A Lower Bound for Immersions of Real Grassmannians

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Abstract

The cohomology of the real Grassmann and flag manifolds is discussed at length, making use of Stiefel-Whitney classes. It is shown that the tangent bundle of the Grassmannian splits into line bundles over the flag manifold. Additionally, it is found that the cohomology of the flag manifold is exactly the polynomial algebra $\mathbb{Z}_2[e_1, \ldots, e_n]$, for one-dimensional classes e_i, modulo the relation $\prod_i (1 + e_i) = 1$. Using facts about this ring to compute the dimension of the Stiefel-Whitney class of the normal bundle to the Grassmannian, we find a lower bound for immersions of certain real Grassmannians. In particular, $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ with $n \leq 2^s \leq k$ and $n+k \leq 2^{1+s}$ cannot be immersed in dimension less than $n\bigl(2^{s+1} - 1\bigr)$.

1 Introduction

An immersion is a smooth map from one manifold into another whose derivative is injective at each point in the domain. Immersions are an important topic of study in the field of geometric topology, so it is natural to seek results on their existence or nonexistence. Here we consider immersions of the Grassmann manifolds $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ into $\mathbb{R}^l$.

A significant amount of background material is provided here for the reader who is not familiar with details of the spaces involved or the notion of characteristic classes. This material is derived from Milnor and Stasheff [4] and comprises much of sections 1–3.

The main result of this paper is Theorem 5.2 showing that $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ where n and k satisfy $n \leq 2^s \leq k$ and $n + k \leq 2^{1+s}$ cannot be immersed in dimension less than $n\bigl(2^{s+1} - 1\bigr)$. This result is proved in [3], and the work here follows that paper closely (although it should be noted that Hiller and Stong write $\mathbf{G}_k\bigl(\mathbb{R}^{k+n}\bigr)$ rather than $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$; the latter is used here to maintain consistency with Milnor and Stasheff). We begin with a few standard results. First, the tangent bundle of $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ is isomorphic to $\hom\bigl(E, E^{\perp}\bigr)$, where E is the universal bundle whose fiber at a plane in the Grassmannian is that plane. Next, the immersion dimension of a manifold in $\mathbb{R}^l$ must be at least the dimension of the manifold plus the dimension of the Stiefel-Whitney class of its normal bundle, which is the multiplicative inverse of the Stiefel-Whitney class of the tangent bundle. Last is that $w(T)w(E \otimes E) = w(E)^{n+k}$, where $T = T\bigl(\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)\bigr)$ is the tangent bundle of the Grassmannian.

Along the way we will use the flag manifold $\mathrm{Flag}(\mathbb{R}^n)$, whose elements are strictly increasing sequences of linear subspaces of $\mathbb{R}^n$. The relevant property is that the universal bundle E pulls back to a direct sum of line bundles under the projection $\mathrm{Flag}\bigl(\mathbb{R}^{n+k}\bigr) \to \mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ sending a flag to its n-dimensional subspace. Of general interest are Theorem 4.1 indicating a presentation of the flag manifold's cohomology (a well-known result), and Theorem 4.3 elaborating its structure (less well-known).

2 The Grassmann manifold

The Grassmann manifold, or Grassmannian, of dimension n over $\mathbb{R}^{n+k}$, is a manifold whose elements are the linear subspaces of $\mathbb{R}^{n+k}$ of dimension n. It is denoted $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$.

2.1 Topology

To produce the topology on $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$, it is most convenient to define it as a quotient space.

Definition 2.1. The Stiefel manifold[1] $\mathbf{V}_n\left(\mathbb{R}^{n+k}\right)$ is the space of all linearly independent ordered sets ("n-frames") of n vectors in R^{n+k}. It is topologized as a subspace of $\mathbb{R}^{n(n+k)} = M(n+k, n)$, consisting of those matrices all of whose $n \times n$ minors have nonzero determinant.

$\mathbf{V}_n\left(\mathbb{R}^{n+k}\right)$ is the preimage of $(\mathbb{R} \setminus \{0\})^{\binom{n+k}{n}}$ under the function sending a matrix to a list containing for each choice of n of $n+k$ columns the determinant of those columns. That function is continuous (in fact, it is a polynomial), so $\mathbf{V}_n\left(\mathbb{R}^{n+k}\right)$ is an open subset of $\mathbb{R}^{n(n+k)}$.

Definition 2.2. The Grassmannian $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ is the set of n-dimensional linear subspaces of $\mathbb{R}^{n+k}$, with the quotient topology derived from the map from $\mathbf{V}_n\left(\mathbb{R}^{n+k}\right)$ to $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ sending each set of vectors to the space it spans.

For convenience, we set $\mathbf{G} = \mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ and $\mathbf{V} = \mathbf{V}_n\left(\mathbb{R}^{n+k}\right)$ for the remainder of this section.

It is clear from this definition that the spaces $\mathbf{G}_1\left(\mathbb{R}^{1+k}\right)$ are identical to $\mathbb{RP}^k$, as $\mathbf{V}_1\left(\mathbb{R}^{1+k}\right)$ is just $\mathbb{R}^{1+k} \setminus \{1\}$, and 1-dimensional subspaces are lines.

Theorem 2.1. $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ *is compact and Hausdorff.*

Proof. The quotient map $\mathbf{V} \to \mathbf{G}$ factors through a map from $\mathbf{V}$ to the set $\mathbf{V}_n^0\left(\mathbb{R}^{n+k}\right)$ (or $\mathbf{V}_0$) of *orthonormal* n-frames given by the Gram-Schmidt procedure. Thus $\mathbf{G}$ can be represented as a topological quotient of $\mathbf{V}_0$, with the subspace topology inherited from $\left(S^{(n+k)-1}\right)^n$. As that space is compact and $\mathbf{V}_0$ is closed in it (being the preimage of $I \in M(n, \mathbb{R})$ under the map $A \mapsto A^T A$), $\mathbf{G}$ is compact.

To show that $\mathbf{G}$ is Hausdorff, it suffices to find a continuous function separating two given elements X, Y in $\mathbf{G}$. Let $w \in X \setminus Y$, and define $f(Z)$ to be the square of the distance from w to $Z \in \mathbf{G}$ (in the Euclidean metric on $\mathbb{R}^{n+k}$). Thus

$$\begin{aligned} f(Z) &= w \cdot w - \pi_Z(w) \cdot \pi_Z(w) \\ &= w \cdot w + \sum_i (z_i \cdot w)^2, \end{aligned}$$

[1] This definition is not universal: many sources define the Stiefel manifold as the space of orthonormal n-frames.

where π_Z is projection onto Z and $\{z_i\}_{i=1}^n$ is any orthonormal basis for Z. Clearly $f(X) = 0 < f(Y)$, and the above formula shows that the induced map is continuous on $\mathbf{V}_0$ and hence f is continuous on $\mathbf{G}$. $\square$

2.2 Topological manifold structure

We will now demonstrate that any point $X \in \mathbf{G}$ has a neighborhood

$$U = \{ Y \in \mathbf{G} \mid Y \cap X^\perp = \{0\} \},$$

which is homeomorphic to the real vector space $\hom(X, X^\perp)$. This proves that $\mathbf{G}$ is a topological manifold of dimension

$$\dim(\hom(X, X^\perp)) = \dim X \cdot \dim(X^\perp) = nk.$$

Let $Y \in U$ for the neighborhood U described above. Thus

$$Y \subset \mathbb{R}^{n+k} = X \oplus X^\perp.$$

Define orthogonal projections $p\colon \mathbb{R}^{n+k} \to X$, $q\colon \mathbb{R}^{n+k} \to X^\perp$. Thus $p(Y) = X$, and since $\dim Y = \dim X$, $p|_Y$ is a linear isomorphism. Then the map

$$q \circ (p|_Y)^{-1}\colon X \to X^\perp$$

is a function in $\hom(X, X^\perp)$ corresponding to Y. Conversely, such a function yields Y, which is its graph: given $f\colon X \to X^\perp$,

$$Y = \{ x + f(x) \mid x \in X \}.$$

The resulting map $Y \to \hom(X, X^\perp)$ is continuous: choose a frame $\{x_i\}_{i=1}^n$ for X (that is, an element in $\mathbf{V}$ that maps to X). For any $Y \in U$, $(p|_Y)^{-1}$ sends the chosen frame to a frame on Y, yielding an injective map λ from U to $\mathbf{V}$ with inverse given by the canonical projection back to $\mathbf{G}$. This map can be shown to be continuous through explicit calculation (noting that λ is continuous if and only if the induced map $\mathbf{V} \to \mathbf{V}$ is continuous). Given any set of n vectors $\{v_i\}_{i=1}^n$, there is a linear map $X \to \mathbb{R}^{n+k}$ sending x_i to v_i. Clearly the assignment of linear maps to sets of vectors is linear in v_i, and so when we restrict to $\operatorname{Im}(\lambda) \subset \mathbf{V}$ we obtain a linear map—call it μ. Finally, there is a continuous function from $\hom(X, \mathbb{R}^{n+k})$ to $\hom(X, X^\perp)$ given by composition with q, the projection onto $X^\perp$. Thus Φ is given by the composition

$$U \xrightarrow{\lambda} \mathbf{V} \xrightarrow{\mu} \hom(X, \mathbb{R}^{n+k}) \xrightarrow{q\circ} \hom(X, X^\perp)$$

and is continuous since each component is continuous.

The inverse map from $\hom(X, X^\perp)$ to U is also continuous, as choosing a frame $\{x_i\}_{i=1}^n$ for X yields a frame $\{y_i\}$ for Y which depends continuously on a function $f\colon X \to X^\perp$, with $y_i = x_i + f(x_i)$. Thus we have constructed a homeomorphism Φ from U to $\hom(X, X^\perp)$.

2.3 Smooth structure and tangent bundle

To show that $\mathbf{G}$ is a smooth manifold, it suffices to show that the maps Φ obtained at each point in $\mathbf{G}$ are smoothly compatible. Consider X and U defined as before, and let $X' \in U$. X' has a neighborhood U', with a homeomorphism $\Phi' \colon U' \to \hom\big(X', X'^{\perp}\big)$. A straightforward computation shows that Φ and Φ' are smoothly compatible, that is,

$$\Phi' \circ \Phi^{-1} \colon \Phi(U \cap U') \to \Phi'(U \cap U')$$

is smooth.

Definition 2.3. The universal bundle E on $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ is the subbundle of the trivial bundle $\mathbb{R}^{n+k}$ obtained by taking at each point in $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ the subspace it corresponds to.

We also define $E^{\perp} \simeq \mathbb{R}^{nk}/E$ to be the subbundle of $\mathbb{R}^{n+k}$ obtained by taking the vector space $X^{\perp}$ to be the fiber over $X \in \mathbf{G}$.

Then, because the fiber of E over X is X and the fiber of $E^{\perp}$ over X is $X^{\perp}$, the tangent space of $\mathbf{G}$ is given by

$$T \simeq \hom\big(E, E^{\perp}\big) \tag{1}$$

as a vector bundle.

3 Stiefel-Whitney Classes

3.1 Definition and basic properties

The Stiefel-Whitney class $w(V)$ of a finite rank real vector bundle V over a space X is an element of $H^*(X;\mathbb{Z}_2)$. We write $w(V) = \bigoplus_{i\in\mathbb{N}} w_i(V)$, where $w_i(V) \in H^i(X;\mathbb{Z}_2)$. The operation w uniquely satisfies:

1. Normalization.
$$w\big(\gamma_1^1\big) = 1 + a$$
Where γ_1^1 is the tautological line bundle over $\mathbb{RP}^1$ and a is the generator of $H^1\big(\mathbb{RP}^1;\mathbb{Z}_2\big)$, so that $H^*\big(\mathbb{RP}^1;\mathbb{Z}_2\big) = \mathbb{Z}_2[a]/\big(a^2\big)$.

2. Rank.
$$\begin{aligned} &w_0(V) = 1 \in H^0(X;\mathbb{Z}_2) \\ &w_i(V) = 0 \in H^i(X;\mathbb{Z}_2), \quad \text{if } \ i > \operatorname{rank}(V). \end{aligned}$$

3. Whitney product formula.
$$w(V \oplus W) = w(V) \smile w(W)$$

4. Naturality.
$$w(f^*V) = f^*w(V)$$
for f^* the pullback by a continuous map $f\colon X' \to X$.

Proofs of the existence and uniqueness of the Stiefel-Whitney classes are given by Milnor and Stasheff [4].

It is clear from the last axiom that given a homeomorphism of spaces, the induced isomorphism in cohomology fixes Stiefel-Whitney classes. Additionally we see that the trivial bundle $\mathbb{R}^n$ over a space has Stiefel-Whitney class zero, since it pulls back from the trivial bundle on a point, which has trivial cohomology.

In the case of Grassmannians, it can be shown that the cohomology ring is generated by the Stiefel-Whitney classes $w_i(E)$, $0 < i \leq n$, of the universal bundle. We refer to again Milnor and Stasheff for a proof [4, p. 83].

3.2 Computation for $T(\mathbb{RP}^n)$

For the real projective space $\mathbb{RP}^n = \mathbf{G}_1\big(\mathbb{R}^{n+1}\big)$, the universal bundle has dimension one and hence has just one nonzero Stiefel-Whitney class, $w_1(E) = a$. Thus the cohomology ring is a quotient of the ring $\mathbb{Z}_2[a]$. As the projective space must have a top-level cohomology class, but no classes of greater degree, its cohomology ring is $\mathbb{Z}_2[a]/\big(a^{n+1}\big)$ (this fact can be shown by more elementary means, but Stiefel-Whitney classes provide a very elegant proof). We will refer to the universal bundle E on $\mathbb{RP}^n$ as the tautological line bundle γ to match convention.

As an aside, it is simple to compute $w(\gamma)$ using only the knowledge of $H^*(\mathbb{RP}^n; \mathbb{Z}_2)$. Let the tautological bundle on $\mathbb{RP}^1$ be γ_1. The inclusion $\mathbb{R}^1 \to \mathbb{R}^n$ induces an inclusion $\iota\colon \mathbb{RP}^1 \to \mathbb{RP}^n$. ι^* pulls γ back to the γ_1, whose Stiefel-Whitney class we know from axiom 1. By naturality, $\iota^*(w(\gamma)) = w(\gamma_1) = 1 + a$, implying that $w_1(\gamma)$ is nonzero. Since $w(\gamma)$ has no terms of degree greater than one, as γ is a one-dimensional bundle, we have $w(\gamma) = 1 + a$.

We now complete our computation of $w(T(\mathbb{RP}^n))$, starting with equation 1:
$$T(\mathbb{RP}^n) \simeq \hom\big(\gamma, \gamma^\perp\big).$$
To simplify, we add the trivial bundle $\mathbb{R} = \hom(\gamma, \gamma)$ to each side, obtaining
$$\begin{aligned} T(\mathbb{RP}^n) \oplus \mathbb{R} &\simeq \hom\big(\gamma, \gamma^\perp \oplus \gamma\big) \\ &= \hom\big(\gamma, \mathbb{R}^{1+n}\big) \\ &= \hom(\gamma, \mathbb{R})^{\oplus 1+n}. \end{aligned}$$
We can further simplify using the fact that $\hom(\gamma, \mathbb{R}) = \gamma$, from Appendix A. Applying w and using the Whitney product formula,
$$w(T(\mathbb{RP}^n))w(\mathbb{R}) = w(\gamma)^{1+n}.$$
Since $w(\mathbb{R}) = 1$, we obtain
$$w(T(\mathbb{RP}^n)) = (1 + a)^{n+1}.$$

3.3 Consequences for immersions

Suppose we have a manifold M of dimension n and an immersion

$$f\colon M \to \mathbb{R}^{n+k}.$$

Then f^* sends $T\big(\mathbb{R}^{n+k}\big)$ to a vector bundle on M, which is necessarily trivial of dimension $n+k$ and contains $T(M)$ as a sub-bundle. Defining the normal bundle

$$N(M) = T(M)^{\perp}$$

as a sub-bundle of the trivial bundle $\mathbb{R}^{n+k}$, we have

$$N(M) \oplus T(M) = \mathbb{R}^{n+k}$$

and consequently

$$w(N(M))w(T(M)) = w\big(\mathbb{R}^{n+k}\big) = 1.$$

If we know the cohomology ring of M and the Stiefel-Whitney class of its tangent space, this places constraints on the dimension of spaces in which M is embedded. The constraints are due to the fact that the dimension of $N(M)$ is k, so that terms of order greater than k must be zero. If $k \geq n$, then there are no restrictions as the cohomology ring of M has no terms of order greater than k.

3.4 Stiefel-Whitney class inverses

The cohomology ring $H^*(M;\mathbb{Z}_2)$ of a space M is a graded algebra over $\mathbb{Z}_2$. We consider only spaces M with finite-dimensional cohomology. Let $A = H^*(M;\mathbb{Z}_2)$ and $A^i = H^i(M;\mathbb{Z}_2)$, so that

$$A = \bigoplus_{i=0}^{n} A^i.$$

Let π_i be projection onto A^i, that is, if $a = \sum_i a_i$ with $a_i \in A^i$, then $\pi_i(a) = a_i$. The Stiefel-Whitney classes occupy a subset W of A with constant coefficient equal to one,

$$W = \{\, w \in A \mid \pi_0(w) = 1 \,\}.$$

We will show that this set is a group under multiplication, and consequently that we can divide as well as multiply the Stiefel-Whitney classes of tangent bundles.

Define the index function $i\colon A \to [0,\infty]$ by

$$i(0) = \infty$$

$$i(a) = \min\{\, i \mid \pi_i(a) \neq 0 \,\}, \quad a \neq 0.$$

Clearly $i(a+b) \geq \min(i(a), i(b))$, and $i(a) > n$ only if $a = 0$. It is easy to show additionally that $i(ab) \geq i(a) + i(b)$.

To show that W is closed under multiplication, let $1 + v, 1 + w \in W$. Then

$$(1 + v)(1 + w) = 1 + v + w + vw$$

$$i((1 + v)(1 + w) - 1) \geq \min(i(v), i(w), i(v) + i(w)) \geq 1,$$

so $(1 + v)(1 + w)$ lies in W. W clearly contains the multiplicative identity 1, and we can demonstrate that it has inverses using the formula

$$\frac{1}{1 + x} = 1 + x + x^2 + x^3 + \cdots$$

where factors of -1 are omitted because A is a $\mathbb{Z}_2$-algebra. Since $i(x) \geq 1$, we have $i(x^k) \geq k$, so x^k is zero for k larger than the maximum grade n. Then

$$(1 + x)\left(1 + x + x^2 + \cdots + x^n\right) = 1 + x^{n+1} = 1.$$

Consequently W is a group under multiplication and the inverse formula given provides the unique inverse. This inverse is unique not only in W but in A because if $a(1 + w) = 1$ for $a \in A$, $1 + w \in W$, then $\pi_0(a)\pi_0(1 + w) = \pi_0(1)$, or $\pi_0(a) = 1$. For a Stiefel-Whitney class $w(V)$, we denote its multiplicative inverse by $\overline{w}(V)$.

3.5 Immersions of projective spaces

We combine the results of the two previous sections:

Theorem 3.1. *A lower bound for the immersion dimension of n-dimensional manifold M is $n + k$, where*

$$k = \dim w(N(M)) = \dim \overline{w}(T(M)).$$

We now specialize to the case of $\mathbb{RP}^n$, with $H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[a]/\left(a^{n+1}\right)$ and $w(T(\mathbb{RP}^n)) = (1 + a)^{n+1}$.

We note that in any algebra over $\mathbb{Z}_2$, $(a+b)^2 = a^2+b^2$, that is, squaring is a homomorphism. We conclude that $(1+a)^{2^l} = 1+a^{2^l}$, which is 1 in $\mathbb{Z}_2[a]/\left(a^{n+1}\right)$ if $2^l > n$. Therefore the inverse of $w(T(\mathbb{RP}^n)) = (1 + a)^{n+1}$ in $H^*(\mathbb{RP}^n; \mathbb{Z}_2)$ is $(1 + a)^{2^l-n-1}$, where $2^l > n$. This polynomial has the same value for all l that satisfy the given inequality by uniqueness of inverses. Its degree is computed most easily if we additionally require that $2^l \leq 2n$, so that $2^l - n - 1 < n$ is the degree. Consequently

Theorem 3.2. *The space $\mathbb{RP}^n$ cannot be immersed in dimension less than 2^l-1, where $2^{l-1} \leq n < 2^l$.*

4 The flag manifold

4.1 Definition

In working with the Grassmannian it is convenient to introduce another quotient of $\mathbf{V}_n^0(\mathbb{R}^n)$, the space of orthogonal n-frames in $\mathbb{R}^n$. Our intention is for the flag manifold, which we denote using $\mathrm{Flag}(\mathbb{R}^n)$, to consist of all complete flags in $\mathbb{R}^n$, that is, all strictly increasing sequences

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{R}^n.$$

As the dimensions of the V_i must be strictly increasing, it follows that $\dim V_i = i$. Noting that a flag is equivalent to an ordered set of orthogonal lines in $\mathbb{R}^n$ (so that V_i is the span of the first i lines), we let

Definition 4.1. The flag manifold $\mathrm{Flag}(\mathbb{R}^n)$ is the quotient of $\mathbf{V}_n^0(\mathbb{R}^n)$ by the relation setting $\{x_i\}_{i=1}^n \sim \{x_i'\}_{i=1}^n$ when $x_i = \pm x_i'$ for all i.

Thus the flag manifold is the quotient of the action on $\mathbf{V}_n^0(\mathbb{R}^n)$ of a discrete group $(\mathbb{Z}^\times)^n$, where $\mathbb{Z}^\times = \{-1, 1\}$ under multiplication (so $\mathbb{Z}^\times$ is cyclic of order 2). $\mathbf{V}_n^0(\mathbb{R}^n)$ is the submanifold of $\left(S^{n-1}\right)^n \subset M(n, \mathbb{R})$ satisfying the (smooth) relation $A^T A = I$, so $\mathrm{Flag}(\mathbb{R}^n)$ inherits its smooth manifold structure.

The flag manifold $\mathrm{Flag}\left(\mathbb{R}^{n+k}\right)$ projects to the Grassmannian $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ by sending a flag to its n-dimensional space.

4.2 Cohomology

Theorem 4.1. *The cohomology ring $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ is generated by n one-dimensional classes $e^1, e^2, \ldots, e^n$, which obey the relation*

$$\prod_{i=1}^n (1 + e_i) = 1.$$

Proof. The desired classes are found using the map $\pi\colon \mathrm{Flag}(\mathbb{R}^n) \to \left(\mathbb{RP}^{n-1}\right)^n$ which sends a flag to the orthogonal lines defining it. The cohomology ring of each copy of $\mathbb{RP}^{n-1}$ is generated by a single one-dimensional class, and we define e_i to be the pullback of this class in the i^{th} copy.

By naturality of Stiefel-Whitney classes, we have

$$e_i = \pi_i^*(w_1(\gamma)) = w_1(\pi_i^*(\gamma)),$$

where π_i is the projection of $\mathrm{Flag}(\mathbb{R}^n)$ onto the i^{th} component. We let $\gamma_i = \pi_i^*(\gamma)$. The bundle $\bigoplus_i \gamma_i$ is the trivial bundle $\mathbb{R}^n$, since the product of all the lines in a flag is always the entire space. The relation of the classes, $\prod_i (1 + e_i) = 1$ is then given by applying w. $\square$

To find nonzero elements in the cohomology of the flag space, we use the Leray-Hirsch theorem [1, p. 432]:

Theorem 4.2. *Let E be a fiber bundle over the space B with fiber F and projection map $p\colon E \to B$, and R be a commutative ring. Assume also:*

(a) *$H^n(F;R)$ is a finitely generated free R-module for each n.*

(b) *There exist classes $c_j \in H^{k_j}(E;R)$ such that $i^*(c_j)$ form a basis for $H^*(F;R)$ for each inclusion $i\colon F \to E$ sending F to a fiber.*

Then the map $\Phi\colon H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$, $\sum_{ij} b_i \otimes i^*(c_j) \mapsto \sum_{ij} c_j \smile p^*(b_i)$ is an isomorphism.

$\mathrm{Flag}(\mathbb{R}^n)$ may be viewed as a tower of fiber bundles where the fiber in each case is a projective space. We can obtain this tower by successively choosing subspaces of $\mathbb{R}^n$—first V_1, then V_2, and so on. Let F_k be the space of flags

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k$$

where $\dim V_k = k$. Clearly our choice for V_1 is any line in $\mathbb{R}^n$, so F_1 is equal to $\mathbb{RP}^{n-1}$, or equivalently a fiber bundle over a single point with fiber $\mathbb{RP}^{n-1}$. Once we have chosen the first $k-1$ subspaces, the choice of V_k is equivalent to picking a line in $\mathbb{R}^n/V_{k-1}$, which is isomorphic to $\mathbb{R}^{n-k+1}$, meaning that F_k is a fiber bundle over F_{k-1} with fiber $\mathbb{RP}^{n-k}$.

Given the fiber bundle $F_k \to F_{k-1}$, we verify the conditions for Leray-Hirsch (with $R = \mathbb{Z}_2$). Since the cohomology ring of the fiber, $H^*\big(\mathbb{RP}^{n-k};\mathbb{Z}_2\big)$, is equal to $\mathbb{Z}_2[a]/\big(a^{n-k+1}\big)$, each cohomology group $H^j\big(\mathbb{RP}^{n-k};\mathbb{Z}_2\big)$ is the free $\mathbb{Z}_2$-module generated by a^j. To generate the group, then (and satisfy condition b), it suffices to show that $i^*(e_k)$ is nonzero in $\mathbb{RP}^{n-k}$ (equivalently, it is equal to a). But this follows because $\pi \circ i$ is the natural inclusion induced by the map $\mathbb{R}^{n-k} \to \mathbb{R}^n$, so it sends the generator of $\mathbb{RP}^n$ to the generator of $\mathbb{RP}^{n-k}$.

This construction makes it clear that a set of elements which generate $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ by addition is given by the products $\prod_i e_i^{k_i}$ satisfying $k_i \leq n-1$ for all i. It happens that the relation $\prod_i(1+e_i) = 1$ allows us to write any polynomial in e_i in terms of such monomials, a fact which we do not prove (or require) in this paper.

4.3 Monomials in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$

Theorem 4.3. *In the cohomology of* $\mathrm{Flag}(\mathbb{R}^n)$*,*

(1) *$e_i^n = 0$ for all i.*

(2) *The top dimension is $\frac{1}{2}n(n-1)$. A monomial $\prod_i e_i^{k_i}$ of that dimension is nonzero if and only if the k_i are a permutation of the set $\{0, 1, \ldots, n-1\}$.*

Proof. (1) is clear simply by noting that $e_i = \pi_i^*(a)$, where a generates the cohomology of $\mathbb{RP}^{n-1}$. In the latter space, $a^n = 0$, so $e_i^n = \pi_i^*(a^n) = 0$.

To show (2), we begin with the above construction of the flag manifold as an iterated fiber bundle. This construction clearly indicates that

$$\dim(\mathrm{Flag}(\mathbb{R}^n)) = \sum_{k=1}^{n} \dim\big(\mathbb{RP}^{n-k}\big) = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}.$$

That the given monomial is nonzero is readily apparent from Leray-Hirsch. For simplicity, we go through the e_i in order. The top-dimensional component added at each step is e_i^{n-i}, so the product of this element and the previous nonzero component is the top-dimensional component in the new flag manifold F_i. Since the cohomology ring of F_0 (a point) is trivial, the term $\prod_{i=1}^n e_i^{n-i}$ is nonzero.

There is a map $\mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ for any permutation $\sigma \in S_n$ which sends a tuple of orthogonal lines to the same lines permuted by σ. Clearly such a map is a homeomorphism. However, this map also rearranges the cohomology classes by sending e_i to $e_{\sigma(i)}$. Consequently the monomial $\prod_i e_{\sigma(i)}^{n-i}$ for any $\sigma \in S_n$ is nonzero, or equivalently $\prod_i e_i^{k_i}$ is nonzero if the k_i are a rearrangement of the set $\{0, 1, \ldots, n-1\}$.

It remains to show that any monomial with dimension $\frac{1}{2}n(n-1)$ not of this form is equal to zero. We will call monomials fitting the form, that is, $\prod_i e_i^{k_i}$ with $\{k_i\} = \{0, 1, \ldots, n-1\}$, decreasing monomials. For an arbitrary top-dimensional monomial expressed as a product $\prod_i e_i^{k_i}$, let K_j, for $0 \le j \le n$, be the set of basis elements e_i for which the corresponding k_i are greater than or equal to j. Define $d_j = |K_j|$ and $p_j = \prod_{i \in K_j} e_i$, so that

$$P = \prod_{j=1}^{n} p_j.$$

Clearly $K_j \subset K_{j-1}$ for all j, so d_j form a non-increasing sequence. For a decreasing monomial, we have $d_j = n - j$, and the sequence (d_j) is strictly decreasing. In fact this property holds only for decreasing monomials: the total dimension of the monomial is

$$\sum_{j \ge 1} d_j = \sum_k \left(d_0 - \sum_{j=1}^{n} d_j - d_{j-1} \right),$$

where d_0 is fixed at n. The maximal value of this expression is attained only when $d_j - d_{j-1}$ is minimized, that is, when it is always equal to 1.

We will show that monomials which are not decreasing are zero by directly using the relation found in Theorem 4.1,

$$\prod_{i=1}^{n} (1 + e_i) = 1.$$

For $j > 0$, the degree j component of the left-hand side is the sum of all j-dimensional monomials which contain at most one copy of each e_i. Thus it contains $\binom{n}{j}$ components corresponding to the choices of j of the e_i. As the j^{th} component of the right-hand side is zero, that polynomial is zero in the cohomology ring of $\mathrm{Flag}(\mathbb{R}^n)$, or in other words a product of j distinct generators is equal to the sum of all other products of j distinct generators. As a special case, we note that the product $\prod_{i=1}^n e_i$ is zero.

$$
\begin{aligned}
\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \circ & \cdot \\ \circ & \circ & \circ & \cdot \end{bmatrix}
&= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \cdot & * \\ \circ & \circ & \circ & \cdot \end{bmatrix}
+ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & * \\ \circ & \circ & \circ & \cdot \end{bmatrix}
+ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \circ & * \\ \circ & \circ & \circ & \cdot \end{bmatrix} \\
&= \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot \\ \circ & \circ & \circ & \circ \end{bmatrix}
+ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot \\ \circ & \circ & \circ & \circ \end{bmatrix}
+ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \circ & \cdot \\ \circ & \circ & \circ & \circ \end{bmatrix}
\end{aligned}
$$

Figure 1: Rewriting a monomial as a sum of vanishing terms. The rows correspond to p_l with p_1 at the bottom, and the columns to variables e_i, with element (l, i) marked if $e_i \in K_l$. Each monomial in the final sum is zero since it contains e_i for each i.

Suppose we have a monomial $P = \prod_i e_i^{k_i}$ for which the first (that is, for minimal j) value of $d_j - d_{j-1}$ not equal to 1 is 0. We will demonstrate by induction that such monomials are zero. If $j = 1$, this is clear, as $d_1 = d_0 = n$ implies that the p_1 contains e_i for each $1 \le i \le n$, and thus is the product of all generators, which we have just noted is zero. Now suppose $j > 1$. By hypothesis $d_j = d_{j-1}$, so $K_j = K_{j-1}$ and $p_j = p_{j-1}$. Now rewrite p_j as the sum of all other products of d_j generators (this procedure is illustrated in figure 1). Consider a single summand P' of the resulting expression for P obtained by replacing p_j with $q_j = \prod_{e_a \in L_j} e_a$. Since L_j is distinct from $K_j = K_{j-1}$, some terms $e_a \in L_j$ are not in K_{j-1}. Consequently for each such term e_a, there is an $l_a < j - 1$ such that $e_a \in K_{l_a} \setminus K_{l_a - 1}$. The l_a are distinct, since the fact that $d_l - d_{l-1} = 1$ for $l < j$ implies that $K_{l_a} \setminus K_{l_a-1}$ has only one element, and the e_a are distinct. Then taking $l = \min_a l_a$, we have $d_k - d_{k-1} = 1$ for all $k < l$, while $d_l - d_{l-1} = 0$, and P' is zero by induction on j.

To complete the proof, consider the monomials for which the first value of $d_j - d_{j-1}$ not equal to 1 is greater than one. We will show that each such monomial is zero in two steps—first, by assuming that $d_j - d_{j-1}$ is exactly 2, and next, by showing that any monomial with a greater difference can be rewritten as a sum of monomials of the first kind. To simplify both proofs, we note a consequence of the result from last paragraph on the rewriting operation performed there: when rewriting a term p_l of P, we can ignore combinations not contained entirely in K_{j-1}. This is due to the fact that if any e_i in the result falls outside of that set, then the first $d_j - d_{j-1}$ not equal to 1 is equal to 0, so the resulting monomial P' is zero. We also define the function

$$h(P) = \sum_l l d_l,$$

a positive integer associated to any given monomial. If we choose k with $K_k = K_{k-1}$, and rewrite p_k, then each summand P' in the result must have lower h than P. This is because some term of L_k (where p_k is moved to $\prod_{e_a \in L_k} e_a$ must lie outside of $K_k = K_{k-1}$, so it moves to K_l with $l < k$. This exchange decreases

h by $k-l$. No term K_l with $l > k$ can increase in size when we rewrite, because any element of L_k in K_l merely fills the value occupied earlier by an element of K_k. Thus $h(P) > h(P')$.

Let $d_j - d_{j-1} = 2$, and $d_l - d_{l-1} = 1$ for $l < j$. Choose $k > j$ such that $d_k - d_{k-1} = 0$ (recall that we are guaranteed at least one such index because the monomial is not a decreasing one). When we rewrite p_k, the terms $\tilde{p}_k$ which lie entirely in K_{j-1} fall in one of the following categories:

- Two elements of $\tilde{p}_k$ lie in $K_j \setminus K_{j-1}$. In this case, the corresponding P' will have $K'_j = K'_{j-1}$ and be zero.
- Exactly one element lies in $K_j \setminus K_{j-1}$. In this case, there is another term which contains the other element and is otherwise equal. These elements are equal, as when we rewrite p_j in the first, the only combination contained in K_{j-1} is the p_j of the second. Thus they annihilate.
- All elements of $\tilde{p}_k$ lie in K_{j-1}. Here we use the fact that $h(P) > h(P')$ (noting that we rewrote a term $p_k = p_{k-1}$). As there is a minimum possible h, P' is zero by induction on h.

Now let $d_j - d_{j-1} > 2$. Again, we choose $k > j$ with $d_k - d_{k-1} = 0$, rewrite p_k, and split the terms contained in K_{j-1} into cases:

- The elements of $\tilde{p}_k$ cover $K_j \setminus K_{j-1}$. Again, we obtain $P' = 0$ as $K'_j = K'_{j-1}$.
- The elements of $\tilde{p}_k$ cover all but one of $K_j \setminus K_{j-1}$. Now we have $d_j - d_{j-1} = 1$, but $d'_{j+1} \leq d_j$, so $d'_{1+j} - d'_j \leq 2$. Thus P' is not a decreasing monomial, and $h(P) > h(P')$.
- The elements of $\tilde{p}_k$ leave at least two elements of $K_j \setminus K_{j-1}$ uncovered. Then P' is not a decreasing monomial, and $h(P') < h(P)$.

To summarize, each monomial which is non-decreasing is either zero or can be written as a sum of non-decreasing monomials with strictly smaller h value. Since h is a positive integer, all non-decreasing top-dimensional monomials are zero. □

Remark. The rewriting procedure described in the above proof can also be used to show explicitly that all decreasing monomials are equivalent. Let P be the decreasing monomial given by the permutation $\sigma \in S_n$, that is,

$$P = \prod_{i=1}^{n} e_{\sigma(i)}^{i-1}, \quad p_j = \prod_{i>j} e_{\sigma(i)}, \quad K_j = \left\{ e_{\sigma(i)} \mid i > j \right\}.$$

Then when we rewrite p_j, the only terms that give a decreasing monomial are those whose elements lie in K_{j-1} (as discussed above) but cover K_{j+1}. The latter condition arises because if an element of K_{j+1} is left out, that element moves from K_{j+1} to K_j, increasing d_j by one so that $d_j = d_{j-1}$. But it is evident that the only polynomial satisfying these conditions other than p_j is the one that exchanges $e_{\sigma(i)}$ for $e_{\sigma(i-1)}$. Since the exponents of those terms differ only by one, this exchange also exchanges the two terms in the full monomial, so that the new monomial is the one given by $\sigma \circ (j-1, j)$. Since S_n is generated by the pair-exchanging permutations $(j-1, j)$, we can obtain all the decreasing monomials using transformations of this form.

5 Immersions of Grassmannians

5.1 The Hsiang-Szczarba formula

This section follows closely the derivation in section 3.2. To compute the Stiefel-Whitney class of the tangent bundle T of $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$, recall (from equation 1) that

$$T \simeq \hom\big(E, E^{\perp}\big),$$

where E is the universal bundle. Adding $\hom(E, E)$ to both sides,

$$\begin{aligned} T \oplus \hom(E, E) &\simeq \hom\big(E, E^{\perp} \oplus E\big) \\ &= \hom\big(E, \mathbb{R}^{n+k}\big) \\ &= \hom(E, \mathbb{R})^{\oplus n+k}. \end{aligned}$$

Since $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ is compact and Hausdorff,

$$\hom(E, \mathbb{R}) = E^{*} \simeq E.$$

We then obtain, by taking Stiefel-Whitney classes of the earlier equation,

$$w(T)w(E \otimes E) = w(E)^{n+k}. \tag{2}$$

This result is sometimes referred to as the Hsiang-Szczarba formula.

We know that $w(E) = 1 + \sum_i x_i$, where x_i generate $H^{*}\big(\mathbf{G}_n\big(\mathbb{R}^{n+k}\big); \mathbb{Z}_2\big)$ as a ring. It remains to find $w(E \otimes E)$.

5.2 The splitting principle

The Stiefel-Whitney class of a tensor product is not readily apparent in the general case. However, there is a simple formula for line bundles: given line bundles L_1, L_2, with $w(L_1) = 1 + l_1$ and $w(L_2) = 1 + l_2$, we have

$$w(L_1 \otimes L_2) = 1 + l_1 + l_2.$$

This is equivalent to a proposition from Hatcher's *Vector Bundles and K Theory*, which states that the function $w_1 \colon \mathrm{Vect}^1(X) \to H^1(X; \mathbb{Z}_2)$ is a group homomorphism, where $\mathrm{Vect}^1(X)$ is the group of line bundles on X under the tensor product [2, p. 86].

For general vector bundles, the splitting principle provides a convenient way to calculate tensor products. The principle states that for any space X with vector bundle V, there is a space Y and map $p\colon Y \to X$ such that the induced homomorphism p^* on cohomology is injective and the vector bundle p^*V splits as a direct sum of line bundles $p^*(V) = \bigoplus_{i=1}^n V_i$. Then given a vector bundle V, the following procedure suffices to compute $w(V \otimes V)$: first select a space Y and map $p\colon X \to Y$ with the above properties and split p^*V into a sum of line bundles V_i. Compute $w(V_i) = 1 + v_i$. Then

$$\begin{aligned}
p^*(w(V \otimes V)) = w(p^*(V \otimes V)) &= w\left(\left(\bigoplus_i V_i\right) \otimes \left(\bigoplus_i V_i\right)\right) \\
&= w\left(\bigoplus_{ij} V_i \otimes V_j\right) \\
&= \prod_{ij} w(V_i \otimes V_j) \\
&= \prod_{ij} (1 + v_i + v_j).
\end{aligned}$$

Assuming we are working in $\mathbb{Z}_2$, we can simplify slightly, noting that $1+v_i+v_i = 1$ and $(1 + v_i + v_j)^2 = 1 + v_i^2 + v_j^2$. We obtain

$$p^*(w(V \otimes V)) = \prod_{i<j} \left(1 + v_i^2 + v_j^2\right).$$

Finally, we must find a cohomology class whose value under p^* is the computed result. Such a class will necessarily be equal to $w(V \otimes V)$ since p^* is injective.

We will not need to prove, or in fact use, the splitting principle in order to carry out this procedure, as it turns out we know a space over which the vector bundle E splits—it is the flag manifold discussed in section 4.

Theorem 5.1. *The map* $\Psi\colon \mathrm{Flag}\left(\mathbb{R}^{n+k}\right) \to \mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$ *sending a flag to the plane in it of dimension* n *splits the universal bundle into line bundles.*

Proof. Recall the map $\pi\colon \mathrm{Flag}\left(\mathbb{R}^{n+k}\right) \to \left(\mathbb{RP}^{(n+k)-1}\right)^{n+k}$ sending a flag to the tuple of orthogonal lines associated with it. Composing π with projection onto the i^{th} factor yields $n+k$ maps $\pi_i\colon \mathrm{Flag}\left(\mathbb{R}^{n+k}\right) \to \mathbb{RP}^{(n+k)-1}$. We claim that

$$\Psi^*(E) = \bigoplus_{i=1}^n \pi_i^*(\gamma)$$

where γ is the tautological line bundle on $\mathbb{RP}^{(n+k)-1}$. But this is clear: the fiber of $\Psi^*(E)$ at a flag containing the plane V_n is simply that plane, and the fiber of the direct sum is the direct sum of the lines $V_1, V_2/V_1, \ldots, V_n/V_{n-1}$, also equal to V_n.

We also need to demonstrate that Ψ^* is injective on cohomology, a fact we will show using the Leray-Hirsch theorem. The isomorphism given in the conclusion of that theorem automatically implies that p^* is injective on cohomology, since $p^*(b) = \Phi(b \otimes 1)$. Thus we need only find a sequence of fiber bundles with initial base space $\mathbf{G}_n(\mathbb{R}^{n+k})$ and final space $\mathrm{Flag}(\mathbb{R}^{n+k})$ each satisfying the conditions of Leray-Hirsch, and for which the resulting projection $\mathrm{Flag}(\mathbb{R}^{n+k}) \to \mathbf{G}_n(\mathbb{R}^{n+k})$ is Ψ. For the first such map, we recall the partial flag manifold F_n from the proof of Theorem 4.1. This manifold may be represented as a bundle over $\mathbf{G}_n(\mathbb{R}^{n+k})$ where p sends a partial flag to its n-dimensional component. The fiber $p^{-1}(V_n)$ for a plane $V_n \in \mathbf{G}_n(\mathbb{R}^{n+k})$ is the set of partial flags ending in V_n, a space which is isomorphic to $\mathrm{Flag}(\mathbb{R}^{n-1})$ via any isomorphism of V_n and $\mathbb{R}$. The remainder of the fiber bundles to get to $\mathrm{Flag}(\mathbb{R}^{n+k})$ are simply those constructed while proving Theorem 4.1, which we verified in the proof to satisfy Leray-Hirsch. Thus we need only show that the first fiber bundle satisfies those conditions. Condition (a) is immediate because $H^*(F_n; \mathbb{Z}_2)$ is a polynomial ring, and condition (b) follows by taking the classes e_i for $1 \leq i \leq n-1$ in F_n as generators for $\mathrm{Flag}(\mathbb{R}^{n-1})$. Clearly the total projection sends a flag in $\mathrm{Flag}(\mathbb{R}^{n+k})$ to its n-dimensional component, that is, it is exactly Ψ. Thus Ψ^* is injective on cohomology. □

5.3 An immersion bound for some Grassmannians

In this section we will combine the Hsiang-Szczarba formula with our earlier work on the flag manifold and some extensive computation to conclude:

Theorem 5.2. *Given n, k such that $n \leq 2^s \leq k$ and $n + k \leq 2^{s+1}$, $\mathbf{G}_n(\mathbb{R}^{n+k})$ cannot be immersed in dimension less than $n(2^{s+1} - 1)$.*

When $n = 1$ this result restricts to the bound of Theorem 3.2 on projective spaces. To prove it we begin with

Lemma 5.3. *In any $\mathbb{Z}_2$-algebra generated by one-dimensional classes e_i, $1 \leq i \leq n$, the identity*

$$\prod_{1 \leq i < j \leq n} (e_i + e_j) = s_{(n-1,n-2,\ldots,1)}(e_1, \ldots, e_n)$$

holds, where s is a monomial symmetric polynomial as defined in Appendix B.

Proof. For $n = 2$, the result is clear, as the left-hand side has only a single term $e_1 + e_2$ and the right-hand side is equal to $s_{(1)}(e_1, e_2) = e_1 + e_2$, the sum of the two monomials with degree one.

Now for a given n denote the left-hand side by A_n. Assume

$$A_n = s_{(n-1,n-2,\ldots,1)}(e_1, \ldots, e_n).$$

Then A_{n+1} is formed by multiplying A_n by all the terms containing e_{n+1}:

$$A_{n+1} = A_n \prod_{i=1}^{n} (e_i + e_{n+1}).$$

The 2^n terms in the product can be divided according to the number of times e_{n+1} appears in them. In each case, the remaining e_i in that term are distinct and taken from the set $\{1, \dots, n\}$. Thus we have

$$\prod_{i=1}^{n}(e_i + e_{n+1}) = \sum_{j=0}^{n} e_{n+1}^{j} s_{(1,\dots,1)}(e_1, \dots, e_n),$$

where the $s_{(1,\dots,1)}$ term on the right-hand side contains $n - i$ ones. In order to multiply this result by A_n, we move A_n inside the sum, and use Corollary B.2's statement that the product $s_{(1,\dots,1)}(e_1, \dots, e_n) \cdot s_{(n-1,n-2,\dots,1)}(e_1, \dots, e_n)$ is equal to $s_{(n,n-1,\dots,j+1,j-1,\dots,0)}(e_1, \dots, e_n)$. Thus

$$A_{n+1} = \sum_{j=0}^{n} e_{n+1}^{j} s_{(n,n-1,\dots,j+1,j-1,\dots,0)}(e_1, \dots, e_n).$$

But the sum on the right simply groups all of the monomial components of $s_{(n,n-1,n-2,1,\dots,0)}(e_1, \dots, e_{n+1})$ by the number of copies of e_{n+1} they contain, so it is equal to that polynomial. The lemma now follows by induction. □

Lemma 5.4. *$w_{n(n-1)}(E \otimes E)$ is nonzero in $H^*\big(\mathbf{G}_n\big(\mathbb{R}^{2n-1}\big); \mathbb{Z}_2\big)$.*

Proof. Since the map $\Psi\colon \mathrm{Flag}\big(\mathbb{R}^{2n-1}\big) \to \mathbf{G}_n\big(\mathbb{R}^{2n-1}\big)$ induces an injective map Ψ^* in cohomology, it suffices to show that the $n(n-1)$-dimensional term of

$$\Psi^*(w(E \otimes E)) = \prod_{1 \le i < j \le n} \Big(1 + (e_i + e_j)^2\Big),$$

where e_i are the generators of $H^*\big(\mathrm{Flag}\big(\mathbb{R}^{2n-1}\big); \mathbb{Z}_2\big)$, is nonzero. The number of terms in the product is $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$, so the term of dimension $n(n-1)$ includes no 1 terms of the product and is equal to

$$\begin{aligned}
\Psi^*\big(w_{n(n-1)}(E \otimes E)\big) &= \prod_{1 \le i < j \le n} (e_i + e_j)^2 \\
&= \big(s_{(n-1,n-2,\dots,1)}(e_1, \dots, e_n)\big)^2 \\
&= s_{(2n-2,2n-4,\dots,2)}(e_1, \dots, e_n)
\end{aligned}$$

(where for the last equality we recall that squaring is a homomorphism). To show that this term is nonzero, we multiply by the value

$$e = e_2 e_3^2 \cdots e_{n-1}^{n-2} e_n^{n-1} e_{n+1}^{n-2} \cdots e_{2n-2}.$$

One term in the product is

$$e \cdot e_1^{2n-2} e_2^{2n-4} \cdots e_n^2 = e_1^{2n-2} e_2^{2n-3} \cdots e_{2n-2}.$$

Because the product is homogeneous and this term has degree $\frac{1}{2}(2n-1)(2n-2)$, the dimension of $\mathrm{Flag}\big(\mathbb{R}^{2n-1}\big)$, we can apply the second fact from Theorem 4.3 to show that the other terms in the product are zero.

Each term in the product is obtained from an assignment of values in the tuple $(2n-2, 2n-4, \ldots, 0)$ to indices $1, \ldots, n$. We will demonstrate that the only ordering which yields a nonzero product is the one placing them in descending order. The result follows from induction on the tuple index: to begin, no term can have degree higher than $2n-2$, so the exponent $2n-2$ must be paired with e_1, which has exponent 0 (the minimum) in e. Next, no remaining term can have degree higher than $2n-3$, so $2n-4$ must be sent to index 2, with exponent 1, the new minimum. Each time we add the term $2n-2j$ at index j in the tuple, the maximum exponent of any remaining e_i is $2n-j-1$ and the minimal remaining exponent in e is the one for e_j, $j-1$. The sum of these exponents is $2n-j-1$, so the j^{th} exponent in the tuple must be paired with e_j. Because $\Psi^*\big(w_{n(n-1)}(E \otimes E)\big)$ yields a nonzero monomial when multiplied by e, $w_{n(n-1)}(E \otimes E)$ is nonzero as desired. $\square$

Lemma 5.5. *Let* $i\colon \mathbf{G}_n\big(\mathbb{R}^{n+k-1}\big) \to \mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ *be the natural inclusion sending a plane* $X \subset \mathbb{R}^{n+k-1}$ *to* $X \times \{0\} \subset \mathbb{R}^{n+k}$*, and consider an arbitrary class* $x \in H^{n(k-1)}\big(\mathbf{G}_n\big(\mathbb{R}^{n+k}\big); \mathbb{Z}_2\big)$*. Then*

$$\big\langle x \smile w_n(E), \big[\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)\big]\big\rangle = \big\langle i^*(x), \big[\mathbf{G}_n\big(\mathbb{R}^{n+k-1}\big)\big]\big\rangle,$$

where $[M]$ *is the fundamental class of the space* M*.*

Proof. The proof here is substantially the same as that of [5]. The result arises from the construction of $\mathrm{Flag}\big(\mathbb{R}^{n+k}\big)$ as an iterated fiber bundle over $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ illustrated in Theorem 5.1. If we decompose the fiber $\mathrm{Flag}(\mathbb{R}^n)$ of the first bundle into an iterated bundle of projective spaces $\mathbb{RP}^{n-1}, \mathbb{RP}^{n-2}, \ldots, \mathbb{RP}^1$, then we obtain a representation of $\mathrm{Flag}\big(\mathbb{R}^{n+k}\big)$ as an iterated fiber bundle over $\mathbf{G}_n\big(\mathbb{R}^{n+k}\big)$ all of whose fibers are projective spaces. Letting u be a top-dimensional form in the Grassmannian (and also its pullbacks into bundles over the Grassmannian, for simplicity), the top-dimensional forms in the resulting bundles over the Grassmannian are

$$ue_1^{n-1},\ ue_1^{n-1}e_2^{n-2},\ ue_1^{n-1}e_2^{n-2}e_3^{n-3},\ \ldots$$

for the first set of bundles. Then if u' denotes the last of these (and pullbacks, again), the top-dimensional forms leading to $\mathrm{Flag}\big(\mathbb{R}^{n+k}\big)$ are

$$u'e_{n+1}^{k-1},\ u'e_{n+1}^{k-1}e_{n+2}^{k-2},\ u'e_{n+1}^{k-1}e_{n+2}^{k-2}e_{n+3}^{k-3},\ \ldots.$$

Note that e_n does not appear in any of the above expressions. This is because the n^{th} plane of a flag in $\mathrm{Flag}\big(\mathbb{R}^{n+k}\big)$ is already determined by the Grassmannian—indeed, it is the plane given by the projection Ψ—so any factors of e_n come from the Grassmannian. We may include a fiber bundle for the n^{th} plane, but its fiber will be $\mathbb{RP}^0$, a single point. From the above, the top-dimensional form derived from u is

$$\Psi^*(u)e_1^{n-1}e_2^{n-2}\ldots e_{n-1}e_{n+1}^{k-1}e_{n+2}^{k-2}\cdots e_{n+k-1}.$$

This class must have the same value on $\left[\mathrm{Flag}\left(\mathbb{R}^{n+k}\right)\right]$ as u on $\left[\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)\right]$.

To obtain our result, first consider $\left\langle i^*(x), \left[\mathbf{G}_n\left(\mathbb{R}^{n+k-1}\right)\right]\right\rangle$. This expression is equal to the value of

$$\Psi^*(i^*(x))e_n^{n-1}\dots e_{n-1}e_{n+1}^{k-1}\dots e_{n+k-2}$$

on $\mathrm{Flag}\left(\mathbb{R}^{n+k-1}\right)$, where $\Psi^* \circ i^*(x) = (i \circ \Psi)^*(x)$ is a symmetric function of $e_{n+1}, \dots, e_{n+k-1}$. The left-hand side $\left\langle x \cdot w_k(E), \left[\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)\right]\right\rangle$ is given by the value of

$$\Psi^*(x)e_1 \dots e_n \cdot e_n^{n-1}\dots e_{n-1}e_{n+1}^{k-1}\dots e_{n+k-1}$$

on the flag manifold, since $\Psi^*(w_n(E)) = e_1 \dots e_n$. Again, $\Psi^*(x)$ is a symmetric polynomial, this time of $e_{n+1}, \dots, e_{n+k}$. The element multiplied by $\Psi^*(x)$ here is exactly $\prod_{i=1}^{n+k-1} e_i$ times the element multiplied by $\Psi^*(i^*(x))$ above, so any summand in $\Psi^*(x)$ containing e_{n+k} will be eliminated by the product. The components that remain arise from a symmetric polynomial in $e_{n+1}, \dots, e_{n+k-1}$. But this polynomial must be equal to $\Psi^*(i^*(x))$, so the two values are the same. □

We note that for the statement proved, it suffices to consider an iterated fiber bundle ending in the partial flag manifold F_n. However, the link between evaluation of a form on a Grassmannian and on the flag manifold is perhaps of more general interest, so it is shown here.

The above lemma may be simplified somewhat for our limited use in this paper.

Corollary 5.6. *If $i^*(x)$ is a nonzero top-level class in $\mathbf{G}_n\left(\mathbb{R}^{n+k}\right)$, then $xw_n(E)$ is a nonzero top-level class in $\mathbf{G}_n\left(\mathbb{R}^{n+k+1}\right)$.*

Finally, we may prove the theorem stated at the beginning of this section.

Proof of Theorem 5.2. Let n, k satisfy $n \le 2^s \le k$ and $n + k \le 2^{s+1}$, and define $l = 2^{s+1} - n - k$. Recall the Hsiang-Szczarba formula (2)

$$w(T)w(E \otimes E) = w(E)^{n+k}.$$

We isolate $\overline{w}(T)$:

$$\overline{w}(T) = w(E \otimes E)\overline{w}(E)^{n+k}.$$

$w(E)^{2^{s+1}}$ is equal to 1, as its pullback to the flag manifold is

$$w(\Psi^*(E))^{2^{s+1}} = \left(\prod_{i=1}^{n} 1 + e_i\right)^{2^{s+1}} = \prod_{i=1}^{n}\left(1 + e_i^{2^{s+1}}\right)$$

And $e_i^{2^{s+1}} = 0$ from the first part of Theorem 4.3. Thus

$$\overline{w}(E)^{n+k} = w(E)^{2^{s+1}-n-k} = w(E)^l$$

and

$$\overline{w}(T) = w(E \otimes E) w(E)^l.$$

By Lemma 5.4, $w_{n(n-1)}(E \otimes E)$ is nonzero in $\mathbf{G}_n\bigl(\mathbb{R}^{2n-1}\bigr)$. Applying Corollary 5.6 l times, we find that $w_{n(n-1)}(E \otimes E) w_n(E)^l$ is nonzero in $\mathbf{G}_n\bigl(\mathbb{R}^{(2n-1)+l}\bigr)$. The exponent of $\mathbb{R}$ is

$$(2n-1) + 2^{s+1} - n - k = 2^{s+1} + n - k - 1 < n + k$$

since $1+2k > 2^{s+1}$, so we can pull this class back to a nonzero class in $\mathbf{G}_n\bigl(\mathbb{R}^{n+k}\bigr)$ via a standard embedding. The dimension of the resulting class (which is the top-level component of $\overline{w}(T)$) is

$$n(n-1) + nl = n(n+l-1) = n\bigl(2^{s+1} - k - 1\bigr),$$

which when added to the dimension nk of the space gives a lower bound on the immersion dimension of $n\bigl(2^{s+1} - 1\bigr)$. □

A Dual vector bundles

Theorem A.1. *For a paracompact Hausdorff smooth manifold M, any real vector bundle over M is isomorphic to its dual.*

Proof. We note for this theorem the well-known fact that a paracompact Hausdorff manifold M admits a smooth partition of unity subordinate to any given open cover of M. Such a partition is a set of smooth functions φ_i, with $\sum_i \varphi_i = 1$, such that any point $x \in M$ has a neighborhood on which all but finitely many φ_i are zero and the support of each φ_i (an open set, since φ_i are smooth) is contained in some set in the given open cover of M.

Let V be a vector bundle over M. A vector bundle isomorphism of V with its dual $V^\perp$ is an element of $\hom\bigl(V, V^\perp\bigr)$ which is everywhere injective (hence bijective, as $\dim V = \dim V^\perp$). Since

$$\hom\bigl(V, V^\perp\bigr) = \hom(V, \hom(V, \mathbb{R})) = \hom(V \oplus V, \mathbb{R}),$$

we may instead find an element f of $\hom(V \oplus V, \mathbb{R})$ with the property that $f(v, w) = 0$ for all w only if $v = 0$. We will satisfy the stronger condition that $f(v, v) > 0$ for all $v \neq 0$ (that is, f is a positive-definite bilinear form on V). This criterion is convex: if both f and g satisfy it, then clearly any linear combination $\alpha f + (1-\alpha) g$ with $0 \le \alpha \le 1$ does as well.

Given an open subset U of M such that V is trivial on U, it is easy to construct a positive-definite form f_U on $V|_U$: letting $v = \dim V$, take a vector bundle isomorphism $V|_U \simeq \mathbb{R}^v$, and use the pullback of the ordinary Euclidean metric $x \cdot y = \sum_i x_i y_i$. Now given an open cover of M such that V is locally trivial in each set in the cover, we take a partition of unity of M into φ_i subordinate to that cover and with φ_i supported on U_i. We can take f_{U_i} to be a positive-definite form on $V|_{U_i}$, and $\varphi_i f_{U_i}$ to be a form on all of M whose restriction to U_i is positive definite, extending by setting it to zero outside of U_i. Then $\sum_i \varphi_i f_{U_i}$ is a positive-definite bilinear form on V, as at each point in M it is a finite linear combination (with coefficients in $[0,1]$) of such forms. □

B Monomial symmetric polynomials

Given a tuple α of nonnegative integers α_i, $1 \le i \le k$, We wish to define the smallest (counted by number of additive terms) symmetric polynomial containing the monomial $\prod_{i=1}^k x_i^{\alpha_i}$. For ease of definition, we require that k is the same as the number of variables n by extending the tuple α_i with zeros. We note that a naive sum over all permutations of the x_i would include multiple copies of terms when the α_i are not distinct. Thus we sum instead over the elements in the orbit of α under the action of the permutation group S_n on the product $\mathbb{N}^n$ (by $(\sigma \cdot \alpha)_i = \alpha_{\sigma^{-1}(i)}$), a set which we denote using $S_n\alpha$.

Definition B.1. The monomial symmetric polynomial $s_{(\alpha_1,\ldots,\alpha_n)}(x_1,\ldots,x_n)$ is the sum over all distinct permutations $\alpha' \in S_n\alpha$ of the monomials $\prod_i x_i^{\alpha'_i}$. The symmetric polynomial $s_{(\alpha_1,\ldots,\alpha_k)}(x_1,\ldots,x_n)$, $k \le n$, is equal to this polynomial with α_i set to zero for $k < i \le n$.

By convention we write the tuple α in descending order. The monomial symmetric polynomials generate all symmetric polynomials under addition alone, a fairly intuitive fact that we will not require in this paper.

For convenience, we will define a few terms before proceeding. Given tuples $x = (x_1,\ldots,x_n)$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$ of the same length, let $x^\alpha = \prod_i x_i^{\alpha_i}$. We will use the previously defined action of S_n, denoting $\sigma \cdot \alpha$ as $\sigma(\alpha)$. Finally, let $S(\alpha)$ be the number of permutations which leave α fixed.

Multiplying two monomial symmetric polynomials to obtain a result expressed as a sum of such polynomials is in general a long computation. We note the simplification that rather than summing the products of all terms in both polynomials, it suffices to keep one of the tuples fixed and add the permutations of the other to it, that is,

Theorem B.1. *The product of two monomial symmetric polynomials s_α and s_β on n variables, where α and β have length n, is*

$$s_\alpha s_\beta = \sum_{\beta' \in S_n\beta} \frac{S(\alpha+\beta')}{S(\alpha)} s_{\alpha+\beta'}.$$

Proof. We begin by defining a polynomial that is easier to work with: for a tuple α of length n and n variables x_i, define

$$m_\alpha(x) = \sum_{\sigma \in S_n} x^{\sigma(\alpha)}.$$

Then m_α includes $S(\alpha)$ copies of each monomial in s_α, so $m_\alpha = S(\alpha)s_\alpha$.

Now

$$\begin{aligned} m_\alpha m_\beta(x) &= \sum_{\rho,\sigma'\in S_n} x^{\rho(\alpha)+\sigma'(\beta)} \\ &= \sum_{\rho,\sigma\in S_n} x^{\rho(\alpha+\sigma(\beta))} \\ &= \sum_{\sigma\in S_n} m_{\alpha+\sigma(\beta)}(x). \end{aligned}$$

Substituting using the formula $m_\alpha = S(\alpha)s_\alpha$, we obtain

$$s_\alpha s_\beta(x) = \frac{1}{S(\alpha)S(\beta)} \sum_{\sigma\in S_n} S(\alpha+\sigma(\beta)) s_{\alpha+\sigma(\beta)}(x).$$

Rather than iterate over all permutations and divide by $S(\beta)$, we can iterate over the distinct permutations of β:

$$s_\alpha s_\beta = \frac{1}{S(\alpha)} \sum_{\beta'\in S_n\beta} S(\alpha+\beta') s_{\alpha+\beta'},$$

the desired result. □

Corollary B.2. *Let β be a tuple of $n-i$ ones followed by i zeros. The following formula holds in n variables with coefficients in $\mathbb{Z}_2$:*

$$s_\beta s_{(n-1,n-2,\ldots,0)} = s_{(n,n-1,\ldots,i+1,i-1,\ldots,0)}.$$

Proof. We will work first in $\mathbb{Z}$ and then reduce to $\mathbb{Z}_2$. Let $\alpha = (n-1, n-2, \ldots, 0)$. Because $S(\alpha) = 1$ (all elements of α are distinct), the preceding formula reduces to

$$s_\alpha s_\beta = \sum_{\beta'\in S_n\beta} S(\alpha+\beta') s_{\alpha+\beta'},$$

and we can immediately reduce to $\mathbb{Z}_2$.

A permutation $\gamma = \alpha+\beta'$ consists of the tuple $(n-1, n-2, \ldots, 0)$ with ones added to $n-i$ of the components. Clearly $\gamma_i \geq \gamma_{i+1}$, and $\gamma_i = \gamma_{i+1}$ if and only if $\beta'_i = 0$ and $\beta'_{i+1} = 1$. Thus each element in γ occurs once or twice, and the number of elements occurring twice is equal to the number of times β' increases. But $S(\gamma)$ is odd only if each element occurs exactly once, in which case $\beta = \beta'$, and $S(\gamma) = 1$, so $s_\alpha s_\beta = s_{\alpha+\beta}$. □

References

[1] Hatcher, Allen. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. xii+544 pp. ISBN: 0-521-79160-X; 0-521-79540-0

[2] Hatcher, Allen. *Vector Bundles and K Theory.* Unpublished online book, version 2.1, 2009.

[3] Hiller, Howard; Stong, R. E. Immersion dimension for real Grassmannians. Math. Ann. 255 (1981), no. 3, 361–367.

[4] Milnor, John W.; Stasheff, James D. *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. vii+331 pp.

[5] Stong, R. E. Cup products in Grassmannians. Topology Appl. 13 (1982), no. 1, 103–113.