambda)^*\;\cong\;\mathrm{Hom}_{var}\big(H+\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)$$

as P-modules, where the P-action on $\mathrm{Hom}_{var}\big(H+\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)$ is given by

$$(p.h)(H_1)=p.\big(h(p^{-1}.H_1)\big)\,.$$

Thus we have isomorphisms of G-equivariant sheaves

$$p_*\,p^*\mathcal{L}_P(\lambda)\;\;\cong\;\;\mathcal{L}_P\Big(\mathrm{Hom}_{var}\big(\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)\Big)$$

and

$$q_*\,\mathcal{A}_P(\lambda)\;\;\cong\;\;\mathcal{L}_P\Big(\mathrm{Hom}_{var}\big(H+\mathfrak{n}_\mathfrak{p},V^P(\lambda)^*\big)\Big)\,.$$

This means that we can interpret a section $u\in p_*\,p^*\mathcal{L}_P(\lambda)$ as follows. Choose $gP\in G/P$; then $u(gP)$ is a variety morphism $T^*(G/P)_{gP}\to\underline{L}_P(\lambda)_{gP}$. Similarly, for $s\in$

$q_* \mathcal{A}_P(\lambda)$, we can interpret $s(gP)$ as a variety morphism $\big(G/L\big)_{gP} \to \underline{L}_P(\lambda)_{gP}$. From here on I will use the less cumbersome notation u_{gP} for $u(gP)$ (resp. s_{gP} for $s(gP)$) and we will interpret these sections as $\underline{L}_P(\lambda)_{gP}$-valued morphisms on the fibers of $T^*(G/P)$ (resp. G/L) over G/P.

Definition 3.3.7. Define a gradation on $p_* p^* \mathcal{L}_P(\lambda)$ by

$$\big(p_* p^* \mathcal{L}_P(\lambda)\big)^n := \mathcal{L}_P\big(S^n(\mathfrak{n_p}^*) \otimes V^P(\lambda)^*\big)$$

for all $n \geq 0$. By remark 3.3.6 above, for $u \in \big(p_* p^* \mathcal{L}_P(\lambda)\big)^n$ and $gP \in G/P$ we can think of u_{gP} as a degree-n map of vector spaces $T^*(G/P)_{gP} \to \underline{L}_P(\lambda)_{gP}$.

Since $H + \mathfrak{n_p}$ is an affine space there is no natural gradation on $\mathbb{C}[H + \mathfrak{n_p}]$. However, for any $f \in \mathbb{C}[H + \mathfrak{n_p}]$ there is a well-defined notion of top-degree monomial, which is a function on $\mathfrak{n_p}$. Thus we can define the **degree** of f to be the degree of this monomial. This gives rise to a natural filtration $\mathbb{C}[H + \mathfrak{n_p}]^{\leq n}$ on $\mathbb{C}[H + \mathfrak{n_p}]$. More generally, although the notion of "top-degree monomial" is not well-defined in $\mathbb{C}[H + \mathfrak{n_p}] \otimes V^P(\lambda)^*$, the notion of degree still is. Thus we obtain a natural filtration $\mathbb{C}[H + \mathfrak{n_p}]^{\leq n} \otimes V^P(\lambda)^*$ on $\mathbb{C}[H + \mathfrak{n_p}] \otimes V^P(\lambda)^*$ which gives rise to a filtration

$$\big(q_* \mathcal{A}_P(\lambda)\big)^{\leq n} := \mathcal{L}_P\big((\mathbb{C}[H + \mathfrak{n_p}]^{\leq n} \otimes V^P(\lambda)^*)\big)$$

on $\mathcal{L}_P\big(\mathbb{C}[H + \mathfrak{n_p}] \otimes V^P(\lambda)^*\big)$. If $s \in \big(q_* \mathcal{A}_P(\lambda)\big)^{\leq n} \setminus \big(q_* \mathcal{A}_P(\lambda)\big)^{\leq n-1}$ we say that the **fiber degree** of s is n.

Given $s \in \big(q_* \mathcal{A}_P(\lambda)\big)^{\leq n}$ and $gP \in G/P$, by remark 3.3.6 we can think of s_{gP} as a regular map from the affine space $\big(G/L\big)_{gP}$ to the vector space $\underline{L}_P(\lambda)_{gP}$ of degree at most n. Thus the fiber degree of s is n iff (1) there exists $gP \in G/P$ such that s_{gP} is a degree-n map, and (2) there does not exist $hP \in G/P$ such that the degree of the map s_{hP} is $> n$.

PROPOSITION 3.3.8. *There is a natural G-equivariant isomorphism*

$$\mathbb{D}^{gr} : gr\ q_* \mathcal{A}_P(\lambda) \xrightarrow{\sim} p_* p^* \mathcal{L}_P(\lambda)$$

of graded sheaves on G/P.

PROOF. We have that

$$\mathrm{gr}\ q_* \mathcal{A}_P(\lambda) \cong \bigoplus_{n \geq 0} \mathcal{L}_P \left(\frac{\left(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right)^{\leq n}}{\left(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right)^{\leq n-1}} \right)$$

and

$$p_* p^* \mathcal{L}_P(\lambda) \cong \mathcal{L}_P \left(\mathbb{C}[\mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right) \cong \mathcal{L}_P \left(S({\mathfrak{n}_\mathfrak{p}}^*) \otimes V^P(\lambda)^*\right)$$

(where $\left(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right)^{\leq -1} = \{0\}$).

Now, for all $n \geq 0$ there is a natural P-equivariant isomorphism

$$\frac{\left(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right)^{\leq n}}{\left(\mathbb{C}[H + \mathfrak{n}_\mathfrak{p}] \otimes V^P(\lambda)^*\right)^{\leq n-1}} \cong S^n({\mathfrak{n}_\mathfrak{p}}^*) \otimes V^P(\lambda)^* .$$

Extending this to a sheaf map gives the desired G-equivariant isomorphism $\mathbb{D}_{gr}$. $\square$

The following easy lemma gives an important tool for computing fiber degrees.

DEFINITION 3.3.9. Let $gP \in G/P$ and let $M(t)$ be a line in the affine space $\left(G/L\right)_{gP}$ (we can write $M(t) = g*(H' + tX')$ for some $H' \in H + \mathfrak{n}_\mathfrak{p}$ and $X' \in \mathfrak{n}_\mathfrak{p}$). For $s \in q_* \mathcal{A}_P(\lambda)$, consider the map $\mathbb{C} \to \underline{L}_P(\lambda)_{gP}$ given by

$$t \mapsto s_{gP}\big(M(t)\big) .$$

This map is a polynomial in t with values in $\underline{L}_P(\lambda)_{gP}$ and we define the **degree of** s **restricted to M** to be the degree of this polynomial.

LEMMA 3.3.10. *Let $gP \in G/P$ and let $s \in q_* \mathcal{A}_P(\lambda)$.*

(i) The degree of s_{gP} on $\big(G/L\big)_{gP}$ is the maximum of the degrees of the restrictions of s_{gP} to lines in $\big(G/L\big)_{gP}$.

(ii) If M_1 and M_2 are parallel lines in $\big(G/L\big)_{gP}$ then the degrees of the restrictions of s to each of these lines are the same.

(iii) The fiber degree of s is the maximum of the degrees of the restrictions of s to all lines contained in fibers of G/L over G/P.

PROOF. (iii) clearly follows from (i). The rest of the proof follows easily from the fact that if $f \in \mathbb{C}[V]$ for some vector space then (i) the degree of f is the maximum of the degrees of f restricted to lines in V, and (ii) the degrees of f restricted to parallel lines are the same. $\square$

DEFINITION 3.3.11. Let $s \in H^0\big(G/P,\ \big(q_* \mathcal{A}_P(\lambda)\big)^{\leq n}\big)$. Motivated by the above lemma, we say that s **attains its fiber degree** on a line $N(t) = g * (H' + tX') \subseteq \big(G/L\big)_{gP}$ (for some $H' \in H + \mathfrak{n}_\mathfrak{p}$ and $X' \in \mathfrak{n}_\mathfrak{p}$) if the degree of the $\underline{L}_P(\lambda)_{gP}$-valued polynomial

$$t \mapsto s_{gP}(g * (H' + tX'))\ =\ s_{gP}\big(N(t)\big)$$

is n.

The next lemma generalizes Brylinski's Lemma 5.6.

LEMMA 3.3.12. *Let $M(t) := g * (H_1 + tZ)$, $t \in \mathbb{C}$, be an affine line in the affine space $\big(G/L\big)_{gP}$. Choose $s \in H^0\left(G/P,\, q_* \mathcal{A}_P(\lambda)\right)$. For each $t \in \mathbb{C}$ let $u'_t \in U_P$ be such that*

$u'_t.H_1 = H_1 + tZ$ and set $u_t = gu'_tg^{-1}$. Then the degree of the map s_{gP} on the line M is the degree of the $\underline{L}_P(\lambda)_{gP}$-valued polynomial $(u_t^{-1}.s)_{gP}\big((g * H_1)\big)$ in t.

PROOF. As noted in the remark above, the degree of this map on the line M is the degree of the $\underline{L}_P(\lambda)_{gP}$-valued polynomial $s_{gP}\big(M(t)\big)$ in t. We have

$$\begin{aligned} (u_t^{-1}.s)_{gP}\big((g * H_1)\big) &= s_{u_tgP}\big(u_tg * H_1\big) \\ &= s_{gu'_tg^{-1}gP}\big(gu'_tg^{-1}g * H_1\big) \\ &= s_{gP}\big(g * u'_t.H_1\big) \\ &= s_{gP}\big(g * (H_1 + tZ)\big) \\ &= s_{gP}\big(M(t)\big) \end{aligned}$$

as desired. □

LEMMA 3.3.13. *Let $s \in H^0\left(G/P,\, q_*\mathcal{A}_P(\lambda)\right)$, $H' \in H + \mathfrak{n}_\mathfrak{p}$, and $X' \in P.X \subseteq \mathfrak{n}_\mathfrak{p}$. Let N be the line $N(t) := H' + tX'$ in $\big(G/L\big)_{eP}$. Then s attains its fiber degree on some G-conjugate of N.*

PROOF. Choose $gP \in G/P$ such that s_{gP} attains its fiber degree on a line in $\big(G/L\big)_{gP}$. For any $Z \in \mathfrak{n}_\mathfrak{p}$ let $d_g(Z)$ be the degree in t of the $\underline{L}_P(\lambda)_{gP}$-valued polynomial

$$t \mapsto s_{gP}\big(g * (H + tZ)\big)\,.$$

It is strightforward to check that the map d_g is upper semicontinuous on $\mathfrak{n}_\mathfrak{p}$ (i.e., $d_g \geq m$ is an open condition on $\mathfrak{n}_\mathfrak{p}$ for all $m \geq 0$), so there is a dense open subset of $\mathfrak{n}_\mathfrak{p}$ where d attains its maximal value. In particular, there is $Z' \in P.X$ such that s attains its fiber degree on the line $g * (H + tZ')$, since $g * P.X$ is a dense subset of $T^*(G/P)_{gP}$ (recall that $P.X$ is dense in $\mathfrak{n}_\mathfrak{p}$ since X is even nilpotent).

Choose $a \in P$ such that $aX' = Z'$. Then $ga.N$ is the desired G-conjugate of N, since $ga.N$ is a line in $\big(G/L\big)_{gP}$ parallel to $g * (H + tZ')$ and the degrees of the restrictions of s to parallel lines in a single fiber are the same.

$\square$

The following result is a basic tool for induced representations; recall that for any algebraic groups $G_1 \subseteq G_2$ and any G_1-module M, we have

$$H^i\left(G_2/G_1,\, \mathcal{L}_{G_2/G_1}(M)\right) \,\cong\, R^i Ind_{G_1}^{G_2} M\,. \tag{3.2}$$

Proposition 3.3.14. *(Frobenius reciprocity) Let $G_1 \subseteq G_2$ be algebraic groups. Let M be a G_1-module and N a G_2-module. Then*

$$Hom_{G_2}(N,\, Ind_{G_1}^{G_2} M) \,\cong\, Hom_{G_1}(Res_{G_1}^{G_2} N,\, M)\,.$$

Definition 3.3.15. Let $\mathcal{W}_\lambda^P(\mu)$ denote the subspace of $V_\lambda(\mu)$ consisting of L-highest weight vectors. Note that, in particular, $\mathcal{W}_\lambda^B(\mu) = V_\lambda(\mu)\,.$

Remark 3.3.16. Let B_L denote the Borel subgroup of L; note that

$$V^P(\mu)^* \,\cong\, H^0\left(L/B_L,\, \mathbb{C}_{-\mu}\right)$$

as L-modules. Hence, by Frobenius reciprocity (Proposition 3.3.14), we have isomorphisms of B_L-modules

$$
\begin{aligned}
Hom_G\Big(V(\mu)^*,\, H^0\left(G/P,\, q_*\,\mathcal{A}_P(\lambda)\right)\Big) \;&\cong\; Hom_G\Big(V(\mu)^*,\, H^0\left(G/L,\, \mathcal{A}_P(\lambda)\right)\Big) \\
&\cong\; Hom_L\big(V(\mu)^*,\, V^P(\lambda)^*\big) \\
&\cong\; Hom_{B_L}\big(V(\mu)^*,\, \mathbb{C}_{-\lambda}\big) \\
&\cong\; Hom_{B_L}\big(\mathbb{C}_\lambda,\, V(\mu)\big) \qquad (3.3) \\
&\cong\; \mathcal{W}^P_\lambda(\mu) \\
&\subseteq\; V(\mu)\,.
\end{aligned}
$$

Define a G-action on $Hom_{vs}\Big(V(\mu)^*,\, H^0\left(G/P,\, q_*\,\mathcal{A}_P(\lambda)\right)\Big)$ by $(g * f)(v) = f(g^{-1}v)$ (where Hom_{vs} stands for vector space homomorphisms). Define a similar G-action (also denoted by $*$) on $Hom_{vs}\big(V(\mu)^*,\, V^P(\lambda)^*\big)$ and $Hom_{vs}\big(V(\mu)^*,\, \mathbb{C}_{-\lambda}\big)$. Define a G-action on $Hom_{vs}\big(\mathbb{C}_\lambda,\, V(\mu)\big)$ by $(g * h)(v) = g.(h(v))$. Then the vector spaces in (3.3) above obtain interpretations as subspaces of G-modules. Furthermore, the isomorphisms in (3.3) commute with this G-action, so that we may think of all of these vector spaces as subspaces of $V(\mu)$. In particular, we will act on elements of these modules by elements of G and consider this action as occuring inside of $V(\mu)$.

Denote by φ the isomorphism

$$
Hom_G\Big(V(\mu)^*,\, H^0\left(G/P,\, q_*\,\mathcal{A}_P(\lambda)\right)\Big) \;\xrightarrow{\sim}\; \mathcal{W}^P_\lambda(\mu) \,\subseteq\, V(\mu)\,.
$$

For $v \in V(\mu)^*$ and $f \in Hom_G\Big(V(\mu)^*,\, H^0\left(G/L,\, \mathcal{A}_P(\lambda)\right)\Big)$ we will denote $f(v) \in H^0\left(G/P,\, q_*\,\mathcal{A}_P(\mu)\right)$ by f_v. Note that there is an explicit isomorphism

$$ev_H : Hom_G\Big(V(\mu)^*,\, H^0\left(G/P,\, q_*\,\mathcal{A}_P(\lambda)\right)\Big) \xrightarrow{\sim} Hom_L\big(V(\mu)^*,\, V^P(\lambda)^*\big)$$

given by

$$\big(ev_H(f)\big)(v) := f_v(eP)(H) \in \underline{A}^P(\lambda)_{eP} \cong V^P(\lambda)^*\,.$$

Definition 3.3.17. We define a filtration $\mathcal{H}$ on $\mathcal{W}^P_\lambda(\mu)$ as follows. Set

$$\mathcal{H}^n\big(\mathcal{W}^P_\lambda(\mu)\big) := \varphi\left(Hom_G\Big(V(\mu)^*,\, H^0\big(G/P,\, \big(q_*\,\mathcal{A}_P(\lambda)\big)^{\leq n}\,\big)\Big)\right) \subseteq \mathcal{W}^P_\lambda(\mu)\,.$$

Theorem 3.3.18. *We have*

$$\mathcal{H}^n\big(\mathcal{W}^P_\lambda(\mu)\big) = \mathcal{F}^n_X\big(\mathcal{W}^P_\lambda(\mu)\big)$$

for all $n \geq 0$ *and* $\lambda,\, \mu \in D$.

Proof. Choose $\bar{f} \in \mathcal{H}^n\big(\mathcal{W}^P_\lambda(\mu)\big) \setminus \mathcal{H}^{n-1}\big(\mathcal{W}^P_\lambda(\mu)\big)$. We need to check that

$$\bar{f} \in \mathcal{F}^n_X\big(\mathcal{W}^P_\lambda(\mu)\big) \setminus \mathcal{F}^{n-1}_X\big(\mathcal{W}^P_\lambda(\mu)\big)\,.$$

Set $f := \varphi^{-1}\bar{f}$. By Remark 3.3.16 we have $(g^{-1} * f)_v = f_{gv} = g.f_v$ for all $g \in G$.

Choose $v \in V(\mu)^*$ and define the line

$$M(t) := H + tX \subseteq \big(G/L\big)_{eP}\,.$$

By Lemma 3.3.13, f_v attains its fiber degree on $g.M$ for some $g \in G$. Since

$$f_{gv}(eP)(M) = g.\big(f_v(g^{-1}P)(g^{-1}M)\big)$$

we have that f_{gv} attains its fiber degree on M. Since $\underline{A}^P(\lambda)_{eP} \cong V^P(\lambda)^*$, by Lemma 3.3.12 this fiber degree is the degree of the $V^P(\lambda)^*$-valued polynomial

$$\big(\exp(-tX).f_{gv}\big)(eP)(H)\,.$$

In particular, this shows that n is the maximum of the degrees of the $V^P(\lambda)^*$-valued polynomials $\big(\exp(-tX).f_v\big)(eP)(H)$ as v runs over the elements of $V(\mu)^*$. By G-equivariance and Remark 3.3.16, for any $v \in V(\mu)^*$ we have

$$\begin{aligned}
\big(\exp(-tX).f_v\big)(eP)(H) &= f_{\exp(-tX).v}(eP)(H) \\
&= \Big(ev_H(f)\Big)(\exp(-tX).v) \\
&= \Big(\exp(tX) * ev_H(f)\Big)(v)\,.
\end{aligned}$$

Hence n is the maximum of the degrees of the $V^P(\lambda)^*$-valued polynomials

$$\Big(\exp(tX) * ev_H(f)\Big)(v)$$

as v varies in $V(\mu)^*$. Now, we may think of $\exp(tX) * \big(ev_H(f)\big)$ as a polynomial in t with values in $Hom_{vs}(V(\mu)^*,\, V^P(\lambda)^*)$, with evaluation at $v \in V(\mu)^*$ given in the obvious fashion. Thus n is clearly the degree of the polynomial $\exp(tX) * \big(ev_H(f)\big)$ in t. Further, this degree is the same as the degree of the polynomial $\exp(tX).\big(\varphi(f)\big)$. Since

$$\exp(tX).\big(\varphi(f)\big) \;=\; (1 + tX + t^2X^2/2 + \dots)\,.\,\varphi(f)$$

we see that $X^n.\big(\varphi(f)\big) \neq 0$ and $X^{n+1}.\big(\varphi(f)\big) = 0$, as desired. $\square$

Proposition 3.3.19. *Let $\lambda,\ \mu \in D$. Let X be an even nilpotent element in good position. Assume that $H^i\big(T^*(G/P),\ p^*\mathcal{L}_P(\lambda)\big) = 0$ for all $i > 0$. Then*

$$r_\mu^{X,\lambda}(q) \;=\; \sum_{n\geq 0} dim\left(\frac{\mathcal{H}^n\big(\mathcal{W}_\lambda^P(\mu)\big)}{\mathcal{H}^{n-1}\big(\mathcal{W}_\lambda^P(\mu)\big)}\right) q^n \;=\; m_\mu^{P,\lambda}(q)\,,$$

where P is the parabolic associated to X.

Proof. The first equality follows immediately from Theorem 3.3.18.

By Proposition 3.3.8 there is a G-equivariant isomorphism

$$\mathrm{gr}\, q_*\, \mathcal{A}_P(\lambda) \xrightarrow{\sim} p_*\, p^*\mathcal{L}_P(\lambda)$$

of graded sheaves on G/P. Furthermore, we have

$$p_*\, p^*\mathcal{L}_P(\lambda) \;\cong\; pr_*\Big(\mathcal{L}_B\big(\lambda\big) \otimes \mathcal{L}_B\big(S(\mathfrak{n_p}^*)\big)\Big)$$

(where $pr : G/B \to G/P$ is the projection map) and hence, for all $n \geq 0$, a short exact sequence

$$0 \;\to\; \big(q_*\, \mathcal{A}_P(\lambda)\big)^{\leq n-1} \;\to\; \big(q_*\, \mathcal{A}_P(\lambda)\big)^{\leq n} \;\to\; pr_*\Big(\mathcal{L}_B\big(\lambda\big) \otimes \mathcal{L}_B\big(S(\mathfrak{n_p}^*)\big)\Big) \;\to\; 0$$

of G-equivariant sheaves on G/P.

As

$$\begin{aligned} H^i\big(T^*(G/P),\ p^*\mathcal{L}_P(\lambda)\big) \;&\cong\; pr_*\Big(\mathcal{L}_B\big(\lambda\big) \otimes \mathcal{L}_B\big(S(\mathfrak{n_p}^*)\big)\Big) \\ &\cong\; H^i\Big(G/B,\ \mathcal{L}_B\big(\lambda\big) \otimes \mathcal{L}_B\big(S(\mathfrak{n_p}^*)\big)\Big) \end{aligned}$$

for all $i \geq 0$, the cohomology vanishing assumption and an easy induction on n give that $H^i\left(G/P,\ \big(q_*\, \mathcal{A}_P(\lambda)\big)^{\leq n}\right) = 0$ for all $i > 0$. This implies that there is an isomorphism of

G-modules

$$\frac{H^0\big(G/P,\ \big(q_*\,\mathcal{A}_P(\lambda)\big)^{\leq n}\big)}{H^0\big(G/P,\ \big(q_*\,\mathcal{A}_P(\lambda)\big)^{\leq n-1}\big)} \;\cong\; H^0\Big(G/B,\ \mathcal{L}_B\big(\lambda\big)\otimes\mathcal{L}_B\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*)\big)\Big)$$

for all $n \geq 0$. Thus we have reduced to showing that

$$\sum_{n\geq 0}\dim\left[\mathrm{Hom}_G\left(V(\mu)^*, H^0\Big(G/B,\ \mathcal{L}_B\big(\lambda\big)\otimes\mathcal{L}_B\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*)\big)\Big)\right)\right]q^n \;=\; m_\mu^{P,\lambda}(q)\,.$$

For any finite-dimensional B-module M set

$$\chi(M) \;:=\; \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,H^i\big(G/B,\ \mathcal{L}_B\,(M)\big)\in X(T)$$

and set

$$\chi\big(\mathrm{ch}\,M\big) \;:=\; \sum_{\substack{\text{weights}\\ \gamma\text{ of }M}}\ \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,H^i\big(G/B,\ \mathcal{L}_B\,(\mathbb{C}_\gamma)\big)\in X(T)\,,$$

where the weights are summed with multiplicity. By the additivity of Euler characteristic we have $\chi(M) = \chi\big(\mathrm{ch}\,M\big)$ for all finite-dimensional B-modules M, and by the vanishing assumption we have

$$\mathrm{ch}\left[H^0\Big(G/B,\ \mathcal{L}_B\big(\lambda\big)\otimes\mathcal{L}_B\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*)\big)\Big)\right] \;=\; \chi\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*)\otimes\mathbb{C}_{-\lambda}\big)$$

for all $n \geq 0$.

For any $n \geq 0$ let $p_q^{n,P}$ be the degree-n coefficient of the polynomial p_q^P and let $m_\mu^{n,P,\lambda}$ be the degree-n coefficient of $m_\mu^{P,\lambda}(q)$ (recall the definitions of p_q^P and $m_\mu^{P,\lambda}(q)$ from section

3.2 above). For all $n \geq 0$ we now obtain

$$
\begin{aligned}
\chi\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*) \otimes \mathbb{C}_{-\lambda}\big) &= \chi\big(\mathrm{ch}\,(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*) \otimes \mathbb{C}_{-\lambda})\big) \\
&= \sum_{\gamma \in \Lambda} p_q^{n,P}(\gamma)\, \chi(\mathbb{C}_{-(\lambda+\gamma)}) \\
&= \sum_{\gamma' \in \Lambda} p_q^{n,P}(\gamma' - \lambda)\, \chi(\mathbb{C}_{-\gamma'}) \qquad\qquad (\text{where } \gamma' = \lambda + \gamma) \\
&= \sum_{\gamma'' \in D} \left(\sum_{w \in W} (-1)^{l(w)} p_q^{n,P}(w * \gamma'' - \lambda) \right) \chi(\mathbb{C}_{-\gamma''}) \\
&= \sum_{\gamma'' \in D} m_{\gamma''}^{n,P,\lambda} \chi(\mathbb{C}_{-\gamma''}) \\
&= \sum_{\gamma'' \in D} m_{\gamma''}^{n,P,\lambda} \mathrm{ch}\big(V(\gamma'')^*\big)\,.
\end{aligned}
$$

That is, for all $\gamma'' \in D$ the multiplicity of $V(\gamma'')^*$ in $H^0\Big(G/B,\ \mathcal{L}_B\big(\lambda\big) \otimes \mathcal{L}_B\big(S^n(\mathfrak{n}_{\mathfrak{p}}{}^*)\big)\Big)$ is $m_{\gamma''}^{n,P,\lambda}$, as desired.

$\square$

The proof of Theorem 3.2.1 is now immediate from Theorem 3.3.18 and Proposition 3.3.19.

CHAPTER 4

Cohomology of Flag Varieties

4.1. Overview

We begin by providing an outline of the results in this section. For any parabolic $P \subseteq G$ let $r : G \times^B \mathfrak{n}_\mathfrak{p} \to G/B$ be the bundle map and consider the cohomology groups

$$H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) := H^i\big(G \times^B \mathfrak{n}_\mathfrak{p},\ r^*\mathcal{L}_B(\lambda)\big)\,.$$

By equation (2.4), when λ is P-dominant we have

$$H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) \;\cong\; H^i\big(T^*(G/P),\ p^*\mathcal{L}_P(\lambda)\big)\,,$$

where $p : T^*(G/P) \to G/P$ is the bundle map.

Broer [**4**] showed that these cohomology groups vanish for $i > 0$ in the following cases (see Theorem 4.2.4 and Propositions 4.2.1 and 4.4.3 below):

- $P = B$ and $\operatorname{cht}(\lambda) = 0$ (see Lemma 4.2.3(i) below for a useful definition of cht). In particular, we have higher cohomology vanishing when $P = B$ and $\lambda \in D$.
- P is a minimal parabolic corresponding to a short simple root and $\lambda \in D$.
- P is any parabolic and λ is any weight such that $\lambda(\alpha^\vee) = -1$ for some $\alpha \in \pi_P$ (in fact, we have cohomology vanishing for all i in this case).
- P is any parabolic and $\mathbb{C}_\lambda$ is a 1-dimensional P-module, i.e. $\lambda(\alpha^\vee) = 0$ for all $\alpha \in \pi_P$.

In this chapter we show that these cohomology groups vanish for $i > 0$ in the following additional cases:

- P is an even parabolic and $\lambda \in D$ is such that $\lambda + 2\rho_P \in D$ (section 4.2).
- P is any minimal parabolic and $\lambda \in D$ (section 4.4).
- P is any parabolic in type A and $\lambda \in D$ is regular (section 4.5).

4.2. Vanishing Results for Flag Varieties

The following result of Broer's ([**4**], Lemma 3.1) is very useful. I've stated it in a more general version than Broer does, although his proof goes through mutatis mutandis. It follows easily from the Borel-Weil-Bott Theorem and the Leray spectral sequence.

Proposition 4.2.1. *(Broer, [**4**]) Let $P \subseteq Q$ be two parabolic subgroups of G, let V be an irreducible P-module, and let M be a Q-module.*

(i) There exists at most one $i \geq 0$ such that

$$H^i\left(Q/P,\, \mathcal{L}_P\left(V^*\right)\right) \neq 0.$$

(ii) If $H^i\left(Q/P,\, \mathcal{L}_P\left(V^*\right)\right) = 0$ for all $i \geq 0$, then for all $i \geq 0$

$$H^i\big(G/P,\, \mathcal{L}_P(V^* \otimes M)\big) = 0.$$

(iii) Suppose that $\tilde{V} := H^\nu\left(Q/P,\, \mathcal{L}_P\left(V^*\right)\right) \neq 0$ for $\nu \geq 0$. Then:

$$H^i\big(G/P,\, \mathcal{L}_P(V^* \otimes M)\big) = \begin{cases} 0 & \text{if } i < \nu; \\ H^{i-\nu}\left(G/Q,\, \mathcal{L}_Q\left(\tilde{V} \otimes M\right)\right) & \text{if } i \geq \nu. \end{cases}$$

Definition 4.2.2. For $\gamma \in \Lambda$ and γ^+ denote the unique dominant weight in the Weyl group orbit of γ. By [**4**], section 3, there is a unique dominant weight $\gamma^\star$ such that (a) $\gamma^\star \geq \gamma$ and (b) $\gamma^\star \leq \mu$ for all $\mu \in D$ such that $\mu \geq \gamma$. We say that the **combinatorial height** of λ is n, and we write $\mathrm{cht}\,(\lambda) = n$, if

$$n = \max\left\{m : \text{there is a chain of dominant weights } \gamma^\star =: \gamma_0 < \gamma_1 < \cdots < \gamma_m := \gamma^+\right\}.$$

The following is a more natural way of classifying the weights of combinatorial height 0. Part (i) is due to Broer ([**5**], Proposition 2), and (ii) follows readily from (i).

Lemma 4.2.3.

(i) Let $\lambda \in \Lambda$. Then $cht\,(\lambda) = 0$ iff $\lambda(\beta^\vee) \geq -1$ for all $\beta \in \Delta^+$. In particular, $cht\,(\lambda) = 0$ for all $\lambda \in D$.

(ii) Let $\lambda \in \Lambda$ with $cht\,(\lambda) = 0$ and let $\mu \in D$. Then $cht\,(\lambda + \mu) = 0$.

Theorem 4.2.4. *(Broer, [**4**])*

(i) Let $P \subseteq G$ be any parabolic subgroup. Then $H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) = 0$ for all $i > 0$ and $\lambda \in D$ such that $\mathbb{C}_\lambda$ is a P-module (i.e., $\lambda(\alpha^\vee) = 0$ for all $\alpha \in \pi_P$).

(ii) Choose $\gamma \in \Lambda$. Then $cht\,(\gamma) = 0$ iff $H^{i,\mathfrak{n}}(\gamma) = 0$ for all $i > 0$.

(iii) Choose $\gamma \in \Lambda$. Then $H^{i,\mathfrak{n}}(\gamma) = 0$ for all $i > cht\,(\gamma)$.

We aim to generalize the theorem above to the case of arbitrary irreducible P-modules with dominant highest weight. The case of minimal parabolics will be considered in section 4.4 below; the following proposition is a more general statement about vanishing for arbitrary parabolics. Recall the notation $\langle M\rangle$ for any subset $M \subseteq \Lambda$ from section 2.1; we have that $2\rho_P = \langle \Delta_P^+\rangle$. Recall also that a parabolic P is called even if it is the parabolic associated to an even nilpotent element. Remark that the proof of the following theorem is adapted from Broer's proof of his theorem 2.2 in [**4**].

THEOREM 4.2.5. *Let P be an even parabolic. Then $H^{i,\mathfrak{n}_\mathfrak{p}}(\lambda) = 0$ for all $i > 0$ and $\lambda \in D$ such that $\lambda + 2\rho_P \in D$.*

PROOF. Let $\mu := \lambda + 2\rho_P \in D$. By 2.4, it suffices to show that

$$H^i\big(G/B,\ \mathcal{L}_B(S(\mathfrak{n}_\mathfrak{p}{}^*) \otimes \mathbb{C}_{-\lambda})\big) = 0 \tag{4.1}$$

for all $i > 0$.

Consider the vector bundle $Y = G \times^B (\mathfrak{n}_\mathfrak{p} \oplus \mathbb{C}_\mu)$ on G/B and its subbundles $Y_1 = G \times^B \mathfrak{n}_\mathfrak{p}$ and $Y_2 = G \times^B \mathbb{C}_\mu$. We have $Y = Y_1 \oplus Y_2$ (where $\oplus$ denotes a direct sum of vector bundles). Let ϕ (resp. ϕ_1, ϕ_2) be the bundle maps from Y (resp. Y_1, Y_2) to G/B (so that we have $\phi = \phi_1 \oplus \phi_2$). There is a natural map $\delta_1 : Y_1 \to \mathcal{N}$ (recall that $\mathcal{N}$ is the nullcone of $\mathfrak{g}$) given by $g * X \mapsto \mathrm{Ad}(g).X$. Let $\delta_2 : Y_2 \to V(\mu)$ be the map $g * v \mapsto g.v$, and let $\delta : Y \to \mathcal{N} \times V(\mu)$ be the map $g * (X, v) \mapsto (g.X, g.v)$.

We claim now that δ has a finite fiber. Now, δ_1 factors as the quotient map

$$Y_1 = G \times^B \mathfrak{n}_\mathfrak{p} \twoheadrightarrow G \times^P \mathfrak{n}_\mathfrak{p}$$

followed by the map $b : G \times^P \mathfrak{n}_\mathfrak{p} \to \mathcal{N}$. Choose an even nilpotent element $r \in \mathfrak{n}_\mathfrak{p}$ such that P is the parabolic associated to r; then

$$b : b^{-1}(\mathfrak{O}_r) \xrightarrow{\sim} \mathfrak{O}_r$$

is a variety isomorphism (this follows from Lemma 2.3.6; recall that $\mathfrak{O}_r$ is the adjoint orbit through r). In particular, $b^{-1}(r)$ is an element of the fiber of $G \times^P \mathfrak{n}_\mathfrak{p}$ over eP. Thus $\delta_1^{-1}(r)$ is contained in the inverse image of φ_1 over $P.eB \subseteq G/B$. That is, $\phi_1(g_1^{-1}(r)) \subseteq P.eB$.

Choose nonzero $v \in V(\mu)_\mu$. Note that $\mu(\alpha^\vee) > 0$ for all $\alpha \in \Delta_P^+$. Thus, for any $p \in P$ with $p \notin B$, we have $p.v \notin V(\mu)_\mu$. Hence

$$\phi_2(\delta_2^{-1}(v)) \cap (P.eB) = \{eB\} \subseteq G/B\,.$$

In particular, this gives that

$$\phi_1(\delta_1^{-1}(r)) \cap \phi_2(\delta_2^{-1}(v)) = \{eB\}\,.$$

As

$$\phi\big(\delta^{-1}(r,\, v)\big) \;\subseteq\; \phi_1\big(\delta_1^{-1}(r)\big) \cap \phi_2\big(\delta_2^{-1}(v)\big)$$

we obtain that $\phi\big(\delta^{-1}(r,\, v)\big) = eB$. Now note that δ restricted to the fiber $\phi^{-1}(eB)$ of Y over eB is an injection, as we have $e * (Z, w) \mapsto (Z, w)$ for all $Z \in \mathfrak{n}_\mathfrak{q}$ and $w \in \mathbb{C}_\mu$. Thus the fiber of δ over (r, v) is a single point, as claimed above. In particular, the map δ is generically finite.

Now, δ factors as $Y \hookrightarrow G \times^B (\mathcal{N} \times V(\mu))$ followed by the projection

$$G \times^B (\mathcal{N} \times V(\mu)) \twoheadrightarrow \mathcal{N} \times V(\mu).$$

As $\mathcal{N} \times V(\mu)$ is a G-variety,

$$G \times^B (\mathcal{N} \times V(\mu)) \cong G/B \times (\mathcal{N} \times V(\mu)).$$

Hence δ is a proper map. By Proposition 2.2.16 (i), since δ has finite fibers generically and $\mathcal{N} \times V(\mu)$ is affine, we obtain that $H^i\,(Y,\, \omega_Y) = 0$ for all $i > 0$.

As Y is a homogeneous bundle,

$$\omega_Y \;\cong\; \phi^*\mathcal{L}_B\big({\textstyle\bigwedge}^{\rm top}(\mathfrak{n}_\mathfrak{p} \oplus \mathbb{C}_\mu)^* \otimes {\textstyle\bigwedge}^{\rm top}(\mathfrak{g}/\mathfrak{b})^*\big).$$

Using the Killing form, we may identify $(\mathfrak{g}/\mathfrak{b})^*$ with $\mathfrak{n}$. Furthermore,

$$\bigwedge^{\mathrm{top}}(\mathfrak{n}_{\mathfrak{p}}^*) \cong \mathbb{C}_{-\langle \Delta^+ \setminus \Delta_P^+ \rangle} = \mathbb{C}_{\langle \Delta_P^+ \rangle - \langle \Delta^+ \rangle}$$

and

$$\bigwedge^{\mathrm{top}}(\mathfrak{n}) \cong \mathbb{C}_{\langle \Delta^+ \rangle} .$$

Thus

$$\omega_Y \;\cong\; \phi^* \mathcal{L}_B\big(\mu - \langle \Delta_P^+ \rangle\big) \;\cong\; \phi^* \mathcal{L}_B\,(\lambda)$$

and we have

$$H^i\,(Y,\, \phi^* \mathcal{L}_B\,(\lambda)) = 0$$

for all $i > 0$.

Since ϕ is an affine map,

$$H^i\,(Y,\, \phi^* \mathcal{L}_B\,(\lambda)) \cong H^i\big(G/B,\, \phi_* \phi^* \mathcal{L}_B\,(\lambda)\big)$$

for all i. By arguments similar to those in the proof of Proposition 2.2.16,

$$\phi_* \phi^* \mathcal{L}_B\,(\lambda) \;\cong\; \mathcal{L}_B\big(S({\mathfrak{n}_{\mathfrak{p}}}^* \oplus \mathbb{C}_{-\mu}) \otimes \mathbb{C}_{-\lambda}\big) .$$

Thus

$$H^i\Big(G/B,\, \mathcal{L}_B\big((S({\mathfrak{n}_{\mathfrak{p}}}^*)) \otimes S(\mathbb{C}_{-\mu}) \otimes \mathbb{C}_{-\lambda}\big)\Big) = 0 \tag{4.2}$$

for all $i > 0$.

Now, $\mathcal{L}_B\big(S(\mathbb{C}_{-\mu})\big)$ is graded by the G-subsheaves $\mathcal{L}_B\big(S^n(\mathbb{C}_{-\mu})\big)$ for $n \geq 0$. Hence the cohomology group in (4.2) above is graded by the G-submodules

$$H^i\Big(G/B,\, \mathcal{L}_B\big((S({\mathfrak{n}_{\mathfrak{p}}}^*)) \otimes S^n(\mathbb{C}_{-\mu}) \otimes \mathbb{C}_{-\lambda}\big)\Big)$$

for $n \geq 0$. In particular, setting $n = 0$, the vanishing in (4.2) implies the desired vanishing in equation (4.1) (recall that $\mathcal{L}_B(\lambda) = \mathcal{L}_B(\mathbb{C}_{-\lambda})$). This completes the proof.

□

Corollary 4.2.6. *Let P be an even parabolic. Recall that w_0^P is the longest element of the Weyl group $W_P \subseteq W$ corresponding to π_P. Choose $\mu \in \Lambda$ such that $w_0^P(\mu) \in D$ and w_0^P is of minimal length with this property. Then $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all $i > l(w_0^P)$.*

Proof. If $\mu(\alpha^\vee) = -1$ for any $\alpha \in \pi_L$ then the result follows immediately from 4.2.1. Thus we may assume that $\mu(\alpha^\vee) \leq -2$ for all $\alpha \in \pi_L$ and hence $w_0^L\mu(\alpha^\vee) \geq 2$ for all $\alpha \in \pi_L$. This implies that

$$\left(w_0^L * \mu\right)(\alpha^\vee) = (\mu - 2\rho_L)(\alpha^\vee) \geq 0$$

for all $\alpha \in \pi_L$ and we get that $w_0^L * \mu \in D$. Thus $H^{i,\mathfrak{n}_\mathfrak{p}}(w_0^P * \mu) = 0$ for all $i > 0$ by Theorem 4.2.5. The result now follows from Proposition 4.2.1.

□

Corollary 4.2.7. *Let $\mu \in \Lambda$ be such that $\mu \notin D$ and $r_\beta(\mu) \in D$ for some short root $\beta \in \pi$. Assume that the minimal parabolic P_β corresponding to β is even. Then $H^{i,\mathfrak{n}}(\mu) = 0$ for all $i > 1$.*

Proof. Let $\mathfrak{p}$ be the parabolic with $\pi_P = \{\beta\}$. Consider the Koszul complex

$$0 \;\to\; S(\mathfrak{n}^*) \otimes \mathbb{C}_{-(\mu+\beta)} \;\to\; S(\mathfrak{n}^*) \otimes \mathbb{C}_{-\mu} \;\to\; S({\mathfrak{n}_\mathfrak{p}}^*) \otimes \mathbb{C}_{-\mu} \;\to\; 0.$$

We use induction on $\mu(\beta^\vee)$.

If $\mu(\beta^\vee) = -1$ then $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all i by Proposition 4.2.1 (ii). Also, $\mu + \beta = r_\beta(\mu) \in D$, so $H^{i,\mathfrak{n}}(\mu + \beta) = 0$ for all $i > 0$ by Theorem 4.2.4 (ii) (this also follows from Theorem 4.2.5). Thus, by the Koszul complex above, $H^{i,\mathfrak{n}}(\mu) = 0$ for all $i > 0$.

If $\mu(\beta^\vee) = -2$ then $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all $i > 1$ by Corollary 4.2.6. Further, as $r_\beta(\mu) \in D$ and $(\mu + \beta)(\beta^\vee) = (r_\beta\mu - \beta)(\beta^\vee) = 0$, we have that $\mu + \beta \in D$. The result now follows from the Koszul complex above.

Now assume that $\mu(\beta^\vee) = -n$ for some $n > 2$. By induction assume that $H^{i,\mathfrak{n}}(\lambda) = 0$ for all $i > 1$ and λ such that $r_\beta(\lambda) \in D$ and $0 < -\lambda(\beta^\vee) < n$.

Note that

$$r_\beta(\mu + \beta) = r_\beta(\mu) - \beta \in D,$$

since

$$(r_\beta(\mu) - \beta)(\beta^\vee) = n - 2 \geq 0$$

and

$$(r_\beta(\mu) - \beta)(\alpha^\vee) \geq r_\beta(\alpha^\vee) \geq 0$$

for any $\alpha \in \pi \setminus \{\beta\}$. Thus, by induction, $H^{i,\mathfrak{n}}(\mu + \beta) = 0$ for all $i > 1$. Furthermore, $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all $i > 1$ by Corollary 4.2.7. The result now follows easily from the Koszul complex above. □

4.3. Some Combinatorics

The following proposition is a handy combinatorial tool.

Proposition 4.3.1. *(Thomsen, [**30**]) Let $\lambda \in \Lambda$, $\alpha \in \pi$, and $\beta \in \Delta^+$.*

(i) If $\lambda(\beta^\vee) \geq 0$ then $\lambda^+ < (\lambda + \beta)^+$.

(ii) *If* $\lambda(\beta^\vee) = -1$ *then* $\lambda^+ = (\lambda+\beta)^+$.

(iii) *If* $\lambda(\beta^\vee) < -1$ *then* $\lambda^+ > (\lambda+\beta)^+$.

(iv) *If* $\lambda(\alpha^\vee) < 0$ *then* $\lambda^\star = (\lambda+\alpha)^\star$.

(v) *If* $\lambda(\alpha^\vee) = -1$ *then* $cht(\lambda) = cht(\lambda+\alpha)$.

(vi) *If* $\lambda(\alpha^\vee) \leq -2$ *then* $cht(\lambda) > cht(\lambda+\alpha)$.

(vii) *If* $\lambda(\alpha^\vee) \leq 0$ *then* $cht(\lambda) \geq cht(s_\alpha\lambda)$.

(viii) *If* $\lambda(\alpha^\vee) \leq -2$ *then* $cht(\lambda) > cht(s_\alpha * \lambda)$.

4.4. Vanishing for Minimal Parabolics

Lemma 4.4.1. *(Broer, [**4**]) Assume that G is simple. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be any (nonempty) collection of short simple orthogonal roots. Then*

$$cht\left(\sum_{i=1}^{k} \alpha_i\right) = k - 1\,.$$

Lemma 4.4.2. *$cht(\beta+\mu) = 0$ for all short roots β and $\mu \in D$.*

Proof. This is immediate from Lemma 4.2.3 and the fact that $\beta(\gamma^\vee) \geq -1$ for all $\gamma \in \Delta^+$. $\square$

The following proposition is implicit in [**4**].

Proposition 4.4.3. *Let G be simple. Let P be a minimal parabolic corresponding to a short simple root α. Let $\mu \in D$. Then $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all $i > 0$.*

Proof. We have a short exact sequence of B-modules

$$0 \;\rightarrow\; \mathfrak{n}_\mathfrak{p} \;\rightarrow\; \mathfrak{n} \;\rightarrow\; \mathbb{C}_\alpha \;\rightarrow\; 0$$

which gives rise (upon taking the dual and tensoring with $\mathbb{C}_{-\mu}$) to a short exact Koszul sequence

$$0 \to S(\mathfrak{n}^*) \otimes \mathbb{C}_{-\alpha-\mu} \to S(\mathfrak{n}^*) \otimes \mathbb{C}_{-\mu} \to S(\mathfrak{n}_{\mathfrak{p}}{}^*) \otimes \mathbb{C}_{-\mu} \to 0.$$

Hence we obtain the short exact sequence

$$0 \to \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\alpha+\mu\right) \to \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\mu\right) \to \mathcal{L}_B\big(S(\mathfrak{n}_{\mathfrak{p}}{}^*)\big) \otimes \mathcal{L}_B\left(\mu\right) \to 0$$

of sheaves on G/B. Using the associated long exact sequence of cohomology, the result now follows from Lemma 4.2.3 (ii), Theorem 4.2.4 (ii), and Lemma 4.4.2. $\square$

We now have the following theorem.

Theorem 4.4.4. *Let P be any minimal parabolic. Let $\mu \in D$. Then $H^{i,\mathfrak{n}_{\mathfrak{p}}}(\mu) = 0$ for all $i > 0$.*

Before we can prove the theorem, we need a combinatorial result. For the rest of this section let $(\cdot\,,\cdot)$ denote a Weyl group-invariant inner product on $\mathfrak{h}$, normalized so that $|\alpha|^2 = 1$ for all short simple roots α. Let $\mathbb{Z}\Delta$ denote the root lattice of T.

The following is a trivial but useful lemma.

Lemma 4.4.5.

(i) $|\lambda|^2 \in \mathbb{Z}^+$ for all $\lambda \in \mathbb{Z}\Delta$.

(ii) Let μ and λ in D. If $\mu < \lambda$ then $|\mu|^2 < |\lambda|^2$.

Recall the combinatorial notation from section 4.3.

Lemma 4.4.6. *Let $\lambda \in \Lambda$.*

(i) $cht(\lambda) \leq |\lambda^{+}|^2 - |\lambda^{\star}|^2$.

(ii) For any $\gamma \in \Lambda$ *such that (1)* $\lambda \leq \gamma \leq \lambda^{\star}$ *and (2)* $cht(\gamma) = 0$, *we have*

$$cht(\lambda) \leq |\lambda|^2 - |\gamma|^2.$$

Proof. (i) For any μ, $\gamma \in D$ with $\mu < \gamma$, we have

$$|\gamma|^2 - |\mu|^2 \; = \; |\gamma - \mu|^2 + 2(\gamma - \mu \, , \mu) \; \geq \; 1 \, .$$

Indeed, $\gamma - \mu \in \mathbb{Z}\Delta$ implies $|\gamma - \mu|^2 \in \mathbb{Z}^{+}$ by 4.4.5 (i); and $(\gamma - \mu \, , \mu) \geq 0$ since $\mu \in D$ and $\gamma - \mu > 0$. The result now follows from the definition of cht.

(ii) By the definition of $\lambda^{\star}$ and the fact that $\lambda \leq \gamma \leq \gamma^{\star}$, we have $\gamma^{\star} \geq \lambda^{\star}$. By the definition of $\gamma^{\star}$ and the fact that $\gamma \leq \lambda^{\star}$ we have $\gamma^{\star} \leq \lambda^{\star}$. Thus $\lambda^{\star} = \gamma^{\star}$. And, since $\mathrm{cht}\,(\gamma) = 0$, we have $\gamma^{+} = \gamma^{\star}$. Hence $\gamma^{+} = \lambda^{\star}$. The result now follows from the Weyl-group invariance of $(\ ,\ )$.

□

We can now prove Theorem 4.4.4.

Proof. By Proposition 4.4.3, we need to check this result for parabolics corresponding to long simple roots. Recall that for any $\lambda \in \Lambda$ and minimal parabolic $\mathfrak{p}$ corresponding to a simple root α we have the Koszul complex

$$0 \; \to \; S(\mathfrak{n}^{*}) \otimes \mathbb{C}_{-\alpha-\lambda} \; \to \; S(\mathfrak{n}^{*}) \otimes \mathbb{C}_{-\lambda} \; \to \; S({\mathfrak{n}_{\mathfrak{p}}}^{*}) \otimes \mathbb{C}_{-\lambda} \; \to \; 0.$$

Thus, by the associated long exact sequence of sheaves on G/B and Theorem 4.2.4, to prove the theorem for $\mathfrak{p}$ it suffices to show that

$$\mathrm{cht}\,(\alpha + \mu) \leq 1$$

for all $\mu \in D$.

We now check the cht condition case by case, and we use the usual labelling of simple roots $\pi = \{\alpha_1, \ldots, \alpha_n\}$.

(1) Type B_n.

We use the standard labelling for the simple roots, where α_n is the unique short simple root. Let $\mu \in D$ and let α be any long simple root. If $\mu(\alpha_n^\vee) > 0$ then $(\mu + \alpha)(\beta^\vee) > -1$ for all $\beta \in \Delta^+$, and hence $\operatorname{cht}(\mu + \alpha) = 0$. Thus we may assume that $\mu(\alpha_n^\vee) = 0$. We may also assume that $\mu \neq 0$ as the higher cohomology vanishing when $\mu = 0$ follows from Theorem 4.2.4 (i).

Set

$$m := \max\{i : \mu(\alpha_i^\vee) > 0\}.$$

We claim that $\operatorname{cht}(\mu + \alpha_l) = 0$ for all $0 < l < m$ (if $m = 1$ this condition is of course empty and there is nothing to check). Set

$$c_1 := \max\{j : j < l \text{ and } \mu(\alpha_j^\vee) > 0\}$$

and set

$$c_2 := \min\{k : l < k \text{ and } \mu(\alpha_k^\vee) > 0\} \leq m\,;$$

if $\mu(\alpha_j^\vee) = 0$ for all $j < m$ then set $c_1 = 0$. Set

$$\gamma := \sum_{i=c_1+1}^{c_2-1} \mu + \alpha_i\,.$$

By Proposition 4.3.1 (v) and an easy induction we have $\operatorname{cht}(\mu + \alpha_l) = \operatorname{cht}(\gamma)$. Furthermore, one checks easily that $\gamma \in D$. This shows the claim, and thus it suffices to consider $\mu + \alpha_k$ for $m \leq k < n$.

For any $k \in \mathbb{N}$ with $m \le k < n$, set

$$\beta_k := \sum_{i=k}^{n} \alpha_i \in \Delta^+ .$$

An easy induction utilizing Proposition 4.3.1 (iv) shows that $(\beta_k + \mu)^\star = (\alpha_k + \mu)^\star$, since $\mu(\alpha_j^\vee) = 0$ for $j > m$. Thus

$$\alpha_k + \mu \ \le\ \beta_k + \mu \ \le\ (\alpha_k + \mu)^\star .$$

Further, β_k is a short root, so $\operatorname{cht}(\beta_k + \mu) = 0$ by Corollary 4.4.2. So, by Lemma 4.4.6, we have

$$\begin{aligned}
\operatorname{cht}(\alpha_k + \mu) \ &\le\ |\alpha_k + \mu|^2 - |\beta_k + \mu|^2 \\
&=\ -\left[\sum_{i=k+1}^{n} |\alpha_i|^2 \ +\ 2\sum_{i=k}^{n-1} (\alpha_i \,, \alpha_{i+1})\right] \\
&=\ -\Big[(2(n-k-1)+1) - (2(n-k))\Big] \\
&=\ 1\,.
\end{aligned}$$

(2) Type C_n.

We use the standard labelling for the simple roots, where α_n denotes the unique long simple root. Let $\mu \in D$. If $\mu(\alpha_{n-1}^\vee) > 0$ then $\operatorname{cht}(\mu + \alpha_n) = 0$, so we may assume that $\mu(\alpha_{n-1}^\vee) = 0$.

By Proposition 4.3.1 (iv), since $(\mu + \alpha_n)\left(\alpha_{n-1}^\vee\right) < 0$, we get $(\alpha_{n-1} + \alpha_n + \mu)^\star = (\alpha_n + \mu)^\star$. Thus

$$\alpha_n + \mu \ \le\ \alpha_{n-1} + \alpha_n + \mu \ \le\ (\alpha_n + \mu)^\star .$$

Further, $\alpha_{n-1}+\alpha_n$ is a short root, so $\operatorname{cht}(\alpha_{n-1}+\alpha_n+\mu)=0$ by Corollary 4.4.2. So, by Lemma 4.4.6, we have

$$\begin{aligned}
\operatorname{cht}(\alpha_n+\mu) &\leq |\alpha_n+\mu|^2-|\alpha_{n-1}+\alpha_n+\mu|^2 \\
&= -\left[\,|\alpha_{n-1}|^2+2(\alpha_{n-1}\,,\alpha_n)\right] \\
&= -(1-2) \\
&= 1\,.
\end{aligned}$$

(3) Type G_2.

We have a short simple root α_1 and a long simple root α_2. If $\mu(\alpha_1^\vee)>1$ then $\operatorname{cht}(\mu+\alpha_2)=0$ by Proposition 4.3.1 (v), so we may assume that $\mu(\alpha_1^\vee)\leq 1$. Since $\alpha_1+\alpha_2$ is a short root, by Corollary 4.4.2 we get that $\operatorname{cht}(\alpha_1+\alpha_2+\mu)=0$. By Proposition 4.3.1 (iv) and the assumption on μ, we have $(\alpha_2+\mu)^\star=(\alpha_1+\alpha_2+\mu)^\star$. Hence

$$\alpha_2+\mu\;<\;\alpha_1+\alpha_2+\mu\;\leq\;(\alpha_2+\mu)^\star$$

and we have

$$\begin{aligned}
\operatorname{cht}(\alpha_2+\mu) &\leq |\alpha_2+\mu|^2-|\alpha_1+\alpha_2+\mu|^2 \\
&= -\left[|\alpha_1|^2+2(\mu\,,\alpha_1)+2(\alpha_1\,,\alpha_2)\right] \\
&= -(1+\mu(\alpha_1^\vee)-3) \\
&= 2-\mu(\alpha_1^\vee)\,.
\end{aligned}$$

Thus, if $\mu(\alpha_1^\vee)=1$, we have $\operatorname{cht}(\alpha_2+\mu)\leq 1$ and we are done.

Now assume that $\mu(\alpha_1^\vee) = 0$. Let $\mathfrak{p}$ be the minimal parabolic associated to α_2. If $\mu = 0$ then

$$H^{i,\mathfrak{n}_\mathfrak{p}}(\mathbb{C}) \;\cong\; H^i\left(T^*(G/P),\, \mathcal{O}_{T^*(G/P)}\right) = 0$$

for $i > 0$, by Theorem 4.2.4.

If $\mu \neq 0$ then $\mu(\alpha_2^\vee) > 0$ and $\mu + \alpha_1 + \alpha_2 \in D$. Let Q be the minimal parabolic corresponding to the short simple root α_1. We have the short exact Koszul complex

$$\begin{aligned} 0 \;\;\rightarrow\;\; & \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\mu + \alpha_1 + \alpha_2\right) \rightarrow \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\mu + \alpha_2\right) \\ \rightarrow\;\; & \mathcal{L}_B\big(S({\mathfrak{n}_\mathfrak{q}}^*)\big) \otimes \mathcal{L}_B\left(\mu + \alpha_2\right) \;\rightarrow\; 0 \end{aligned}$$

of sheaves on G/B. Since $\mu + \alpha_1 + \alpha_2 \in D$ we have

$$H^{i,\mathfrak{n}}(\mu + \alpha_1 + \alpha_2) = 0$$

for all $i > 0$. Note also that $r_{\alpha_1} * \mu = \mu + \alpha_1 + \alpha_2$ and that $\left(\mu + \alpha_1 + \alpha_2\right)(\alpha_1^\vee) = 0$. Thus

$$H^{i,\mathfrak{n}_\mathfrak{q}}(r_{\alpha_1} * \mu) = 0$$

for all $i > 1$ by Proposition 4.2.1 and Theorem 4.2.4. Thus, by the short exact sequence above,

$$H^{i,\mathfrak{n}}(\mu + \alpha_2) = 0$$

for all $i > 1$. From the short exact Koszul complex

$$\begin{aligned} 0 \;\;\rightarrow\;\; & \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\mu + \alpha_2\right) \rightarrow \mathcal{L}_B\big(S(\mathfrak{n}^*)\big) \otimes \mathcal{L}_B\left(\mu\right) \\ \rightarrow\;\; & \mathcal{L}_B\big(S({\mathfrak{n}_\mathfrak{p}}^*)\big) \otimes \mathcal{L}_B\left(\mu\right) \;\rightarrow\; 0 \end{aligned}$$

and the fact that $\mu \in D$ we now obtain that $H^{i,\mathfrak{n}_\mathfrak{p}}(\mu) = 0$ for all $i > 0$, as desired.

(4) Type F_4.

We have 4 simple roots α_i, $i = 1, 2, 3, 4$, where α_3 and α_4 are the long simple roots. Fix $\mu \in D$.

We claim that if $\mu(\alpha_2^\vee) > 0$ then $\mathrm{cht}\,(\alpha_i + \mu) = 0$, $i = 3, 4$. In the following we use Proposition 4.3.1 (v) extensively. If $\mu(\alpha_3^\vee) > 0$ and $\mu(\alpha_4^\vee) > 0$ then $\mu + \alpha_3 \in D$ and $\mu + \alpha_4 \in D$. If $\mu(\alpha_3^\vee) = 0$ and $\mu(\alpha_4^\vee) > 0$ then $\mu + \alpha_3 \in D$ and $\mathrm{cht}\,(\mu + \alpha_4) = \mathrm{cht}\,(\mu + \alpha_3 + \alpha_4) = 0$ as $\mu + \alpha_3 + \alpha_4 \in D$. If $\mu(\alpha_4^\vee) = 0$ and $\mu(\alpha_3^\vee) > 0$ then $\mu + \alpha_4 \in D$ and $\mathrm{cht}\,(\mu + \alpha_3) = \mathrm{cht}\,(\mu + \alpha_3 + \alpha_4) = 0$ as $\mu + \alpha_3 + \alpha_4 \in D$. Finally, if $\mu(\alpha_3^\vee) = \mu(\alpha_4^\vee) = 0$ we have that $\mathrm{cht}\,(\mu + \alpha_3) = \mathrm{cht}\,(\mu + \alpha_3 + \alpha_4) = \mathrm{cht}\,(\mu + \alpha_4) = 0$ as $\mu + \alpha_3 + \alpha_4 \in D$. This shows the claim and hence we may assume that $\mu(\alpha_2^\vee) = 0$.

Fix $k = 3$ or 4. If $\mu(\alpha_3^\vee) > 0$ then $\mu + \alpha_4 \in D$ so we may assume that $\mu(\alpha_i^\vee) = 0$ for $1 < i < k$. Set

$$\beta_k := \sum_{i=2}^{k} \alpha_i .$$

Then β_k is a short root and $\mathrm{cht}\,(\beta_k + \mu) = 0$ by Corollary 4.4.2. Also, by our assumptions on μ, $(\beta_k + \mu)^\star = (\alpha_k + \mu)^\star$ by Proposition 4.3.1 (iv). Hence

$$\alpha_k + \mu \;<\; \beta_k + \mu \leq (\alpha_k + \mu)^\star$$

and we have (using $\delta_{k,4}$ to denote the terms that occur only when $k = 4$):

$$\begin{aligned}
\mathrm{cht}\,(\alpha_k + \mu) &\leq |\alpha_k + \mu|^2 - |\beta_k + \mu|^2 \\
&= -\left[\,|\alpha_2|^2 + \delta_{k,4}|\alpha_3|^2 + 2(\alpha_2\,,\alpha_3) + 2\delta_{k,4}(\alpha_3\,,\alpha_4)\right] \\
&= -\Big[1 + 2\delta_{k,4} - 2 - 2\delta_{k,4}\Big] \\
&= 1\,.
\end{aligned}$$

□

4.5. Vanishing in Type A

As pointed out by S. Kumar, in type A, Frobenius splitting methods easily imply that for any regular dominant weight μ and any standard parabolic P we have $H^{i,\mathfrak{n}_\mathfrak{P}}(\mu) = 0$ for all $i > 0$. For completeness we outline this proof. The main reference for this whole section is [**2**].

Frobenius splitting.

We now introduce the main technical tool in this section. Fix a prime p; for the rest of this section we assume that all schemes are $\bar{\mathbb{F}}_p$-schemes unless otherwise specified, where $\mathbb{F}_p$ is the finite field with p elements and $\bar{\mathbb{F}}_p$ is its algebraic closure.

Let X be a scheme. We define a morphism F_X of schemes over $\mathbb{F}_p$ as follows. Set $F_X(x) = x$ for all $x \in X$ and define $F_X^{\#} : \mathcal{O}_X \to F_{X*}\mathcal{O}_X$ to be the p^{th} power map $f \mapsto f^p$; this is clearly an $\mathbb{F}_p$-linear map (although it is *not* an $\bar{\mathbb{F}}_p$-linear map). This morphism is called the **absolute Frobenius morphism**. Generally when the context is clear we'll drop the subscript and just write F. Note that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the sheaf $F_*\mathcal{F}$ on X is isomorphic to $\mathcal{F}$ as a sheaf of abelian groups, but the $\mathcal{O}_X$-module structure is twisted: for $m \in F_*\mathcal{F}$ and $f \in \mathcal{O}_X$ we have the twisted action $f * m = f^p\, m$.

Definition 4.5.1. We say that X is **Frobenius split** if there is an $\mathcal{O}_X$-linear map $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^{\#}$ is the identity map on $\mathcal{O}_X$.

We will black-box the proof of the next result, which is the essential result we require from the theory of Frobenius splitting.

Proposition 4.5.2. *([**2**], Lemma 1.2.7) Let X be a Frobenius split scheme and let $\mathcal{L}$ be an invertible sheaf on X. Then for all $i \geq 0$ there is an injection*

$$H^i(X,\, \mathcal{L}) \hookrightarrow H^i(X,\, \mathcal{L}^p)$$

(as $\mathbb{F}_p$-vector spaces). In particular, if $H^i(X,\, \mathcal{L}^n) = 0$ for all $n \gg 0$ then $H^i(X,\, \mathcal{L}) = 0$ also.

Corollary 4.5.3. *([**2**], Theorem 1.2.8) Assume that there is a proper morphism from X to an affine variety. Let $\mathcal{L}$ be an ample invertible sheaf on a Frobenius split variety X. Then $H^i(X,\, \mathcal{L}) = 0$ for all $i > 0$.*

Application to cohomology of cotangent bundles of flag varieties.

We now apply Frobenius splitting methods to flag varieties in type A. The following result is due to Mehta and van der Kallen, cf [**26**], Theorem 3.8 and [**2**], Exercise 5.1.E.6.

Theorem 4.5.4. *Choose any prime p. Let $\mathcal{G}$ be a semisimple algebraic group over $\bar{\mathbb{F}}_p$. Let $\mathcal{X}$ be the full flag variety $\mathcal{G}/\mathcal{B}$ of $\mathcal{G}$. If all components of $\mathcal{G}$ are of type A, then the bundles $\mathcal{G} \times^{\mathcal{B}} (\mathfrak{n}_{\mathfrak{p}})_{\bar{p}}$ on $\mathcal{X}$ are Frobenius split, where $(\mathfrak{n}_{\mathfrak{p}})_{\bar{p}}$ is the nilradical of any standard parabolic subgroup of Lie$(\mathcal{G}) =: \mathfrak{g}_{\bar{p}}$.*

We now come to the main result, which is a modification of Theorem 5.2.11 in [**2**].

Theorem 4.5.5. *Let G be a semisimple algebraic group over $\mathbb{C}$ with all components of type A. For every standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ let $p_{\mathfrak{p}} : G \times^B \mathfrak{n}_{\mathfrak{p}} \to G/B$ be the bundle map. Then*

$$H^i\left(G \times^B \mathfrak{n}_{\mathfrak{p}},\, p_{\mathfrak{p}}^* \mathcal{L}_B(\lambda)\right) = 0$$

for all regular $\lambda \in D$ and $i > 0$. Hence

$$H^i\big(T^*(G/P),\, p^*\mathcal{L}_P(\lambda)\big) = 0$$

for all $i > 0$ and regular $\lambda \in D$ (recall that $p : T^*(G/P) \to G/P$ is the bundle map).

Proof. For the moment, assume that G is a semisimple algebraic group over $\bar{\mathbb{F}}_p$ for any prime p with all components of type A. Consider the inclusion

$$G \times^B \mathfrak{n}_\mathfrak{p} \hookrightarrow G \times^B \mathfrak{g} \cong G/B \times \mathfrak{g}\,.$$

Let $q : G/B \times \mathfrak{g} \to G/B$ be projection onto the first coordinate. Since $\mathfrak{g}$ is affine, the bundle $q^*\mathcal{L}_B\,(\lambda)$ is ample on $G/B \times \mathfrak{g}$, since $\mathcal{L}_B\,(\lambda)$ is ample on G/B. Note that the restriction of $q^*\mathcal{L}_B\,(\lambda)$ to $G \times^B \mathfrak{n}_\mathfrak{p}$ is $p_\mathfrak{p}^*\,\mathcal{L}_B\,(\lambda)$; hence $p_\mathfrak{p}^*\,\mathcal{L}_B\,(\lambda)$ is ample on $G \times^B \mathfrak{n}_\mathfrak{p}$.

Note that there is a proper morphism $G \times^B \mathfrak{n}_\mathfrak{p} \to \mathfrak{g}$ given by

$$G \times^B \mathfrak{n}_\mathfrak{p} \;\hookrightarrow\; G/B \times \mathfrak{g} \;\twoheadrightarrow\; \mathfrak{g}\,.$$

By Theorem 4.5.4 and Corollary 4.5.3, we have that

$$H^i\left(G \times^B \mathfrak{n}_\mathfrak{p},\, p_\mathfrak{p}^*\,\mathcal{L}_B\,(\lambda)\right) = 0$$

for all $i > 0$. The result in characteristic 0 now follows from base change, cf [**2**], section 1.6.

$\square$

CHAPTER 5

Examples

In this section we explicitly compute some examples to illustrate Theorem 3.2.1. All three examples will be generalized BK-filtrations on L-highest weight subsets of $V_0(\mu)$ for some $\mu \in D$; we will always consider the 0 weight space since it doesn't make the examples any more interesting to consider other weight spaces.

5.1. Example 1

Let G be of type A_3. Recall our method of constructing orbits from Section 2.3. Using the methods of that section one can easily check that there are 5 partitions of $n = 4$ and hence 5 nilpotent orbits in G. All but one are even; the non-even orbit corresponds to the partition $[2, 1, 1]$. Recall that we set $\pi = \{\alpha_1, \alpha_2, \alpha_3\}$.

Let's pick the nilpotent orbit corresponding to the partition $[3, 1]$. From Section 2.3 we see that $X := X_{\alpha_1} + X_{\alpha_2}$ is an orbit representative. However, one may check (cf [**8**]) that a semisimple element H in an sl_2 triple containing X is given by $H = 2\chi_1^\vee + 2\chi_2^\vee - 2\chi_3^\vee$, where the $\chi_i^\vee$ are fundamental coweights. As H is not dominant we should conjugate the triple to obtain a representative of our nilpotent orbit in good position.

Choose a representative $\dot{r}_{\alpha_3} \in N(T)$ of $r_{\alpha_3} \in W$. We may choose $\dot{r}_{\alpha_3}$ so that $\dot{r}_{\alpha_3}(X) = X_{\alpha_1} + X_{\alpha_2+\alpha_3}$. Then $H' := \dot{r}_{\alpha_3}(H) = 2\chi_1^\vee + 2\chi_3^\vee$, which is dominant. Set $Z := \dot{r}_{\alpha_3}(X)$; then Z is in good position.

Now, considering H', we see that the standard parabolic P corresponding to Z is the parabolic corresponding to $\{\alpha_2\} \subseteq \pi$. Explicitly, on the Lie algebra level, we have that the Levi of $\mathfrak{p} = \mathrm{Lie}(P)$ is

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}$$

and the nilradical of $\mathfrak{p}$ is the $\mathfrak{b}$-subalgebra of $\mathfrak{n}$ with weights

$$\{\alpha_1,\, \alpha_3,\, \alpha_1 + \alpha_2,\, \alpha_2 + \alpha_3,\, 2\rho = \alpha_1 + \alpha_2 + \alpha_3\}\,.$$

Now set $\mu := \chi_2 + 2\chi_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \in D$ and consider the weight-0 subspace $\mathcal{W}_0^P(\mu) \subseteq V_0(\mu) \subseteq V(\mu)$. Recall that we have the tangent bundle $p : T^*(G/P) \to G/P$ of G/P. By Theorem 4.2.4 we have

$$H^i\big(T^*(G/P),\, p^*\mathcal{L}_P(\mathbb{C}_0)\big) \;\cong\; H^i\left(T^*(G/P),\, \mathcal{O}_{T^*(G/P)}\right) = 0$$

for all $i > 0$. Thus, by Theorem 3.2.1, the jump polynomial for the generalized BK-filtration on $\mathcal{W}_0^P(\mu)$ is given by the P-analog Kazhdan-Lusztig polynomial $m_\mu^{P,0}(q)$. We now compute this polynomial.

Recall that

$$m_\mu^{P,0}(q) = \sum_{w \in W} p_q^P(w * \mu)\,,$$

where $p_q^P(\lambda)$ is the coefficient of e^λ in

$$\prod_{\beta \in \Delta^+ \setminus \Delta_P^+} (1 - qe^\beta)^{-1}.$$

Recall too that $p_q^P(\lambda)$ counts the number of ways of writing λ as a sum of roots from $\Delta^+ \setminus \Delta_P^+$.

Note that $p_q^P(\lambda) = 0$ whenever λ is not in the $\mathbb{Z}^+$-span of $\Delta^+ \setminus \Delta_P^+$ (the weights of $\mathfrak{n}_\mathfrak{p}$). Hence we only need to restrict our attention to $w \in W$ such that $w * \mu$ is in this

$\mathbb{Z}^+$-span. A quick computation verifies that the only W $*$-translates of μ that qualify are μ, $r_{\alpha_1} * \mu = 2\alpha_2 + 2\alpha_3$, and $r_{\alpha_2} * \mu = \alpha_1 + 2\alpha_3$.

We now compute $p_q^P(\mu)$, $p_q^P(r_{\alpha_1} * \mu)$, and $p_q^P(r_{\alpha_2} * \mu)$. We can write

$$\mu = (\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_2 + \alpha_3) \;=\; (\alpha_1) + 2(\alpha_2 + \alpha_3) \;=\; (\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3)\,.$$

Hence $p_q^P(\mu) = 2q^2 + q^3$. Similarly, we can write $2\alpha_2 + 2\alpha_3 = 2(\alpha_2 + \alpha_3)$ and this is the only way; hence $p_q^P(r_{\alpha_1} * \mu) = q^2$. Finally, there is one way of writing $\alpha_1 + 2\alpha_3$, and $p_q^P(r_{\alpha_2} * \mu) = q^3$. Thus

$$m_\mu^{P,0}(q) = p_q^P(\mu) - p_q^P(r_{\alpha_1} * \mu) - p_q^P(r_{\alpha_2} * \mu) = q^2\,.$$

This says that $\mathcal{W}_0^P(\mu)$ is 1-dimensional and that for any nonzero $v \in \mathcal{W}_0^P(\mu)$ we have $Z^3.v = 0$ and $Z^2.v \neq 0$.

5.2. Example 2

Let G be of type G_2. Let α denote the short simple root and β the long simple root. We have

$$\Delta^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

and

$$\begin{aligned} W \;&=\; \{e,\, r_\alpha,\, r_\beta,\, r_\alpha r_\beta,\, r_\beta r_\alpha,\, r_\alpha r_\beta r_\alpha,\, r_\beta r_\alpha r_\beta,\, r_\alpha r_\beta r_\alpha r_\beta, \\ &\qquad r_\beta r_\alpha r_\beta r_\alpha,\, r_\alpha r_\beta r_\alpha r_\beta r_\alpha,\, r_\beta r_\alpha r_\beta r_\alpha r_\beta,\, r_\alpha r_\beta r_\alpha r_\beta r_\alpha r_\beta\}\,. \end{aligned}$$

Recall that the **subregular** nilpotent orbit $\mathfrak{O}_{subreg}$ in $\mathfrak{g}$ is the orbit that is dense in $\mathcal{N} \setminus \mathfrak{O}_{reg}$, where $\mathfrak{O}_{reg}$ is the regular nilpotent orbit. By Example 8.2.13 in [**8**], the subregular orbit in G is even and the corresponding weighted Dynkin diagram has 2 on the node corresponding to α and 0 on the node corresponding to β. Equivalently, set $H := 2\chi_\alpha \in \mathfrak{h}$; then there is an sl_2-triple $\{X, Y, H\}$ with $X \in \mathfrak{O}_{subreg} \cap \mathfrak{n}_\mathfrak{p}$, and X is clearly in good position.

The corresponding parabolic P is the minimal parabolic corresponding to the long simple root β with Levi factor and unipotent radical (on the Lie algebra level) given by

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}$$

and

$$\mathfrak{n}_\mathfrak{p} = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_L^+} \mathfrak{g}_\beta \,,$$

respectively, where

$$\Delta^+ \setminus \Delta_L^+ = \{\alpha,\, \alpha+\beta,\, 2\alpha+\beta, 3\alpha+\beta,\, 3\alpha+2\beta\} \,.$$

Consider the adjoint representation $V := V(\mu)$ of G where $\mu = 3\alpha + 2\beta = \chi_\beta$. We compute the generalized BK-filtration on the subspace $\mathcal{W}_0^P(\mu) \subseteq V_0(\mu) \subseteq V(\mu)$, i.e. we compute $m_\mu^{P,0}(q)$. Hence we must compute $p_q^P(w * \mu - 0) = p_q^P(w * \mu)$ for each $w \in W$. One verifies easily that for $w \in W$, $w * \mu \in \mathbb{Z}^+(\Delta^+ \setminus \Delta_L^+)$ iff $w \in \{e, r_\alpha, r_\beta\}$. Explicitly, we have $e * \mu = \mu$, $r_\alpha * \mu = \mu - \alpha = 2(\alpha+\beta)$, and $r_\beta * \mu = 3\alpha$.

Now, we can write $\mu = 3\alpha + 2\beta = \alpha + 2(\alpha+\beta) = (\alpha+\beta) + (2\alpha+\beta)$. Hence

$$p_q^P(e * \mu) = q + q^2 + q^3 \,.$$

The only way we can write $2\alpha + 2\beta$ as a positive sum of roots of $\Delta^+ \setminus \Delta_P^+$ is as $2(\alpha+\beta)$ and thus

$$p_q^P(r_\alpha * \mu) = q^2 \,.$$

Similarly, there is only one way to write 3α and we have $p_q^P(r_\beta * \mu) = q^3$. Hence we obtain

$$m_\mu^{P,0}(q) = q + q^2 + q^3 - q^2 - q^3 = q\,.$$

5.3. Example 3

Let G be of type C_3. Set $\pi = \{\alpha_1, \alpha_2, \beta\}$ where β is the unique long simple root. We have

$$\begin{aligned}\Delta^+ &= \{\alpha_1,\, \alpha_2,\, \alpha_1+\alpha_2,\, \beta,\, \alpha_2+\beta,\, \alpha_1+\alpha_2+\beta, \\ &\qquad 2\alpha_2+\beta,\, \alpha_1+2\alpha_2+\beta,\, 2\alpha_1+2\alpha_2+\beta\}\,.\end{aligned}$$

One may verify (see chapter 5 of [**8**]) that

$$X := X_{\alpha_1} + X_{2\alpha_2+\beta} + X_\beta$$

is a standard even nilpotent and that $H = 2\chi_{\alpha_1} + 2\chi_\beta$ is the unique standard semisimple element in an sl_2-triple containing X. Hence the weighted Dynkin diagram of X gives weight 2 to the nodes corresponding to α_1 and β, and weight 0 to the node corresponding to α_2. The associated parabolic P is defined by $\pi_P = \{\alpha_1, \beta\}$; we have $\Delta_P^+ = \{\alpha_1, \beta\}$ and the weights of the nilradical $\mathfrak{n}_\mathfrak{p}$ of $\mathfrak{p}$ are

$$\Delta^+ \setminus \Delta_P^+ = \{\alpha_2,\, \alpha_1+\alpha_2,\, \alpha_2+\beta,\, \alpha_1+\alpha_2+\beta, 2\alpha_2+\beta,\, \alpha_1+2\alpha_2+\beta,\, 2\alpha_1+2\alpha_2+\beta\}\,.$$

Set

$$\mu = 2\chi_{\alpha_1} = 2\alpha_1 + 2\alpha_2 + \beta\,,$$

the highest root in Δ^+, and consider the adjoint representation $V(\mu)$ of G. We consider the generalized BK-filtration on $\mathcal{W}_0^P(\mu)$ corresponding to X. It is straightforward to verify that for $w \in W$, $w * \mu \in \mathbb{Z}^+(\Delta^+ \setminus \Delta_P^+)$ iff $w \in \{e,\, r_\beta\}$.

Note that we can write $\mu = (2\alpha_1 + 2\alpha_2 + \beta) = (\alpha_1 + \alpha_2 + \beta) + (\alpha_1 + \alpha_2)$. Thus $p_q^P(e * \mu) = p_q^P(\mu) = q + q^2$. We have $r_\beta * \mu = 2\alpha_1 + 2\alpha_2$ and the only way of writing this is as $2(\alpha_1 + \alpha_2)$. Thus $p_q^P(r_\beta * \mu) = q^2$ and we obtain $m_\mu^{P,0}(q) = q$.

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