Caustics and the Indefinite Signature Schrödinger Equation, Linear and Nonlinear

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ABSTRACT

BENJAMIN DODSON: Caustics and the Indefinite Signature Schrödinger Equation,

Linear and Nonlinear

(Under the direction of Professor Michael Taylor)

The evolution of surface waves in deep water is given by a Schrodinger-like equation. In deep water surface water waves evolve under the nonlinear equation

$$2iu_t = \frac{1}{4}(u_{xx} - 2u_{yy}) + q|u|^2 u \tag{0.0.1}$$

Where x, y are coordinates in $\mathbf{R}^2$, q is a constant. The techniques for the Schrodinger equation can be used in the study of the evolution of (1.1.19), although the behavior is often quite different.

This thesis will focus on three main areas. First, it will concentrate on the behavior the linear Schrödinger equation $iu_t + \Delta u = 0$, in particular, on the asymptotic behavior of $e^{it\Delta}u_0$ as $t \searrow 0$. This has a connection to the asymptotic behavior of the pointwise Fourier inversion $S_R f$ as $R \nearrow \infty$.

Secondly, this thesis will address the behavior of the indefinite signature Schrödinger equation $iu_t + Lu = 0$, where

$$L_{(p,q)} = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2},$$

as well as $iu_t + Lu = F(u)$, where F is a nonlinearity.

Finally, some supercritical local existence results will be obtained for a power-type nonlinearity,

$$iu_t + \Delta u = |u|^\alpha u$$

(0.0.2)

$$u(0,x) = \chi_{B(0;1)},$$

and a global existence result for $\alpha = \frac{4}{n-2\rho}$, $u_0 \in H^{\rho+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,1}(\mathbf{R}^n)$, u_0 radial.

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CHAPTER 1

Introduction

1.1. Introduction

The linear Schrödinger equation on Euclidean space $\mathbf{R}^n$ is the partial differential equation

$$iu_t + \Delta u = 0 \tag{1.1.1}$$
$$u(0,x) = u_0(x).$$

In phase space this has the representation

$$i\frac{\partial}{\partial t}\hat{u}(t,\xi) - |\xi|^2\hat{u}(t,\xi) = 0, \tag{1.1.2}$$

so the solution to (1.1.1) is given by the Fourier multiplier

$$\mathcal{F}(e^{it\Delta}u_0) = e^{-it|\xi|^2}\hat{u}_0(\xi). \tag{1.1.3}$$

Linear Schrödinger operator as $t \searrow 0$:

Since $|e^{-it|\xi|^2}| = 1$,

$$\|e^{it\Delta}u_0\|_{H^{\sigma,2}(\mathbf{R}^n)} = \|u_0\|_{H^{\sigma,2}(\mathbf{R}^n)}.$$

By the Lebesgue dominated convergence theorem, when $u_0 \in H^{\sigma,2}$,

$$\lim_{t\searrow 0}\|e^{it\Delta}u_0 - u_0\|_{H^{\sigma,2}} = 0. \tag{1.1.4}$$

By the Sobolev embedding theorem $H^{\sigma,2}(\mathbf{R}^n) \subset L^\infty(\mathbf{R}^n)$ for $\sigma > n/2$, so if $u_0 \in H^{\sigma,2}(\mathbf{R}^n)$

$$\lim_{t\searrow 0}\|e^{it\Delta}u_0 - u_0\|_{L^\infty(\mathbf{R}^n)} = 0. \tag{1.1.5}$$

(1.1.5) is not true in general for $\sigma \leq \frac{n}{2}$. This failure gives rise to the Gibbs phenomenon. If $u_0 \in L^1(\mathbf{R}^n)$, $e^{it\Delta}u_0$ is continuous for all $t > 0$. Thus if $e^{it\Delta}u_0 \to u_0$ uniformly, then u_0 is continuous. Therefore, u_0 discontinuous forces

$$\|e^{it\Delta}u_0 - u_0\|_{L^\infty(\mathbf{R}^n)} \geq c > 0 \tag{1.1.6}$$

for all $t > 0$.

Definition 1.1.1. Suppose u_0 is discontinuous in a neighborhood of $x_0 \in \mathbf{R}^n$. The Gibbs phenomenon is the failure of uniform convergence, (1.1.6), in a neighborhood of x_0.

Remark: The Gibbs phenomenon was originally described in the context of Fourier inversion. Let

$$\begin{aligned} S_R f(x) &= D_n^R * f(x), \\ D_n^R(x) &= (2\pi)^{-n/2} \int_{|\xi| \leq R} e^{ix\cdot\xi} d\xi. \end{aligned} \tag{1.1.7}$$

Again by Plancherel's theorem

$$\lim_{R\to\infty} \|S_R f(x) - f(x)\|_{L^2(\mathbf{R}^n)} = 0. \tag{1.1.8}$$

However, if $f \in L^1(\mathbf{R}^n)$, $S_R f$ is continuous for every $R < \infty$, so when f is discontinuous

$$\lim_{R\to\infty} \|S_R f - f\|_{L^\infty(\mathbf{R}^n)} \geq c > 0. \tag{1.1.9}$$

See [**29**] for a detailed examination of this topic.

In contrast to the Gibbs phenomenon, the Pinsky phenomenon is a non-local phenomenon. Let χ_Ω denote the characteristic function of a region $\Omega \subset \mathbf{R}^n$. When $n \geq 2$,

$$|e^{it\Delta}\chi_{B(0;1)}(0) - \chi_{B(0;1)}(0)| = C(t)t^{(2-n)/2} + O(t^{(3-n)/2}), \tag{1.1.10}$$

$|C(t)| = c > 0$, $C(t)$ is an oscillatory function. $B(0;1)$ is the ball centered at $0 \in \mathbf{R}^n$ of radius 1. This is despite the fact that $\chi_{B(0;1)}$ is C^∞ at the origin. A similar effect arises in Fourier inversion. When $n \geq 3$,

$$|S_R\chi_{B(0;1)} - \chi_{B(0;1)}| = C(R)R^{(n-3)/2} + O(R^{(n-4)/2}), \tag{1.1.11}$$

with $|C(R)| = c' > 0$, $C(R)$ is an oscillatory function. There is, in fact, a connection between the asymptotics of the Fourier inversion at a point x_0 as $R \nearrow \infty$ and the asymptotics of the Schrödinger operator as $t \searrow 0$ at that same point.

The behavior of the wave equation gives good intuition for the Pinsky phenomenon. For the wave equation

$$\begin{aligned} \partial_{tt}u - \Delta u = 0, \\ u(0,x) = \chi_{B(0;1)}; u_t(0,x) = 0, \end{aligned} \tag{1.1.12}$$

the singularities at the boundary $|x| = 1$ will flow to the center and will focus at time $t = 1$. By the work of [**29**], this effect explains the failure of convergence for (1.1.11) in dimensions $n \geq 3$. A similar focusing effect arises for the Schrödinger equation.

THEOREM 1.1.2. *Let f(x) be a compactly supported function, and suppose f(x) is* C^∞ *in some neighborhood of* x_0. *If*

$$S_R f(x_0) = f(x_0) + \sum_{k=1}^{n} c_k e^{iRt_k} R^\alpha + O(R^{\alpha-1/2}), \tag{1.1.13}$$

as $R \to \infty$, *then*

$$e^{it\Delta} f(x_0) = f(x_0) + O(t^{-\alpha-1/2}). \tag{1.1.14}$$

THEOREM 1.1.3. *Suppose the Schrödinger equation has the asymptotic expansion*

$$e^{it\Delta} f(x) = f(x) + O(t^\alpha) e^{i\beta/t} + O(t^{\alpha+1/2}). \tag{1.1.15}$$

Additionally suppose that $f(x)$ *is smooth in a neighborhood of* x_0 *and is compactly supported. Then there is the pointwise Fourier convergence,*

$$S_R f(x) = f(x) + O(R^{-1/2-\alpha}). \tag{1.1.16}$$

By inspecting (1.1.10) it is quite clear that in $\mathbf{R}^3$, $e^{it\Delta}\chi_{B(0;1)}$ fails to have a uniform L^∞ bound. In fact, as $t \searrow 0$, all blowup is nonlocal for any χ_Ω, where Ω is a manifold with corners.

DEFINITION 1.1.4. In $\mathbf{R}^2$ a two dimensional manifold with corners is a two dimensional manifold with boundary, $\Omega \subset \mathbf{R}^2$, where $\partial\Omega = \cup_{i=1}^N \gamma_i$, where $\gamma_i : [0,1] \to \mathbf{R}^2$ are smooth on [0,1].

Inductively, define $\Omega \subset \mathbf{R}^m$ to be an n-dimensional manifold with corners if Ω is n-dimensional and $\partial\Omega = \cup_{i=1}^N K_i$, where K_i is an n - 1 dimensional manifold with corners embedded in $\mathbf{R}^m$.

Choose $\eta \in C_0^\infty$, $\delta > 0$ depends on Ω,

$$\eta(x) = \begin{cases} 1, & |x| < \delta; \\ 0, & |x| > 2\delta. \end{cases}$$

THEOREM 1.1.5. *There exists a constant* $0 < C < \infty$ *such that*

$$|e^{it\Delta}(\eta(x - x_0)\chi_\Omega)(x_0)| \leq C < \infty \tag{1.1.17}$$

for any point $x_0 \in \mathbf{R}^n$, $t \in (0, \infty)$, Ω *is a manifold with corners.*

This is proved by applying the methods used to prove Theorem [1.1.2] and Theorem [1.1.3],

THEOREM 1.1.6. *There exists a constant* $0 < C < \infty$ *such that*

$$|S_R(\eta(x - x_0)\chi_\Omega)(x_0)| \leq C < \infty, \tag{1.1.18}$$

for any point $x_0 \in \mathbf{R}^n$, $R \in (0, \infty)$, Ω *is a manifold with corners.*

Indefinite Signature Schrödinger Equation

The evolution of surface waves in deep water is given by a Schrödinger-type equation On deep water surface water waves evolve under the nonlinear equation

$$2iu_t = \frac{1}{4}(u_{xx} - 2u_{yy}) + q|u|^2u, \tag{1.1.19}$$

where x, y are coordinates in $\mathbf{R}^2$.

This motivates the study of the linear equation

$$iu_t + Lu = 0. \tag{1.1.20}$$

One could also choose an operator L with signature (1,1),

$$L = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}, \tag{1.1.21}$$

More generally take

$$L_{p,q} = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_n^2}. \tag{1.1.22}$$

Both the Gibbs phenomenon and the Pinsky phenomenon will be different for an indefinite signature Schrödinger equation than they are for the Schrödinger equation

$$iu_t + \Delta u = 0.$$

Gibbs Phenomenon: Let $A \in O(2)$, the group of orthogonal 2×2 matrices. Define the unitary operator

$$(T_A f)(x) = f(A^{-1}x).$$

$$(1.1.23) \qquad e^{it\Delta}(T_A f)(x) = T_A(e^{it\Delta} f)(x).$$

This is not true when L has signature (1,1). When L has signature (p,q) let e_k be the vector corresponding to the differential operator $\frac{\partial}{\partial x_k}$. Decompose $\mathbf{R}^n = V_p \oplus V_q$, $V_p = \text{span } \{e_1, ..., e_p\}$ and $V_q = \text{span } \{e_{p+1}, ..., e_{p+q}\}$. Define the group

$$(1.1.24) \qquad G_{p,q} = \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in O(p), B \in O(q)\}.$$

If $g \in G \subset O(n)$,

$$(1.1.25) \qquad e^{itL}(T_g f)(x) = T_g(e^{itL} f)(x).$$

But $e^{itL}(T_h f)(x) \neq T_h(e^{itL} f)(x)$ in general when $h \in O(n)$, $h \notin G$. So the Gibbs phenomenon for $e^{itL}\chi_\Omega$ depends on both the shape and position Ω.

The Pinsky Phenomenon: Define a generalized ball in $\mathbf{R}^n$,

$$(1.1.26) \qquad \Omega = \{x : |\langle x, x\rangle| \leq 1\},$$

where $\langle , \rangle$ is the inner product with signature (p,q). The region Ω is bounded by the hyperboloids

$$x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 = \pm 1$$

So the Pinsky phenomenon for $e^{itL}\chi_{B(0;1)}$ is weaker than for $e^{it\Delta}\chi_{B(0;1)}$. In particular, suppose L has signature (p,q) with $p \geq 2$, $p \geq q$. Then $e^{itL}\chi_{B(0;1)}$ has divergence of order $O(t^{(2-p)/2})$ at $0 \in \mathbf{R}^n$. However, there is a price to pay.

THEOREM 1.1.7. *Let L be the differential operator*

$$L = \frac{1}{a_1}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \frac{1}{a_2}\sum_{j=p+1}^{n}\frac{\partial^2}{\partial x_j^2}. \tag{1.1.27}$$

Make the decomposition $\mathbf{R}^n = V_p \oplus V_q$, $x = (x_p, x_q)$, $(x_p, 0) \in V_p$, $(0, x_q) \in V_q$. There is focusing of type $C(t)O(t^{(2-p)/2})$ along the axis $x_p = 0$ when $|x_p| < \frac{|a_1-a_2|}{|a_1|}$ and pointwise convergence when $|x_p| > \frac{|a_1-a_2|}{|a_1|}$. In particular, as $a_1 \to a_2$ the focusing concentrates to the center. C(t) is a function of the form $Ce^{ia/t}$, $a \in \mathbf{R}$. When $|C(t)| = 1$, but when $a \neq 0$, C oscillates more and more rapidly as $t \to 0$.

This type of focusing gives improved nonlinear results.

THEOREM 1.1.8. *When $n = 1, 2$, the equation*

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0,x) &= \chi_{B(0;1)}, \end{aligned} \tag{1.1.28}$$

with $F \in C^\infty$, $F(0) = F'(0) = 0$, $F : \mathbf{C} \to \mathbf{C}$, has a local solution on $[0,T]$ for some $T > 0$.

Proof: See [**27**].

The proof makes heavy use of the uniform estimate $\|e^{it\Delta}\chi_{B(0;1)}\|_{L^\infty(\mathbf{R}^n)} \leq C$. This estimate is not available when $n \geq 3$ due to the Pinsky phenomenon. However, this estimate is available when L has signature $(2,1)$, since $\|e^{itL}\chi_{B(0;1)}\|_{L^\infty(\mathbf{R}^3)} \leq C < \infty$.

THEOREM 1.1.9. *Let $F \in C^\infty$, $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(0) = F'(0) = 0$ be the nonlinearity. Then the equation*

$$\begin{aligned} iu_t + Lu &= F(u), \\ u(0) &= u_0 = \chi_{B(0;1)} \end{aligned} \tag{1.1.29}$$

is locally well-posed on some time interval $[0, T_0)$.

Schrödinger Equation for Power-type nonlinearities The Schrodinger equation

$$\begin{aligned} iu_t + \Delta u &= \pm |u|^{\frac{4}{n-2\rho}} u, \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1.30}$$

is called the defocusing Schrodinger equation when the sign is $+$ and the focusing Schrodinger equation when the sign is $-$. Solving (1.1.30) gives an entire class of solutions due to scaling. If $u(t,x)$ is a solution on some interval $[t_-, t_+] \subset \mathbf{R}$ then

$$\lambda^{n/2-\rho} u(\lambda^2 t, \lambda x)$$

is also a solution on the interval $[\lambda^{-2} t_-, \lambda^{-2} t_+]$.

$$\|\lambda^{n/2-\rho} u(0, \lambda x)\|_{\dot{H}^\rho(\mathbf{R}^n)} = \|u(0,x)\|_{\dot{H}^\rho(\mathbf{R}^n)} \tag{1.1.31}$$

(1.1.30) is called a $\dot{H}^{\rho}(\mathbf{R}^n)$ - critical nonlinear Schrodinger equation. When $\alpha = \frac{4}{n}$ (1.1.30) is called an L^2 - critical nonlinear Schrödinger equation, and when $\alpha = \frac{4}{n-2}$, (1.1.30) is a $\dot{H}^1$ - critical nonlinear Schrödinger equation. Such equations are particularly important, because the mass

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)), \tag{1.1.32}$$

and energy

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx \pm \frac{1}{2+\alpha}\int |u(t,x)|^{2+\alpha} dx, \tag{1.1.33}$$

are conserved. The sign depends on the sign in (1.1.30). For the defocusing Schrödinger equation, using quite different methods than the ones that will be used in this thesis, substantial global well-posedness results have been obtained in each special case.

THEOREM 1.1.10. *The energy critical Schrödinger equation is globally well-posed and scatters for* $u_0 \in \dot{H}^1(\mathbf{R}^n)$.

Proof: See [**17**] when $n = 3$, [**34**] when $n = 4$, and [**35**] for $n \geq 5$.

THEOREM 1.1.11. *The mass critical Schrödinger equation is globally well-posed and scatters for* $u_0 \in L^2(\mathbf{R}^n)$, *radial.*

Proof: See [**36**] when $n = 2$, [**40**] for $n \geq 3$.

LEMMA 1.1.12. *If* $u_0 \in \dot{H}^\rho(\mathbf{R}^n)$ *for some* $0 \leq \rho < \frac{n}{2}$, $n \geq 3$, *then (1.1.30) has a solution for some interval* $[0,T]$, *where* $T(u_0) > 0$. *If* $u_0 \in H^{\rho+\epsilon}(\mathbf{R}^n)$, *then (1.1.30) is locally well-posed on some interval* $[0,T]$, $T(\|u_0\|_{H^{\rho+\epsilon}(\mathbf{R}^n)}) > 0$.

Combining this lemma with the conservation of $L^2(\mathbf{R}^n)$ and conservation of $\dot{H}^1$ in the defocusing case gives a global solution when $u_0 \in H^1(\mathbf{R}^n)$, $\frac{4}{n} \leq \alpha < \frac{4}{n-2}$, simply by iterating the local solutions.

The first improvement of these results came in [**1**], which extended global well-posedness to

$$\begin{gathered} iu_t + \Delta u = |u|^2 u, \\ u(0,x) \in H^s(\mathbf{R}^2), s > 3/5. \end{gathered} \tag{1.1.34}$$

This method motivated [**20**], [**18**], [**19**], and [**32**] to extend $H^1(\mathbf{R}^n)$ global well-posedness results to lower regularity $u_0 \in H^s(\mathbf{R}^n)$, $s < 1$, via the I-method. If $u(t)$ solves (1.1.30), then $I_N u(t)$ solves the equation

$$iI_N u_t + I_N \Delta u = I_N(|u|^\alpha u). \tag{1.1.35}$$

Where I_N is a smooth Fourier multiplier,

$$\widehat{I_N u}(\xi) = m_N(|\xi|)\hat{u}(\xi), \tag{1.1.36}$$

$$m_N(|\xi|) = \begin{cases} 1, & |\xi| \leq N; \\ |\xi|^{s-1}, & |\xi| \geq 2N. \end{cases} \tag{1.1.37}$$

$$\|I_N u_0\|_{H^1(\mathbf{R}^n)} \leq N^{1-s} \|u_0\|_{H^1(\mathbf{R}^n)}.$$

However, $I_N(|u|^\alpha u) \neq |I_N u|^\alpha (I_N u)$, so to prove global existence one must endeavor to control the modified energy

$$E(I_N u(t)) = \frac{1}{2} \int |\nabla I_N u(t,x)|^2 dx + \frac{1}{2+\alpha} \int |I_N u(t,x)|^{2+\alpha} dx. \tag{1.1.38}$$

Currently the best results are global well-posedness for $s > 1/3$ for

$$iu_t + \Delta u = |u|^2 u \tag{1.1.39}$$

in $\mathbf{R}^2$ (see [**11**]) and $s > 4/5$ in $\mathbf{R}^3$ (see [**18**]). For the L^2-critical nonlinear Schrodinger equation

$$iu_t + \Delta u = |u|^{4/n} u, \tag{1.1.40}$$

there is global well-posedness for $s > \frac{\sqrt{7}-1}{3}$ when $n = 3$, and $s > \frac{-(n-2)+\sqrt{(n-2)^2+8(n-2)}}{4}$ for $n \geq 4$ (see [**32**]).

The method used in this thesis was inspired by the I-method, and seeks to take advantage of the special structure of many types of solutions to (1.1.30). For many types of u_0, (1.1.30) has a local solution with some additional structure that can be exploited to prove global existence.

(1.1.30) fails to be locally well-posed for $u_0 \in H^s(\mathbf{R}^n)$, $s < \rho$. See [**21**] and [**22**]. Nevertheless, as was demonstrated in [**27**], when $n = 1, 2$,

$$iu_t + \Delta u = |u|^{2k} u$$

has a local solution for some interval $[0, T]$. In particular, when $n = 2$, there is a local solution for any $\dot{H}^1$ - subcritical Schrödinger equation. This can be extended to higher dimensions ($n \geq 3$) for a $\dot{H}^1$ subcritical Schrödinger equation.

THEOREM 1.1.13. *The nonlinear Schrödinger equation*

$$\begin{aligned} iu_t + \Delta u &= |u|^{\alpha} u, \\ u_0 &= \chi_{\Omega}, \end{aligned} \tag{1.1.41}$$

has a local solution as long as $\alpha < \frac{4}{n-2}$. Ω *is a smoothly bounded region in* $\mathbf{R}^n$.

For this initial data, the Duhamel term smooths the local solution, and thus it is of the form

$$\begin{aligned} &u(t, x) + v(t, x), \\ &v(t, x) \in L_t^{\infty} H_x^1([0, T] \times \mathbf{R}^n). \end{aligned} \tag{1.1.42}$$

When $n = 3$,

$$
\begin{aligned}
& iu_t + \Delta u = |u|^2 u, \\
& u(0,x) = \chi_\Omega,
\end{aligned}
\tag{1.1.43}
$$

has a solution of the form

$$e^{it\Delta}\chi_\Omega + w(t,x),$$

$w(t,x) \in L_t^\infty H_x^1([0,T] \times \mathbf{R}^3)$. It is clear that the equation

$$
\begin{aligned}
& iv_t + \Delta v = |v|^2 v, \\
& v(T,x) = w(T,x),
\end{aligned}
\tag{1.1.44}
$$

has a global solution since $v \in H^1(\mathbf{R}^3)$. But since

$$\|\nabla e^{it\Delta}\chi_\Omega\|_{L_x^\infty(\mathbf{R}^3)} \leq \frac{1}{t^{3/2}},$$

then on $[T,\infty)$, $e^{iT\Delta}\chi_\Omega + w(T,x)$ can be treated very effectively as a perturbation of $w(T,x)$.

Theorem 1.1.14. *(1.1.30) has a global solution for $u_0 = \chi_\Omega$ when Ω is a compact region in $\mathbf{R}^n$ with smooth boundary and $\alpha = \frac{4}{n-2\rho}$, $1 \leq \rho < \frac{4}{n-2}$.*

Secondly, after making two more restrictions on the initial data (u_0 radial and $u_0 \in H^{1/2+\epsilon,1}(\mathbf{R}^n)$), there are global existence results.

Theorem 1.1.15. *(1.1.30) has a global solution for u_0 radial,*

$$u_0 \in H^{\rho+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,1}(\mathbf{R}^n). \tag{1.1.45}$$

Remark: This method can also be applied to equations with combined power-type nonlinearities.

THEOREM 1.1.16. *If* $u_0 \in H^{\rho+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,1}(\mathbf{R}^n)$ *then there is a global solution to*

$$iu_t + \Delta u = \sum_{i=1}^{k} c_i |u|^{\alpha_i} u \tag{1.1.46}$$

With $\frac{4}{n} \leq \alpha_i \leq \frac{4}{n-2\rho}$ *and* $c_i > 0$.

CHAPTER 2

Positive Definite Signature

2.1. The Fourier Transform

The Fourier transform is an essential tool to the study of the linear Schrödinger equation, and the free solution of the Schrodinger equation sheds light on the inverse Fourier transform. Consider the function $f : \mathbf{R}^n \to \mathbf{C}$.

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi} dx. \tag{2.1.1}$$

Since $|e^{-ix\cdot\xi}| = 1$, this function is well defined for $f \in L^1(\mathbf{R}^n)$. We have

$$\mathcal{F} : L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n). \tag{2.1.2}$$

Definition 2.1.1. The Schwartz function space is a Frechet space of functions with bounded seminorms

$$\mathcal{S}(\mathbf{R}^n) = \{f : \|x^\alpha \partial_x^\beta f\|_\infty \le C(\alpha, \beta) < \infty\} \tag{2.1.3}$$

Here $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multiindices.

Let $\mathcal{F}$ denote the Fourier transform operator

$$\mathcal{F} : f(x) \to \hat{f}(\xi). \tag{2.1.4}$$

The inverse of the Fourier transform is denoted $\mathcal{F}^{-1}$. This operator will be denoted $\check{f}$.

$$\mathcal{F}^{-1} : \hat{f}(\xi) \to f(x), \tag{2.1.5}$$

$$\check{g}(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} g(\xi) e^{ix\cdot\xi} d\xi. \tag{2.1.6}$$

This function is well defined on $\mathcal{S}(\mathbf{R}^n)$ and in fact defines a 1-1 isomorphism.

$$\begin{gathered}\mathcal{F} : \mathcal{S} \to \mathcal{S}, \\ \mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}.\end{gathered} \tag{2.1.7}$$

This can be proved using the integration by parts identities.

$$\int_{\mathbf{R}^n} \partial_{x_i} f(x) e^{-ix\cdot\xi} dx = -\int_{\mathbf{R}^n} f(x) \partial_{x_i} e^{-ix\cdot\xi} dx$$

$$= -i\xi_i \int_{\mathbf{R}^n} f(x) e^{-ix\cdot\xi} dx = -i\xi_i \hat{f}(\xi),$$

$$\int_{\mathbf{R}^n} -ix_i f(x) e^{-ix\cdot\xi} dx = \int f(x) \frac{\partial}{\partial \xi_i} e^{-ix\cdot\xi} dx = \frac{\partial}{\partial \xi} \hat{f}(\xi).$$

Then by the fact that $(1+|x|^2)^{-n} \in L^1$ and the commutator relations of ∂_x^α and the $x^\alpha \cdot f$ multiplier, (2.1.6) is well-defined on $\mathcal{S}(\mathbf{R}^n)$. The proof that $\mathcal{F}^{-1}$ is the proper inverse can be found in [**26**], for example.

Then by a change in the order of integration, for $f, g \in \mathcal{S}(\mathbf{R}^n)$,

Theorem 2.1.2.

$$\int_{\mathbf{R}^n} \hat{f}(\xi) g(\xi) d\xi = \int_{\mathbf{R}^n} f(x) \check{g}(x) dx. \tag{2.1.8}$$

This proves, in particular that $\mathcal{F}^{-1}$ is the adjoint of $\mathcal{F}$.

$$\|\hat{f}\|^2_{L^2(\mathbf{R}^n)} = \langle \mathcal{F}(f), \mathcal{F}(f)\rangle = \langle f, \mathcal{F}^{-1}\mathcal{F}(f)\rangle = \|f\|^2_{L^2(\mathbf{R}^n)}$$

This proves Plancherel's theorem.

THEOREM 2.1.3.

$$\|f\|_{L^2(\mathbf{R}^n)} = \|\hat{f}\|_{L^2(\mathbf{R}^n)}. \tag{2.1.9}$$

This in turn proves $\mathcal{F}$ is an isomorphism on $L^2(\mathbf{R}^n)$. In order to make sure $\hat{f} \in L^1(\mathbf{R}^n)$, it was necessary to integrate by parts, which required some degree of smoothness of f. Thus there is no guarantee that this inversion formula will work for any old $f(x) \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$. Certainly it will work if $\hat{f}(\xi) \in L^1$, and then $f(x) \in L^\infty$. However, there does not exist an L^p space that will guarantee that $\hat{f} \in L^1$.

Consider $f(x) = \chi_{B(0;1)}$ in $\mathbf{R}$, where χ_Ω be the characteristic function of the set $\Omega \subset \mathbf{R}^n$, and $B(x;\delta)$ is the ball of radius δ centered at $x \in \mathbf{R}^n$.

$$\hat{f}(\xi) = \int_{-1}^{1} e^{-ix\cdot\xi} = \frac{1}{-i\xi}[e^{-i\xi} - e^{i\xi}] = \frac{2\sin(-\xi)}{-\xi} = \frac{2\sin(\xi)}{\xi}.$$

$\hat{f}$ is not integrable even though $f \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$.

Parseval's theorem and Plancherel's theorem will often arise in the analysis of the linear Schrödinger equation. Let $\check{g}(\xi) = \mathcal{F}^{-1}(g)$.

Take the Cauchy Schwartz inequality.

$$\int_{\mathbf{R}^n} f(x)g(x)dx \leq \|f\|_{L^2(\mathbf{R}^n)}\|g\|_{L^2(\mathbf{R}^n)}. \tag{2.1.10}$$

Now, $(1+|x|)^{-\alpha} \in L^2(\mathbf{R}^n)$ for all $\alpha > n/2$. This leads to the definition of the L^2 based Sobolev spaces.

DEFINITION 2.1.4. The Sobolev space $\dot{H}^\alpha$ is the space of functions such that

$$|\xi|^\alpha \hat{f}(\xi) \in L^2(\mathbf{R}^n), \tag{2.1.11}$$

and H^α is the space of functions such that

$$(1+|\xi|)^\alpha \hat{f}(\xi) \in L^2(\mathbf{R}^n). \tag{2.1.12}$$

Then $H^\alpha \subset L^\infty(\mathbf{R}^n)$ for $\alpha > n/2$. In fact, f is also continuous by the dominated convergence theorem.

When f does not lie in $H^{n/2+\epsilon,2}(\mathbf{R}^n)$, pointwise Fourier inversion is not so easy. So instead, consider a family of partial Fourier inverses for some $f(x) \in L^1(\mathbf{R}^n)$. Since $\hat{f}(\xi) \in L^\infty(\mathbf{R}^n)$, $\hat{f}\chi_{B(0;R)} \in L^1$. So define the partial Fourier inverse

$$S_R f(x) = (2\pi)^{-n} \int_{|\xi| \leq R} \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{2.1.13}$$

It has been a subject of some discussion whether or not $S_R f(x) \to f(x)$ as $R \to \infty$, and if it does, what the rate of convergence is. This phenomenon is closely related to the formation of caustics in the wave equation. See for example, [**10**], [**29**]. For a general introduction to the theory of Fourier transforms, see [**8**], [**26**], Chapter three.

Given some extra symmetry, some additional estimates can be made in $\mathbf{R}^n$.

LEMMA 2.1.5. *For* $|t| < 1$, $|x| > 1$.

$$(2.1.14)\quad t^{-n/2}\int_{|r|=1} e^{-ix\cdot r/2t}d\sigma(r) = t^{-1/2}|x|^{-(n-1)/2}[C_1 e^{-i|x|r/2t} + C_2 e^{i|x|r/2t}] + O(|x|^{-n/2}).$$

Proof: After a rotation of coordinates, let $x = (0, ..., 0, 1)$. $x\cdot r = |x||r|\cos(\theta) = |x|\cos(\theta)$, where θ is the angle between $(0, ..., 0, 1)$ and r. Rewrite the integral in polar coordinates.

$$\int_{|r|=1} e^{-ix\cdot r/2t}d\sigma(r) = \int_{-\pi/2}^{\pi/2} e^{-i|x|r\sin(\theta)/2t}(\cos(\theta))^{n-2}d\theta.$$

Make a change of variables, $u = \sin(\theta)$, $du = \cos(\theta)d\theta$.

$$= \int_{-1}^{1} e^{-i|x|u/2t}(1-u^2)^{(n-3)/2}du.$$

If n is odd, integrate by parts $\frac{n-1}{2}$ times. If n is even, integrate by parts $\frac{n-2}{2}$ times and then let $x = v^2$, $dx = 2vdv$,

$$\int_0^1 x^{-1/2}e^{ix/t}dx = \frac{1}{2}\int_0^1 e^{iv^2/t}dv = e^{i\pi/4}t^{1/2} + O(t).$$

This completes the proof $\square$.

THEOREM 2.1.6. *If* u *is a radial function,* $u \in H^{1/2+\epsilon,2}(\mathbf{R}^n)$,

$$(2.1.15)\qquad \||x|^{(n-1)/2-\epsilon}u\|_{L^\infty(\mathbf{R}^n)} \lesssim \|u\|_{H^{1/2+\epsilon,2}(\mathbf{R}^n)}$$

Proof: Make the Fourier inversion,

$$u(x) = \int_{|\xi|\leq\frac{1}{|x|}} e^{ix\cdot\xi}\hat{f}(\xi)d\xi + \int_{|\xi|>\frac{1}{|x|}} e^{ix\cdot\xi}\hat{f}(\xi)d\xi.$$

$$\int_{|\xi|=r} e^{ix\cdot\xi} d\sigma(\xi) d\xi \sim (\frac{1}{r|x|})^{(n-1)/2}.$$

Then split the integral into two pieces.

$$\int_{|\xi|\geq 1/|x|} e^{ix\cdot\xi}\hat{f}(\xi)d\xi \sim \int_{1/|x|}^{\infty} r^{n-1}(\frac{1}{r|x|})^{(n-1)/2}\hat{f}(r)dr = \frac{1}{|x|^{(n-1)/2}}\int_{1/|x|}^{\infty}\hat{f}(r)r^{(n-1)/2}dr.$$

$$\int_0^{\infty} |\hat{f}(r)r^{1/2+\epsilon}|^2 r^{n-1}dr + \int_0^{\infty}|\hat{f}(r)|^2 r^{n-1}dr.$$

$$\int_{1/|x|}^{\infty} |\hat{f}(r)| r^{(n-1)/2} dr$$

$$\leq (\int_{1/|x|}^{\infty} |\hat{f}(r)|^2 r^{n-1}(1+r^{1+2\epsilon}dr)^{1/2}(\int_{1/|x|}^{\infty}(1+r)^{-1-2\epsilon}dr)^{1/2} \leq \|f\|_{H^{1/2+\epsilon}(\mathbf{R}^n)}.$$

Now for the second piece.

$$\int_{|\xi|\leq\frac{1}{|x|}} |\hat{f}(\xi)|d\xi \lesssim (\int_0^{1/|x|}|\hat{f}(r)|^2 r^{(n-1)}(1+r^{1+2\epsilon})dr)^{1/2}(\int_0^{1/|x|}(1+r^{1+2\epsilon})dr)^{1/2}$$

$$\lesssim |x|^{-(n-1)/2}\|f\|_{H^{1/2+\epsilon}(\mathbf{R}^n)}.$$

2.2. The Free Schrodinger Equation

The solution for the linear Schrödinger equation.

$$iu_t = \Delta u$$

(2.2.1)

$$u(0,x) = u_0(x)$$

is given by the function

$$u(x,t) = e^{-it\Delta}u_0. \tag{2.2.2}$$

By the Fourier transform identities of section 1.

$$\Delta f = -(\xi_1^2 + \xi_2^2)\hat{f}(\xi).$$

More generally for the equation

$$(a+ib)u_t = \Delta u, \tag{2.2.3}$$

where $a \geq 0$ and $|a+ib| = 1$ the solution is given by

$$e^{t\Delta/(a+ib)}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^2/(a+ib)}\hat{u}_0(\xi)) \tag{2.2.4}$$

$$\hat{u}(x,t) = e^{-t(|\xi|^2/(a+ib))}\hat{f}(\xi) \tag{2.2.5}$$

$$u(x,t) = \frac{1}{4\pi^2}\int\int e^{-t|\xi|^2/(a+ib)}f(y)e^{i(x-y)\cdot\xi}dyd\xi. \tag{2.2.6}$$

By a well-known approximation argument, found in [**26**], for example, the order of integration can be switched for $f \in L^2(\mathbf{R}^n)$.

$$u(x,t) = \int f(y)\int e^{-t|\xi|^2/(a+ib)}e^{i(x-y)\cdot\xi}d\xi dy. \tag{2.2.7}$$

Now complete the square.

$$t\xi_2^2 + (x_2 - y_2)\xi_2 = t(\xi_2 + \frac{(x_2 - y_2)}{2t})^2 - \frac{(x_2 - y_2)^2}{4t},$$

$$t\xi_1^2 - (x_1 - y_1)\xi_1 = t(\xi_1 - \frac{(x_1 - y_1)}{2t})^2 - \frac{(x_1 - y_1)^2}{4t},$$

$$\frac{t}{a+ib}|\xi|^2 - (x-y)\cdot\xi = \frac{t}{a+ib}|\xi - \frac{(a+ib)}{2t}(x-y)|^2 - \frac{a+ib}{4t}|x-y|^2.$$

Changing the contour gives the identities.

$$\int_{-\infty}^{\infty} e^{ix^2} dx = \sqrt{\pi} e^{i\pi/4},$$

$$\int_{-\infty}^{\infty} e^{-ix^2} dx = \sqrt{\pi} e^{-i\pi/4}.$$

So $e^{\frac{t\Delta}{a+ib}}$ has convolution kernel

(2.2.8) $$K(x,y) = \frac{(-a-ib)^{n/2}}{(4\pi t)^{n/2}} e^{(a+ib)\frac{|x-y|^2}{4t}}.$$

Theorem 2.2.1. *Since $|e^{it|\xi|^2}| = 1$ and $e^{it\Delta}$ is a Fourier multiplier, by Plancherel's theorem there is an L^2 identity.*

(2.2.9) $$\|e^{it\Delta} f\|_{L^2(\mathbf{R}^n)} = \|f\|_{L^2(\mathbf{R}^n)}.$$

Also, by properties of convolution

(2.2.10) $$\|e^{it\Delta} f\|_{L^\infty(\mathbf{R}^n)} \leq \frac{C(n)}{t^{n/2}} \|f\|_{L^1(\mathbf{R}^n)}.$$

These identities can be interpolated for $2 < p < \infty$.

$$\|e^{it\Delta} f\|_{L^p(\mathbf{R}^n)} \leq \frac{C(n,p)}{t^{n(1/2-1/p)}} \|f\|_{L^{p'}(\mathbf{R}^n)}. \tag{2.2.11}$$

Here p' is the dual Hölder exponent to p.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Remark: $e^{it\Delta}$ is in fact an $H^{s,2}(\mathbf{R}^n)$ isometry for any $H^{s,2}(\mathbf{R}^n)$ space of functions for any s.

$$e^{it\Delta} u_0(x) = C(n) \int_{\mathbf{R}^n} e^{i|x-y|^2/4t} u_0(y) dy \tag{2.2.12}$$

2.3. The Free Wave Operator

For the free wave equation

$$\begin{aligned} (\partial_{tt} - \Delta) u &= 0, \\ u(0,x) &= f(x), \\ u_t(0,x) &= 0, \end{aligned} \tag{2.3.1}$$

the solution is given by

$$\cos(t\sqrt{\Delta}) f(x) = u(t,x), \tag{2.3.2}$$

$$\cos(t\sqrt{-\Delta}) = \frac{1}{2}(e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}}), \tag{2.3.3}$$

which in turn is given by the Fourier multiplier

$$\cos(t|\xi|) = \frac{1}{2}(e^{it|\xi|} + e^{-it|\xi|}). \tag{2.3.4}$$

Now by a change of variables

$$\int_{-\infty}^{\infty} e^{-iu^2} e^{-2iut|\xi|} e^{-it^2|\xi|^2} du = C,$$

For some constant C.

$$Ce^{it^2|\xi|^2} = \int_{-\infty}^{\infty} e^{-iu^2} e^{-2iut|\xi|} du.$$

On the other hand

$$\int_0^{\infty} \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx$$

$$= \int_1^{\infty} \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx + \int_0^1 \alpha^{1/2} e^{\pm i(x-1/x)^2} dx.$$

(Let $u = x - \frac{1}{x} \Rightarrow du = dx + \frac{1}{x^2}dx$) Then the previous quantity

$$= \int_0^{\infty} \alpha^{1/2} e^{\pm i\alpha u^2} du - \int_1^{\infty} \frac{\alpha^{1/2}}{x^2} e^{\pm i\alpha(x-1/x)^2} dx + \int_0^1 \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx.$$

After a change of variables $x \mapsto \frac{1}{x}$, $dx \mapsto -\frac{1}{x^2}dx$.

$$\int_0^1 \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx = \int_1^\infty \frac{\alpha^{1/2}}{x^2} e^{\pm i\alpha(x-1/x)^2} dx$$

$$Ce^{\pm i\pi/4} = \int_0^\infty \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx \tag{2.3.5}$$

So,

$$\int_0^\infty \alpha^{1/2} e^{\pm i\alpha(x-1/x)^2} dx = \int_0^\infty \alpha^{1/2} e^{\pm i(\alpha^{1/2}x - \frac{\alpha^{1/2}}{x})^2} dx \tag{2.3.6}$$

$$= \int_0^\infty e^{\pm i(x-\frac{\alpha}{x})^2} dx \tag{2.3.7}$$

$$Ce^{\pm \frac{i\pi}{4}} = \int_0^\infty e^{\pm iu^2} e^{\mp 2i\alpha} e^{\pm i\frac{\alpha^2}{u^2}} du. \tag{2.3.8}$$

Now let $t|\xi| = \alpha$.

$$Ce^{\pm \frac{i\pi}{4}} e^{\pm 2it|\xi|} = \int_0^\infty e^{\pm iu^2} e^{\pm i\frac{t^2|\xi|^2}{u^2}} du. \tag{2.3.9}$$

$$Ce^{\pm 2it|\xi|} = e^{\mp \frac{i\pi}{4}} \int_0^\infty e^{\pm iu^2} e^{\pm i\frac{t^2|\xi|^2}{u^2}} du. \tag{2.3.10}$$

$$2C\cos(2t|\xi|) = \int_0^\infty [e^{-\frac{\pi i}{4}} e^{iu^2} e^{i\frac{t^2|\xi|^2}{u^2}} + e^{\frac{\pi i}{4}} e^{-iu^2} e^{-i\frac{t^2|\xi|^2}{u^2}}] du. \tag{2.3.11}$$

Next, an identity derived in [**29**] will be applied to (2.3.11). For the reader's convenience, teh derivation of the identity will be written below. Let φ be a Borel function of the operator A.

$$\varphi(A) = \int_{-\infty}^{\infty} \varphi(\lambda) dE_\lambda, \tag{2.3.12}$$

Where E_λ is the spectral resolution of A, dE_λ is the spectral measure of A. If there is a sequence of Borel functions $\varphi_\nu \to \varphi$ a.e. with respect to dE_λ, $|\varphi_\nu| \leq C$,

$$\varphi(A)f = \lim_{\nu \to \infty} \int_{-\infty}^{\infty} \varphi_\nu(\lambda) dE_\lambda f. \tag{2.3.13}$$

Suppose that both $\varphi, \hat{\varphi} \in L^1(\mathbf{R})$.

$$\varphi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{it\lambda} dt.$$

From the definition of the spectrum,

$$\int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda d\lambda = e^{itA}.$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{it\lambda} dE_\lambda dt d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{itA} dt.$$

If φ is an even function,

$$\varphi(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\varphi}(t) \cos(tA) dt. \tag{2.3.14}$$

In particular, for the Fourier multiplier

$$S_R(A) = \begin{cases} 1, & |A| \leq R; \\ 0, & |A| > R. \end{cases}$$

$$\hat{\varphi}(t) = \int_{-R}^{R} e^{-itx} dx = 2\frac{\sin(Rt)}{t}.$$

$$S_R f = \lim_{\nu \to \infty} \frac{1}{\pi} \int_{-\nu}^{\nu} \frac{\sin Rt}{t} \cos(tA) f dt. \tag{2.3.15}$$

Combine (2.3.11) and (2.3.15),

$$S_R f = \lim_{\nu \to \infty} \frac{1}{\pi} \int_{-\nu}^{\nu} \int_{0}^{\infty} \frac{\sin(Rt)}{t} [e^{-\frac{\pi i}{4}} e^{iu^2} e^{\frac{-it^2\Delta}{4u^2}} + e^{\frac{\pi i}{4}} e^{-iu^2} e^{\frac{it^2\Delta}{4u^2}}] f(x) du dt. \tag{2.3.16}$$

This establishes a close connection between the behavior of $S_R f$ as $R \to \infty$ and $e^{it\Delta} f$ as $t \searrow 0$.

2.4. Gibbs Phenomenon on R

Because $e^{it\Delta}$ is a Fourier multiplier with $|e^{-it|\xi|^2}| \leq 1$, if $u_0 \in H^s$ for some $s \geq 0$, then by the Lebesgue dominated convergence theorem there will be convergence in H^s.

$$\|e^{it\Delta} u_0 - u_0\|_{H^s}^2 = \int |e^{it|\xi|^2} - 1|^2 |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi \to 0, \tag{2.4.1}$$

as $t \to 0$.

Similarly, for Fourier inversion, $|S_R(\xi)| \leq 1$ so

$$\|S_R f(x) - f(x)\|_{H^s(\mathbf{R}^n)}^2 = \int_{|\xi| \geq R} |\hat{f}(\xi)|^2 \langle \xi \rangle^{2s} d\xi, \tag{2.4.2}$$

which will converge by the Lebesgue dominated convergence theorem.

If $s > n/2$, then by the Sobolev embedding theorem

$$\lim_{R \to \infty} \|S_R f - f\|_{L^\infty(\mathbf{R}^n)} = 0,$$

$$\lim_{t \to 0} \|e^{it\Delta} f - f\|_{L^\infty(\mathbf{R}^n)} = 0.$$

Moreover if $\frac{1}{p} \geq \frac{1}{2} - \frac{s}{n}$,

$$\lim_{R \to \infty} \|S_R f - f\|_{L^p(\mathbf{R}^n)} = 0,$$

$$\lim_{t \to 0} \|e^{it\Delta} f - f\|_{L^p(\mathbf{R}^n)} = 0.$$

On the other hand, by (2.2.12) and the dominated convergence theorem, if $f \in L^1(\mathbf{R}^n)$ and $t > 0$, then $e^{it\Delta} f$ is a continuous function. If $f \in L^1(\mathbf{R}^n)$, $|\hat{f}(\xi)| \leq C\|f\|_{L^1(\mathbf{R}^n)}$, so $S_R f$ is also continuous by the dominated convergence theorem. If $f \in L^2(\mathbf{R}^n)$ then by (2.4.2), $e^{it\Delta} f \to f$ and $S_R f \to f$ almost everywhere.

If $e^{it\Delta} f \to f$ uniformly as $t \to 0$, or $S_R f \to f$ uniformly as $R \to \infty$, then f is a continuous function. So for f discontinuous the best that can be hoped for is pointwise convergence.

Definition 2.4.1. Suppose for some $x \in \mathbf{R}^n$,

$$\lim_{t \searrow 0} |e^{it\Delta} u_0(x) - u_0(x)| = 0 \tag{2.4.3}$$

Then $e^{it\Delta}u_0(x)$ converges to $u_0(x)$.

If for every $x \in \mathbf{R}^n$, (2.4.3) holds, then $e^{it\Delta}u_0 \to u_0$ pointwise.

Remark: The rate of convergence may depend on x.

Example: Let

$$u_0 = \chi_{[-1,1]} \begin{cases} 1, & |x| < 1; \\ 1/2, & |x| = 1; \\ 0, & |x| > 1. \end{cases}$$

Suppose without loss of generality that $x \geq 0$.

$$t^{-1/2} \int_{-1}^{1} e^{i(x-y)^2/t} dy = t^{-1/2} \int_{-1+x}^{1+x} e^{iy^2/t} dy = \int_{(-1+x)/t^{1/2}}^{(1+x)/t^{1/2}} e^{iy^2} dy, \tag{2.4.4}$$

$$= \int_{(-1+x)t^{-1/2}}^{\infty} e^{iy^2} dy - \int_{(1+x)t^{-1/2}}^{\infty} e^{iy^2} dy$$

$$= \int_{(-1+x)t^{-1/2}}^{\infty} e^{iy^2} dy + O(t^{1/2}). \tag{2.4.5}$$

Here the estimate

$$\int_A^\infty e^{ix^2} dx = \int_A^\infty \frac{1}{2ix} \frac{d}{dx}(e^{ix^2}) dx = \frac{1}{2ix} e^{ix^2} |_A^\infty + \int_A^\infty \frac{1}{2ix^2} e^{ix^2} dx = O(\frac{1}{A}) \tag{2.4.6}$$

has been used. Fix an x. If $x > 1$, (2.4.5) $\to 0$ as $t \to 0$. If $x < 1$, (2.4.5) $\to \sqrt{\pi}e^{i\pi/4}$. If $x = 1$, (2.4.5) $= \frac{\sqrt{\pi}}{2}e^{i\pi/4} + O(t^{1/2})$, so there is pointwise convergence. However, uniform convergence fails.

Consider the sequence (x_n, t_n) where $t_n \to 0$, $x_n = 1 + Ct_n^{1/2}$ for some $C \neq 0$,

$$e^{it_n\Delta}u_0(x_n) = \int_C^\infty e^{iy^2}dy + O(t_n^{1/2}).$$

Therefore

$$\lim_{n\to\infty} e^{it_n\Delta}u_0(x_n) = \int_C^\infty e^{iy^2}dy, \tag{2.4.7}$$

and $\int_C^\infty e^{iy^2}dy$ will not equal 0 for a generic $C > 0$, nor will it equal $\sqrt{\pi}e^{i\pi/4}$ for a generic $C < 0$.

Similarly, for Fourier inversion,

$$\int_{-1}^{1} e^{-iy\xi}dy = \frac{2\sin(\xi)}{\xi},$$

$$S_R u_0(x) = \int_{-R}^{R} e^{ix\xi}\frac{2\sin(\xi)}{\xi},$$

take x close to one.

$$2\int_{-R}^{R} e^{i(x+1)\xi}\frac{1}{\xi}d\xi = 2\int_{-\frac{R}{(x+1)}}^{\frac{R}{x+1}} \frac{e^{i\xi}}{\xi}d\xi, \tag{2.4.8}$$

$$= 2\int_{-R}^{R} \frac{e^{i(x+1)\xi}}{\xi} d\xi = 2\int_{-\infty}^{\infty} \frac{e^{i\xi}}{\xi} d\xi + O(\frac{1}{R}). \tag{2.4.9}$$

Using the estimate

$$\int_{K}^{\infty} \frac{e^{i\xi}}{\xi} d\xi = \frac{1}{i\xi} e^{i\xi}|_{K}^{\infty} - \int_{K}^{\infty} \frac{1}{i\xi^2} e^{i\xi} d\xi = O(\frac{1}{K}).$$

For x close to one,

$$2\int_{-R}^{R} \frac{e^{i(x-1)}}{\xi} d\xi = 2\int_{-R(x-1)}^{R(x-1)} \frac{e^{ix\xi}}{\xi} d\xi.$$

This time take a sequence $x_n = 1 + \frac{K}{R_n}$, $R_n \to \infty$. This will exhibit the same type of Gibbs phenomenon as is exhibited for the Schrödinger equation.

Remark: Let $R_n = \frac{1}{t_n}$. Then $(S_{R_n} f)(1+\frac{C}{R_n})$ approaches a constant, as does $(e^{i\Delta/R_n} f)(1+\frac{C}{R_n^{1/2}})$. So x_n can approach 1 faster in the case of pointwise Fourier inversion than in the case of the Schrödinger equation. This phenomenon is not unique to this particular example.

2.5. The Pinsky Phenomenon

The Pinsky phenomenon is perhaps the most intuitively obvious in the case of the wave equation, where it arises as a perfect focus caustic. The wave front set of a function, $WF(u_0)$, is important to understanding the formation of caustics for the wave equation.

DEFINITION 2.5.1. Choose $(x_0, \xi_0) \in \mathbf{R}^n \times \mathbf{R}^N \setminus \{0\}$. If there exists $\beta \in C_0^\infty$ such that $\beta \equiv 1$ in a neighborhood of x_0, and for ξ in some conic neighborhood of ξ_0,

$$|\widehat{\beta u_0}(\xi)| \leq C_M(1+|\xi|)^{-M}, \tag{2.5.1}$$

for each $M \in \mathbf{Z}$, then $(x_0, \xi_0) \notin WF(u_0)$. Let $B(x_0)$ be the collection of all $\beta \in C_0^\infty(\mathbf{R}^n)$ such that $\beta \equiv 1$ near x_0.

$$WF(u_0) = \bigcup_{x_0 \in \mathbf{R}^n} \bigcap_{B(x_0)} (\mathbf{R}^n \times \mathbf{R}^N \setminus \{0\}) \setminus \{(x_0, \xi) : |\widehat{\beta u_0}(\xi)| \leq C_M(1+|\xi|)^{-M}\}. \tag{2.5.2}$$

For example, if $u_0 = \delta_0$, then $WF(u_0) = (0, \mathbf{R}^N \setminus \{0\})$. If $u_0 = \chi_{B(0;1)}$,

$$WF(u_0) = \{(x, \xi) : x \in S^{n-1}, \xi \cdot x = \pm|\xi|\}.$$

Definition 2.5.2. The set of singularities of u is the projection of $WF(u) \subset \mathbf{R}^n \times \mathbf{R}^N \setminus \{0\}$ onto $\mathbf{R}^n$.

$$Sing(u) = \{x : \exists (x, \xi) \in WF(u)\}. \tag{2.5.3}$$

Let $u(t, x)$ solve the wave equation.

$$\begin{aligned} u_{tt} - \Delta u &= 0, \\ u(0, x) &= u_0, \\ u_t(0, x) &= 0. \end{aligned} \tag{2.5.4}$$

The singularities at (x_0, ξ_0) will flow in the direction ξ_0 at velocity one. Let $u_0 = \chi_{B(0;1)}$. For t small, $Sing(u(0)) \mapsto Sing(u(t))$ is a 1-1 differentiable mapping. But at t = 1 the mapping is no longer 1-1, since the flow under $\xi_0 \cdot x = -|\xi_0|$ will map S^{n-1} to the origin. This produces a perfect focus caustic.

Remark: When inverting the Fourier transform of $u_0 = \chi_{B(0;1)}$, the Fourier transform will converge pointwise at the origin when $n \leq 3$. For $n = 3$ $S_R u_0 - u_0$ is a bounded, oscillatory function of R. When $n > 3$, $|S_R u_0 - u_0| = O(R^{(n-3)/2})$. The connection between the analysis of pointwise Fourier inversion and the formation of caustics can be found in [**29**].

LEMMA 2.5.3. *Let* $u_0 = \chi_{B(0;1)}$ *in* $\mathbf{R}^{2k+1}$. *As* $R \to \infty$,

$$S_R \chi_{B(0;1)} = O(R^{k-1}). \tag{2.5.5}$$

Proof: Take the spherical averages about a point

$$\bar{f}_x(|t|) = \int_{S^{n-1}} f(x + |t|\sigma) d\sigma, \tag{2.5.6}$$

$$\cos t\sqrt{-\Delta} f(x) = C_k t (\frac{1}{t}\frac{\partial}{\partial t})^k (t^{2k-1} \bar{f}_x(|t|)). \tag{2.5.7}$$

Then the pointwise Fourier inversion operator is defined:

$$S_R f(x) = \int_{|\xi| \leq R} \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \tag{2.5.8}$$

$$
(2.5.9) \qquad \begin{aligned} S_R f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(Rt)}{t} u(t,x)dt, \\ u(t,x) &= \cos(t\sqrt{-\Delta})f(x), \end{aligned}
$$

$$
S_R f(0) = C_k \int_{-\infty}^{\infty} \frac{\sin(Rt)}{t} t(\frac{1}{t}\frac{\partial}{\partial t})^k (t^{2k-1}\bar{f}_x(|t|))dt.
$$

Suppose $f(x) = \chi_{B(0;1)}$ in $\mathbf{R}^{2k+1}$. $u(t,x)$ is symmetric about t = 0.

$$
S_R f(0) = 2C_k \int_0^{\infty} \sin(Rt) t^{k-1} \delta_{t=1}^{(k-1)}(t) + l.o.t.
$$

$$
S_R f(0) = 2C_k R^{k-1} \begin{cases} (-1)^{(k-1)/2}\sin(R), & \text{k is odd;} \\ (-1)^{(k-2)/2}\cos(R), & \text{k is even.} \end{cases} + l.o.t.
$$

A similar calculation can be made for $n = 2k$ $\square$.

A similar phenomenon also arises for the Schrödinger operator $e^{it\Delta}$ as $t \searrow 0$. However in this case $e^{it\Delta}u_0 - u_0$ is oscillatory for $n = 2$ and blows up for $n \geq 3$.

$$
(2.5.10) \qquad \begin{aligned} e^{it\Delta}u_0(0) &= \frac{C(n)}{t^{n/2}} \int_{B(0;1)} e^{i|y|^2/t} dy = \frac{C(n)}{t^{n/2}} \int_0^1 r^{n-1} e^{ir^2/t} dr \\ &= \frac{C(n)}{t^{n/2}} \int_0^1 r^{(n-2)/2} e^{ir/t} dr = \frac{C(n)}{t^{n/2-1}} e^{i/t} - \begin{cases} C(n), & \text{if n = 2;} \\ 0, & \text{if } n > 2. \end{cases} + l.o.t. \end{aligned}
$$

The similarities between (2.5.5) as $R \to \infty$ and (2.5.10) as $t \searrow 0$ are evident. For simplicity take $n = 2k$.

$$
e^{it\Delta}f(x) = \frac{C(n)}{t^{n/2}} \int e^{i|x-y|^2/t} f(y)dy = \int_0^{\infty} R^{n-1} e^{iR^2/t} \bar{f}_x(R)dR.
$$

(Make a change of variables $u = R^2$)

$$= C_k \int_0^\infty (\frac{d}{du})^k (e^{iu/t}) u^{k-1} \bar{f}_x(\sqrt{u}) du$$

$$= C_k \int_0^\infty e^{iu/t} (\frac{d}{du})^k (u^{k-1} \bar{f}_x(\sqrt{u})) du + f(x)$$

$$= C_k \int_0^\infty e^{iR^2/t} (\frac{1}{R}\frac{d}{dR})^k (R^{2k-2} \bar{f}_x(R)) R dR + f(x).$$

Lemma 2.5.4. *If*

$$u_0 = \begin{cases} 1, & \text{if } |x| < 1; \\ 2 - |x|, & \text{if } 1 \le |x| \le 2; \\ 0, & \text{if } |x| > 2. \end{cases},$$

$$\lim_{t \to 0} \|e^{it\Delta} u_0 - u_0\|_{L^\infty(|x| > 1/2)} = 0 \tag{2.5.11}$$

But at the origin

$$|e^{it\Delta} u_0| \sim t^{-n/2+2}.$$

*This lemma is in response to a question asked during my defense by Mark Williams.

Proof: By Theorem [2.1.6], since $u_0 \in H^{3/2-\epsilon,2}(\mathbf{R}^n)$, (4.8.15) is immediate. In fact, $\|e^{it\Delta} u_0 - u_0\|_{L^\infty(|x|>1/2)} \lesssim t^{1/3-\epsilon}$. However, to compute the focusing at the origin,

$$t^{-n/2} \int_0^2 \frac{t}{2ir} r^{n-1} (\frac{d}{dr} e^{ir^2/t}) f(r) dr = -t^{-n/2} \int_0^2 \frac{-t}{2i} \frac{d}{dr} (f(r) r^{n-2}) e^{ir^2/t} dr,$$

$$= -C_1 t^{-n/2+1} \int_1^2 r^{n-2} e^{ir^2/t} dr + C_2 t^{-n/2+1} \int_0^2 r^{n-3} e^{ir^2/t} f(r) dr,$$

$$= -C_1 t^{-n/2+2} r^{n-3} e^{ir^2/t}|_1^2 + C_2 t^{-n/2+1} \int_0^2 r^{n-3} e^{ir^2/t} f(r) dr,$$

$$\sim t^{-n/2+2} [2^{n-3} e^{4i/t} - e^{i/t}] + O(t^{-n/2+3}).$$

So it is possible to have a Pinsky phenomenon without a Gibbs phenomenon □.

2.6. Pointwise Fourier Inversion: a Schrödinger Equation approach

The close correspondence between the asymptotics of the Schrödinger equation and for pointwise Fourier inversion for $\chi_{B(0;1)}$ was established in the previous section. This connection can be extended to more general situations.

Theorem 2.6.1. *Let f(x) be a compactly supported function, and suppose f(x) is* C^∞ *in some neighborhood of* x_0. *If*

$$S_R f(x_0) = f(x_0) + \sum_{k=1}^{n} c_k e^{iRt_k} R^\alpha + O(R^{\alpha-1/2}), \tag{2.6.1}$$

as $R \to \infty$, *then*

$$e^{it\Delta} f(x_0) = f(x_0) + O(t^{-\alpha-1/2}). \tag{2.6.2}$$

Proof: The solution to Schrödinger equation can be expressed as a superposition of the solutions of wave equations.

$$\int_{-\infty}^{\infty} e^{-ix^2} e^{-2ixt|\xi|} e^{-it^2|\xi|^2} dx = C$$

This constant is independent of ξ.

$$Ce^{it^2|\xi|^2} = e^{it^2|\xi|^2} \int_{-\infty}^{\infty} e^{-ix^2} e^{-2ixt|\xi|} e^{-it^2|\xi|^2} dx$$

$$= \int_{-\infty}^{\infty} e^{-ix^2} e^{-2ixt|\xi|} dx = \int_{0}^{\infty} e^{-ix^2} \cos(2xt|\xi|) dx$$

Since f is smooth in a neighborhood of x_0, f can be modified by a C_0^∞ function h so that $f(x) - h(x) = 0$ in a neighborhood of x_0. Since h is a C_0^∞ function, $h \in \mathcal{S}(\mathbf{R}^n)$.

Suppose $f(x) - h(x)$ is supported on $\frac{2}{C} \leq |x| \leq \frac{C}{2}$. Let $\chi \in C_0^\infty$ be a smooth cutoff such that $\chi \equiv 1$ on $\frac{1}{C} \leq |x| \leq C$ and χ is supported on $\frac{1}{2C} \leq |x| \leq 2C$. Set

(2.6.3) $$\chi_t(x) = \chi(xt).$$

For now, suppose

(2.6.4)
$$\int_0^\infty \frac{\sin(Ry)}{y}(1-\chi(y))u(y,x)dy + \int_0^\infty \frac{\sin(Ry)}{y}\chi(y)u(y,x)dy = \sum_{k=1}^{n} c_k e^{iRt_k} R^\alpha + O(R^{\alpha-1}),$$

$u(y,x) = u(-y,x)$, so

$$\int_{-\infty}^{\infty} \cos(Ry)\frac{\chi(y)}{y}u(y,x_0)dy = 0.$$

Take the Fourier transform of

$$\frac{\chi(y)}{y}u(y,x_0),$$

$$\int_{-\infty}^{\infty} \frac{\sin(Ry)}{y}(1-\chi(y))u(y,x_0)dy + \int_{-\infty}^{\infty} \sin(Ry)\frac{\chi(y)}{y}u(y,x_0)dy = CS_Rf(x_0),$$

for some constant C.

$$\frac{\chi(y)}{y}u(y,x_0) = \int_0^{\infty} S_Rf(x_0)\sin(Ry)dR - \frac{1-\chi(y)}{y}u(y,x_0).$$

Introduce another cutoff function $\eta(y) \in C_0^{\infty}$, η is supported on $\frac{1}{4C} \leq |y| \leq 4C$ and $\eta(y) = y$ on the support of $\chi(y)$, $\frac{1}{2C} \leq |y| \leq 2C$.

$$\int_0^{\infty} e^{-iy^2}\frac{\chi(2ty)\eta(2ty)}{2ty}u(2ty,x_0)dy = \frac{1}{2t}\int_0^{\infty} e^{-iy^2/4t^2}\frac{\chi(y)}{y}\eta(y)u(y,x_0)dy$$

$$= \frac{1}{2t}\int_0^{\infty} e^{-iy^2/4t^2}\eta(y)\int_0^{\infty} S_Rf(x_0)\sin(Ry)dRdy$$

$$-\frac{1}{2t}\int_0^{\infty} e^{-iy^2/4t}\eta(y)\frac{(1-\chi(y))}{y}dy$$

$$= \sum_{k=1}^{n}\frac{1}{2t}\int_0^{\infty} c_k e^{iRt_k}\int_0^{\infty} e^{-iy^2/4t^2}\sin(Ry)R^{\alpha}\eta(y)dydR,$$

$$+\frac{1}{2t}\int_0^{\infty}\int_0^{\infty} e^{-iy^2/4t^2}\sin(Ry)\eta(y)O(R^{\alpha-1})dydR,$$

$$-\frac{1}{2t}\int_0^\infty e^{-iy^2/4t}\eta(y)\frac{(1-\chi(y))}{y}dy.$$

Assume that the quantity is bounded by

$$\frac{1}{t}\int_0^\infty e^{-iy^2/4t^2}\sin(Ry)\eta(y)dy = e^{iR^2t^2}\eta(2Rt^2)+O(t) \tag{2.6.5}$$

when $\frac{1}{10Ct^2}\leq R\leq\frac{10C}{t^2}$, and bounded by and is $O(R^{-N}t^N)$ when $R\notin[\frac{1}{10Ct^2},\frac{10C}{t^2}]$.

$$\int_0^\infty c_k e^{iR^2t^2}e^{iRt_k}\eta(2Rt^2)R^\alpha dR = O(t^{-2\alpha-1}),$$

$$\int_{c_1/t^2}^{c_2/t^2} R^\alpha t dR = O(t^{-2\alpha-1}),$$

$$\int_{\frac{1}{10Ct^2}}^{\frac{10C}{t^2}} O(R^{\alpha-1/2})dR = O(t^{-2\alpha-1}).$$

This proves the theorem, assuming Lemma [2.6.2] is true $\square$.

LEMMA 2.6.2. *Suppose $u(t,x_0)$ is the function just described. Then*

$$\int_0^\infty(1-\chi(t))\frac{sin(Rt)}{t}u(t,x_0)dt = O(R^{-N}), \tag{2.6.6}$$

$$\int_0^\infty e^{iy^2/t^2}(1-\frac{\chi(y)\eta(y)}{y})u(y,x_0)dy = O(t^N), \tag{2.6.7}$$

$$(2.6.8) \qquad \frac{1}{t}\int_0^\infty e^{-iy^2/4t^2}\sin(Ry)\eta(y)dy = e^{iR^2t^2}\eta(2Rt) + O(t),$$

when $\frac{1}{10Ct^2} \leq R \leq \frac{10C}{t^2}$, *and*

$$(2.6.9) \qquad \frac{1}{t}\int_0^\infty e^{-iy^2/4t^2}\sin(Ry)\eta(y)dy = O(t^N R^{-N/2})$$

outside this region.

Proof: When $|t| \leq \frac{1}{C}$, by Huygens principle, $u(t, x_0) = 0$ in even or odd dimensions. When n is odd, $u(t, x_0) = 0$ for $|t| > \frac{C}{2}$, which takes care of (2.6.6), (2.6.7). When $n = 2k$,

$$u(t, 0) = \frac{1}{t}(\frac{\partial}{\partial t}\frac{1}{t})^{k-1} t^{n-1}\int_{|y|\leq 1}\frac{f(ty)}{\sqrt{1-|y|^2}}dy.$$

For $|t| \geq C$,

$$\int_{|y|\leq 1}\frac{f(ty)}{\sqrt{1-|y|^2}}dy = \frac{1}{t^n}\int_{|y|\leq t}\frac{f(y)}{\sqrt{1-\frac{|y|^2}{t^2}}}dy = \frac{1}{t^n}\int_{|y|\leq C/2}\frac{f(y)}{\sqrt{1-\frac{|y|^2}{t^2}}}dy.$$

$$(\frac{\partial}{\partial t})^N \frac{1}{t^n}\int_{|y|\leq C/2}\frac{f(y)}{\sqrt{1-\frac{|y|^2}{t^2}}}dy \lesssim t^{-n-N}.$$

Apply integration by parts.

$$\int_C^\infty (1-\chi(t))\frac{sin(Rt)}{t}u(t, x_0)dt = \int_C^\infty (1-\chi(t))\frac{u(t, x_0)}{t}\frac{1}{R^{4N}}(\frac{\partial}{\partial t})^{4N} sin(Rt)dt,$$

$$= R^{-4N}\int_C^\infty \sin(Rt)(\frac{\partial}{\partial t})^{4N}((1-\chi(t))\frac{u(t, x_0)}{t})dt \lesssim O(R^{-4N}).$$

Similarly for the second integral, define the operator $\mathcal{L}f = \frac{\partial}{\partial y}(\frac{1}{2iy}f)$,

$$\int_C^\infty e^{iy^2/t^2}(1 - \frac{\chi(y)\eta(y)}{y})u(y,x_0)dy,$$
$$= (-1)^N t^{2N}\int_C^\infty e^{iy^2/t^2}\mathcal{L}^N((1 - \frac{\chi(y)\eta(y)}{y})u(y,x_0))dy \lesssim t^{2N}.$$

Now for the third integral:

$$\int_{-\infty}^\infty \eta(y)e^{-iy^2/4t^2}e^{iRy}dy = e^{-iR^2t^2/4}\int_{-\infty}^\infty \eta(y)e^{-i(y/t-Rt/2)^2}.$$

This has a root $R = 2y/t^2$. Take $\frac{1}{10Ct^2} \leq R \leq \frac{10C}{t^2}$.

$$e^{-iR^2t^2/4}\int_{-\infty}^\infty \eta(\frac{Rt^2}{2})e^{-i(y/t-Rt/2)^2}dy = Ce^{-iR^2t^2/4}\eta(\frac{Rt^2}{2}).$$

The remainder can be estimated by Taylor's inequality.

$$\eta(ty) - \eta(\frac{Rt^2}{2}) = (ty - \frac{Rt^2}{2})\int_0^1 \eta'(\frac{Rt^2}{2} + \tau(ty - \frac{Rt^2}{2}))d\tau.$$

By a change of variables,

$$\int_{-\infty}^\infty \eta(y)e^{-i(y/t-Rt/2)^2} = t\int_{-\infty}^\infty \eta(ty)e^{-i(y-Rt/2)^2}dy.$$

Let $u = (y - Rt/2)^2$.

$$t\int_{-\infty}^\infty [\eta(ty) - \eta(\frac{Rt^2}{2})]e^{-i(y-Rt/2)^2}dy$$
$$= \frac{t^2}{2}\int_{-\infty}^\infty \frac{d}{dy}(e^{i(y-Rt/2)^2})\int_0^1 \eta'(\frac{Rt^2}{2} + \tau(ty - \frac{Rt^2}{2}))d\tau dy,$$

$$= t^3\int_{-\infty}^\infty e^{i(y-Rt/2)^2}\int_0^1 \tau\eta''(\frac{Rt^2}{2} + \tau(ty - \frac{Rt^2}{2}))d\tau dy \lesssim O(t^2)$$

When $R \geq \frac{10C}{t^2}$, $\eta(y)$ is supported on $\frac{1}{4C} \leq y \leq 4C$,

$$\frac{1}{t}\int_{1/4C}^{4C} e^{-i(y/2t-tR)^2}\eta(y)dy =$$

$$\frac{1}{t}\int_{1/4C}^{4C} (\frac{1}{2i(y/2t-Rt)}\frac{\partial}{\partial y})^N e^{-i(y/2t-tR)^2}\eta(y)dy;$$

$$= \frac{1}{t}\int_{1/4C}^{4C} e^{-i(y/2t-tR)^2}(\frac{\partial}{\partial y}\frac{1}{y/2t-Rt})^N\eta(y)dy = O(t^N R^{-N/2}).$$

When $R \geq \frac{10C}{t^2}$, $Rt-\frac{y}{2t} \geq \frac{R}{2}t+\frac{C}{2t} \geq \sqrt{RC}+\frac{C}{2t}$. Similarly, if $R \leq \frac{1}{10Ct^2}$, $y/2t-Rt \geq \frac{1}{2Ct}+R$, and again integrate by parts.

N can be arbitrarily large $\Box$.

Thus if the convergence of $S_R f(x_0)$ is $O(R^{-\alpha})$ then the convergence of $e^{it\Delta}f(x_0)$ is no worse than $O(R^{-\alpha+1/2})$, where $t=\frac{1}{R}$.

A lemma will be needed to go in the converse direction.

LEMMA 2.6.3.

$$\int_{1/2}^{M} e^{iR(x^2+\frac{1}{x^2})}dx = O(R^{-1/2}), \tag{2.6.10}$$

for any $M \in [1/2,\infty]$.

Proof:

$$2(2.6.10) = \int_{1/2}^{M} e^{iR(x^2+x^{-2})}(1+\frac{1}{x^2})dx + \int_{1/2}^{M} e^{iR(x^2+x^{-2})}(1-\frac{1}{x^2})dx$$

$$\int_{1/2}^{M} e^{iR(x^2+x^{-2})}(1-\frac{1}{x^2})dx = \int_{1/2}^{M} \frac{1}{2iR}\frac{(1-x^{-2})}{(x-x^{-3})}\frac{d}{dx}(e^{iR(x^2+x^{-2})})dx$$

$$= \frac{1}{2iR}\frac{1}{x}(e^{iR(x^2+x^{-2})})|_{1/2}^{M} - \int_{1/2}^{M} \frac{1}{2iR} x^{-2} e^{iR(x^2+x^{-2})} dx = O(R^{-1}).$$

Estimate the first integral by a change of variables.

$$\int_{1/2}^{M} e^{iR(x^2+x^{-2})}(1+x^{-2})dx$$

$$= e^{2iR} \int_{1/2}^{M} e^{iR(x-1/x)^2}(1+x^{-2})dx = \int_{-3/2}^{M-1/M} e^{iRu^2} du = O(R^{-1/2}).$$

This proves the lemma □.

The converse will utilize the expression of a wave operator as a superposition of Schrödinger operators. Combining (2.3.11) with the fact that $\mathcal{F}(\Delta f) = -|\xi|^2 \hat{f}(\xi)$,

$$\begin{aligned} 2S_R f(x) =& e^{-\pi i/4} \int_0^\infty \frac{\sin(Rt)}{t} \int_0^\infty e^{iu^2} e^{-i\Delta t^2/4u^2} f(x) du dt \\ &+ e^{\pi i/4} \int_0^\infty \frac{\sin(Rt)}{t} \int_0^\infty e^{-iu^2} e^{i\Delta t^2/4u^2} f(x) du dt. \end{aligned} \tag{2.6.11}$$

THEOREM 2.6.4. *Suppose the Schrödinger equation has the asymptotic expansion*

$$e^{it\Delta} f(x) = f(x) + O(t^{\alpha}) e^{i\beta/t} + O(t^{\alpha+1/2}). \tag{2.6.12}$$

Additionally suppose that $f(x)$ is smooth in a neighborhood of x_0 and is compactly supported. Then there is the pointwise Fourier convergence,

$$S_R f(x) = f(x) + O(R^{-1/2-\alpha}). \tag{2.6.13}$$

Proof: As in the other case, let $h(x) \in C_0^\infty$, $h(x) = f(x)$ in a neighborhood of x_0. Subtract off $h(x)$ so that $f(x_0) - h(x_0) = 0$. Write the integral in polar coordinates. Let $t = cu$.

$$r^2 = t^2 + u^2 = (c^2+1)u^2.$$

For a fixed θ, c is fixed.

$$rdr = (c^2+1)udu.$$

$$\int_0^\infty f(r,\theta)rdr = (c(\theta)^2+1)\int_0^\infty f(c(\theta),u)udu.$$

$$dc = \sec^2\theta d\theta,$$

$$d\theta = \cos^2(\theta)dc = \frac{dc}{1+c^2}.$$

By (2.6.4),

$$\int_0^\infty \frac{\sin(Rt)}{t}u(t,x_0)dt = O(R^{-N}) + \int_0^\infty \frac{\sin(Rt)}{t}\chi(t)u(t,x_0)dt.$$

So without loss of generality evaluate the integral.

$$\int_0^\infty\int_0^\infty \chi(cu)\frac{\sin(Rcu)}{c}e^{iu^2}(e^{ic^2\Delta/4}f)(x)dudc. \tag{2.6.14}$$

The free Schrödinger operator is a convolution operator

$$S_R f(x) = (K_{1,R}(x,y) * f)(x) + (K_{2,R}(x,y) * f)(x) + (K_{3,R}(x,y) * f)(x)$$

$$K_{1,R}(x,y) = \int_{\kappa R^{-1/2}}^{\infty} \chi(cu) \int_0^{\infty} \frac{\sin(Rcu)}{c^{n+1}} e^{iu^2} e^{-i|x-y|^2/c^2} du dc \tag{2.6.15}$$

$$K_{2,R}(x,y) = \int_{\delta R^{-1/2}/2}^{\kappa R^{-1/2}} \int_0^{\infty} \chi(cu) \frac{\sin(Rcu)}{c^{n+1}} e^{iu^2} e^{-i|x-y|^2/c^2} du dc \tag{2.6.16}$$

$$K_{3,R}(x,y) = \int_0^{\delta R^{-1/2}/2} \int_0^{\infty} \chi(cu) \frac{\sin(Rcu)}{c^{n+1}} e^{iu^2} e^{-i|x-y|^2/c^2} du dc, \tag{2.6.17}$$

for some κ. Consider the first integral. Take a C^∞ cutoff,

$$\psi(x) = \begin{cases} 1, & x \geq 1/C; \\ 0, & x < 0. \end{cases}$$

Integrating by parts,

$$\int_0^{\infty} \psi(cu) e^{iu^2} e^{iRcu} du \lesssim R^{-N},$$

$$\int_{\kappa R^{-1/2}}^{\infty} R^{-N} \frac{1}{c^{n+1}} dc = O(R^{-N}).$$

When $c > \kappa R^{-1/2}$,

$$\int_{-\infty}^{\infty} \psi(cu) e^{iu^2} e^{-iRcu} du = \sqrt{\pi} e^{i\pi/4} e^{-iR^2c^2/4} + O(R^{-N}),$$

$$\frac{d}{dc}[\frac{R^2c^2}{4}] + \frac{d}{dc}[\frac{|x-y|^2}{c^2}] = \frac{R^2c}{2} - \frac{2|x-y|^2}{c^3}.$$

It suffices to take η a C^∞ cutoff, $\eta \equiv 0$ on $[0, \kappa/2)$ and $\eta \equiv 1$ on $[\kappa, \infty)$, and evaluate the integral

$$\int_0^\infty \eta(R^{-1/2}c)c^{-n-1}e^{-iR^2c^2/4}e^{-i|x-y|^2/c^2}dc.$$

Since $f(x)$ is compactly supported, $|x-y|$ is bounded, so when κ is sufficiently large.

$$\frac{R^2c}{2} - \frac{2|x-y|^2}{c^3} \sim R^{3/2}.$$

Then applying integration by parts,

$$\int_0^\infty \eta(R^{-1/2}c)c^{-n-1}e^{-iR^2c^2/4}e^{-i|x-y|^2/c^2}dc = O(R^{-N}).$$

This gives an L^∞ bound on the kernel.

(2.6.18) $$\|K_1(x,y)\|_\infty \le O(R^{-N}),$$

(2.6.19) $$\|K_1 * f\|_\infty \le O(R^{-N}),$$

when $f \in L^1(\mathbf{R}^n)$.

Next analyze the third piece. When c is small, $\chi(cu) = 0$ unless u is very large. When $c < \delta R^{-1/2}$ for some δ, $(u - Rc/2)^2$ has the root $u = \frac{Rc}{2}$, and $cu = \frac{Rc^2}{2} < \frac{\delta^2}{2}$. For δ sufficiently small the integral can be regularized:

$$\int_0^\infty e^{iu^2}e^{iRcu}\chi(cu)du = e^{-iR^2c^2/4}\int_0^\infty e^{i(u-Rc/2)^2}\chi(cu)du$$

$$= \int_0^\infty (\frac{1}{2u-Rc}\frac{\partial}{\partial u})^N e^{i(u-Rc/2)^2}\chi(cu)du$$

($u \sim c^{-1}$ on the support of $\chi(cu)$)

$$= \int_0^\infty e^{i(u-Rc/2)^2}(\frac{\partial}{\partial u}\frac{1}{2u-Rc})^N \chi(cu)du \lesssim c^N,$$

for any N. Then integrate:

$$\int_0^{\delta R^{-1/2}} c^N dc = O(R^{N/2}).$$

Thus (2.6.18) and (2.6.19) hold for $K_3(x,y)$.

Finally, analyze the second piece. Recall that the convergence of

$$e^{it\Delta}(f-h) = Ce^{i\beta/t}t^{-\alpha} + O(t^{-\alpha+1/2}), \tag{2.6.20}$$

as $t \searrow 0$.

$$e^{-ic^2\Delta/4}(f-h) = Ce^{-4i\beta/c^2}c^{-2\alpha} + O(c^{-2\alpha+1}).$$

Evaluate the integral

$$\int_{\delta R^{-1/2}}^{\kappa R^{-1/2}}\int_0^\infty \sin(Rcu)e^{iu^2}e^{i\alpha/c^2}c^{-2\beta}dudc.$$

$$\int_{\delta R^{-1/2}}^{\kappa R^{-1/2}}\int_0^\infty \chi(cu)\sin(Rcu)e^{iu^2}e^{-4i\beta/c^2}c^{-2\alpha}dudc$$
$$\sim R^\alpha \int_{\delta R^{-1/2}}^{\kappa R^{-1/2}}\int_0^\infty \chi(cu)\sin(Rcu)e^{iu^2}e^{-4i\beta/c^2}dudc$$

$$\sim R^{\alpha}\int_{\delta R^{-1/2}}^{\kappa R^{-1/2}} e^{-iR^2c^2/4}e^{-4i\beta/c^2}dc = R^{\alpha-1/2}\int_{\delta}^{\kappa} e^{-iRc^2/4}e^{-4i\beta/c^2}dc.$$

After a change of variables

$$\int_{\delta}^{0.5} e^{iR(c^2+\frac{1}{c^2})}dc = O(R^{-1}),$$

$$\int_{1.5}^{\kappa} e^{iR(c^2+\frac{1}{c^2})}dc = O(R^{-1}),$$

$$\int_{0.5}^{1.5} e^{iR(c^2+c^{-2})}dc = O(R^{-1/2}).$$

The first two identities follow from taking the derivative of the phase function. The third identity follows from Lemma [2.6.3]. This completes the proof in the opposite direction □.

Remark: If $f(x)$ has only jump discontinuities, is compactly supported, and f is smooth in a neighborhood of x_0, then $f(x)$ satisfies the hypotheses for Theorem [2.6.1] and Theorem [2.6.4].

2.7. A Manifold with corners

Now consider $e^{it\Delta}\chi_{\Omega}$, where Ω is a more general region. The methods of the Schrodinger operator can be carried over to simplicies, complexes, or manifolds with corners.

DEFINITION 2.7.1. In $\mathbf{R}^2$ a two dimensional manifold with corners is a two dimensional manifold with boundary, $\Omega \subset \mathbf{R}^2$, where $\partial\Omega = \cup_{i=1}^{N}\gamma_i$, where $\gamma_i : [0,1] \to \mathbf{R}^2$ are smooth on [0,1].

Inductively, define $\Omega \subset \mathbf{R}^m$ to be an n-dimensional manifold with corners if Ω is n-dimensional and $\partial\Omega = \cup_{i=1}^N K_i$, where K_i is an n - 1 dimensional manifold with corners embedded in $\mathbf{R}^m$.

Although the manifold does not have smooth boundary, and thus the normal vector field will not vary smoothly along $\partial\Omega$, $\partial\Omega$ is a finite union of smooth pieces, like a soccer ball.

There is a maximum sectional curvature for the manifold Ω that has corners. Call this curvature κ. Take a smooth cutoff

$$\chi(x_1)\cdots\chi(x_n),$$

$\chi \in C_0^\infty$, $\chi \equiv 1$ for $|x| < \frac{1}{4\kappa}$ and $\chi = 0$ for $|x| > \frac{1}{2\kappa}$. Define

$$\eta(x) = \chi(x_1)\cdots\chi(x_n).$$

A lemma will be needed.

Lemma 2.7.2. *Suppose* $q = k - (\alpha_1 + ... + \alpha_l)$. *Then*

$$\int_{\mathbf{R}^{n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q} dx = O(1) \tag{2.7.1}$$

Proof: Let $\epsilon = \min\{\epsilon_1, ..., \epsilon_l\}$ and make the change of variables $x \mapsto \epsilon x$.

$$(2.7.1) = \int_0^\infty r^{n-l-1} r^{-2q} \frac{\epsilon_1}{(\epsilon_1^2 + r^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + r^2)^{\alpha_l}} dr$$

$$= \int_0^\infty r^{2(\alpha_1 + ... + \alpha_l) - l - 1} \frac{\epsilon_1}{(\epsilon_1^2 + r^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + r^2)^{\alpha_l}} dr$$

$$= \int_0^\infty \frac{(\epsilon_1/\epsilon)}{((\epsilon_1/\epsilon)^2 + r^2)^{\alpha_1}} \cdots \frac{(\epsilon_l/\epsilon)}{((\epsilon_l/\epsilon)^2 + r^2)^{\alpha_l}} r^{2(\alpha_1 + \ldots + \alpha_l) - l - 1} dr.$$

Now suppose without loss of generality $1 = \epsilon_1 < \epsilon_2 < \ldots < \epsilon_l$.

$$\int_{\epsilon_j}^{\epsilon_{j+1}} \frac{\epsilon_1 \cdots \epsilon_l}{r^{2\alpha_1 + \ldots + 2\alpha_j}} \frac{r^{2\alpha - l - 1}}{\epsilon_{j+1}^{2\alpha_{j+1}} \cdots \epsilon_l^{2\alpha_l}} dr$$

$$\frac{\epsilon_1 \cdots \epsilon_l}{\epsilon_{j+1}^{2\alpha_{j+1}} \cdots \epsilon_l^{2\alpha_l}} \epsilon_j^{2\alpha_{j+1} + \ldots + 2\alpha_l - l}$$

$$= \epsilon_1 \cdots \epsilon_j \epsilon_{j+1}^{-2\alpha_{j+1}+1} \cdots \epsilon_l^{-2\alpha_l + 1} \epsilon_j^{-j} \epsilon_j^{2\alpha_{j+1} - 1} \cdots \epsilon_j^{2\alpha_l - 1}.$$

This quantity is uniformly bounded. As is

$$= \epsilon_1 \cdots \epsilon_j \epsilon_{j+1}^{-2\alpha_{j+1}+1} \cdots \epsilon_l^{-2\alpha_l + 1} \epsilon_{j+1}^{-j} \epsilon_{j+1}^{2\alpha_{j+1} - 1} \cdots \epsilon_{j+1}^{2\alpha_l - 1},$$

$$\int_{\epsilon_l}^\infty \epsilon_1 \cdots \epsilon_l r^{-l-1} dr = O(1)$$

as long as $l > 0$. Since the integrand is uniformly bounded on $B(0;1)$, this completes the proof $\square$.

THEOREM 2.7.3. *There exists a constant $0 < C < \infty$ such that*

$$|e^{it\Delta}(\eta(x - x_0)u_0)(x_0)| \leq C < \infty \tag{2.7.2}$$

for any point $x_0 \in \mathbf{R}^n$, $t \in (0, \infty)$, $u_0 = \chi_\Omega$.

Without loss of generality make a translation such that $x_0 = 0$.

$$(2.7.3) \qquad e^{it\Delta}(\eta u_0)(0) = t^{-n/2} \int_{\Omega} e^{i|x|^2/t} \eta(x) dx.$$

Define the operator

$$(2.7.4) \qquad \mathcal{L} = \sum_{i=1}^{n} \frac{t}{2i} \frac{x_i}{|x|^2} \frac{\partial}{\partial x_i}.$$

$$\int \mathcal{L}(e^{i|x|^2/t}) dx = \int e^{i|x|^2/t} dx.$$

Divide the analysis into two cases. First suppose $t = 2k$ for some k. Also define the operator $\mathcal{R}$.

$$(2.7.5) \qquad \mathcal{R}f = \frac{t}{i} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\frac{x_j}{|x|^2} f).$$

$$(2.7.6) \qquad (2.7.3) = t^{-n/2+1} \int_{\partial\Omega} \frac{(x \cdot \overrightarrow{n})}{|x|^2} (\eta(x) d\sigma(x) + t^{-n/2+1} \int_{\Omega} \mathcal{R}(\eta(x)) dx,$$

where $d\sigma(x)$ is the measure along $\partial\Omega$. Since Ω is a manifold with corners, $\partial\Omega = \cup_{j=1}^{N_{n-1}} K_{j,n-1}$, where each $K_{j,n-1}$ is also an n - 1 dimensional region with corners, and in fact is the graph of an n - 1 dimensional subset of $\mathbf{R}^{n-1}$ with corners. Locally $K_{j,n-1}$ has the expression

$$\partial\Omega = \{(x, \epsilon_1 + f(x)) : x \in K_{j,n-1} \subset \mathbf{R}^{n-1}\},$$

(2.7.7)

$$f(0) = \nabla f(0) = 0, \|\frac{\partial^2 f}{\partial x_i \partial x_j}\| \leq \kappa.$$

Where $\|(a_{ij})\|$ denotes the norm of a matrix. Now return to (2.7.6). The integral over Ω can obviously be regularized again. On the other hand, the integral over $K_{j,n-1}$ can be regularized as well. In the case where n/2 = k, we regularize k - 1 times. For the $\mathcal{R}$ operator,

$$\int_{K_{j,n-1}} \frac{\partial}{\partial x_j}(O(|x|^{-2l-1})\frac{\epsilon}{\epsilon^2 + |x|^2})dx$$

$$= \int_{K_{j,n-1}} (O(|x|^{-2l-2})\frac{\epsilon}{\epsilon^2 + |x|^2})dx + \int_{K_{j,n-1}} O(|x|^{-2l})\frac{\epsilon}{(\epsilon^2 + |x|^2)^2}dx.$$

Each regularization involves a boundary restriction and a regularization derivative. So if the regularization operator is applied k - 1 times, there will be one term that involves an integral over Ω, and all the other terms will be some type of integral over a lower dimensional space of the form

(2.7.8)
$$\frac{1}{t}\int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q} e^{i|x|^2/t} dx,$$

where $q = k - 1 - (\alpha_1 + ... + \alpha_l)$.

$$0 \leq q \leq k - 1 - l.$$

This integral can be regularized again without fear. Suppose $\cup_{j'} K_{j',n-l-1}$ make up the boundary of $K_{j,n-l}$.

$$(2.7.8) \leq \sum_{j'} \int_{K_{j',n-l-1}} \frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}} \frac{\epsilon_{l+1}}{(\epsilon_{l+1}^2+|x|^2)} |x|^{-2q}$$

$$+ \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}} |x|^{-2q-2} dx$$

$$+ \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1+1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}} |x|^{-2q} dx +$$

$$... + \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l+1}} |x|^{-2q} dx.$$

All of these integrals will converge by Lemma [2.7.2].

Similarly let $n = 2k+1$. In this case, the integral can be regularized k times. There will be one term that involves an integral over Ω, and all the other terms will be an integral over a lower dimensional region. So all the other terms will be an integral over a lower dimensional region of the form,

$$\frac{1}{t^{1/2}} \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}} |x|^{-2q} e^{i|x|^2/t} dx, \tag{2.7.9}$$

where $q = k - (\alpha_1 + ... + \alpha_l)$.

$$0 \leq q \leq k - l$$

Define a C_0^∞ cutoff

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1; \\ 0, & \text{if } |x| > 2. \end{cases} \tag{2.7.10}$$

Now apply the estimate

$$\frac{\epsilon}{\epsilon^2 + x^2} \leq \frac{1}{|x|}.$$

(This can be seen by checking $|x| < \epsilon$ and $|x| > \epsilon$ separately).

$$t^{-1/2} \int_{\mathbf{R}^{n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q} \phi(t^{-1/2}x) dx$$

$$\lesssim t^{-1/2} \int_0^{t^{1/2}} \frac{r^{n-1}}{r^{n-1}} dr = O(1).$$

So it remains to consider

$$\text{(2.7.11)} \qquad \frac{1}{t^{1/2}} \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} (1 - \phi(t^{-1/2}x)) |x|^{-2q} e^{i|x|^2/t} dx.$$

We have

$$(2.7.11) \leq t^{1/2} \sum_{j'} \int_{K_{j',n-l-1}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} \frac{\epsilon_{l+1}}{(\epsilon_{l+1}^2 + |x|^2)} |x|^{-2q} \phi(t^{-1/2}x) dx$$

$$+ t^{1/2} \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q-2} \phi(t^{-1/2}x) dx$$

$$+ t^{1/2} \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1+1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q} \phi(t^{-1/2}x) dx +$$

$$... + t^{1/2} \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l+1}} |x|^{-2q} \phi(t^{-1/2}x) dx$$

$$+ \int_{K_{j,n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q-1} \phi'(t^{-1/2}x) dx.$$

These integrals are $O(1)$ also.

$$t^{1/2}\int_{t^{1/2}}^{\infty} r^{l+1}r^{n-l-2}\frac{1}{r^{n+1}}dr = t^{1/2}\int_{t^{1/2}}^{\infty} r^{-2}dr = O(1),$$

$$t^{1/2}\int_{t^{1/2}}^{\infty} r^{l}r^{n-l-1}\frac{1}{r^{n+1}}dr = O(1).$$

Finally $\phi'(t^{-1/2}x)$ is supported on a ball of radius $t^{1/2}$. The integrand is

$$\sim \frac{1}{r^{n-l}} \sim t^{-(n-l)/2}$$

on the support of the integrand.

Finally, the integral over Ω. When n = 2k,

$$\frac{1}{t}\int_{\Omega} O(|x|^{-2k+2})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx \leq$$

$$\sim \int_{\partial\Omega}\frac{\epsilon}{(\epsilon^2+|x|^2)^k}dx$$

$$-\int_{\Omega} O(|x|^{-2k})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx \tag{2.7.12}$$

$$-t^{-1/2}\int_{\Omega} O(|x|^{-2k+1})\phi'(t^{-1/2}x)e^{i|x|^2/t}dx.$$

The first integral is $O(1)$ by Lemma [2.7.2]. In the third integral, $|x| \sim t^{-n/2+1/2}$ on the support of $\phi'(t^{-1/2}x)$.

$$\int_{|x|\leq t^{1/2}} t^{-n/2}dx = O(1).$$

Take two separate cases for (2.7.12). If $\epsilon \leq \frac{1}{2}t^{1/2}$,

$$(2.7.12) \sim -t\int_{\partial\Omega} \frac{1}{(|x|^2+\epsilon^2)^k}\frac{\epsilon}{\epsilon^2+|x|^2}(1-\phi(2t^{-1/2}x))e^{i|x|^2/t}dx$$

$$+t\int_{\Omega} O(|x|^{-2k-2})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx + t^{1/2}\int_{\Omega} O(|x|^{-2k-1})\phi'(t^{-1/2}x)e^{i|x|^2/t}dx.$$

The first integral is an integral over $\mathbf{R}^{n-1}$ with integrand bounded by $\frac{1}{r^{n+1}}$. The second integral is an integral over $\mathbf{R}^n$ with integrand bounded by $\frac{1}{r^{n+2}}$.

$$t\int_{t^{1/2}}^{\infty} \frac{r^{n-2}}{r^{n+1}}dr = O(1),$$

$$t\int_{t^{1/2}}^{\infty} \frac{r^{n-1}}{r^{n+2}}dr = O(1).$$

Finally the last integral is supported on $B(0;t^{1/2}) \subset \mathbf{R}^n$ with integrand bounded by $t^{-n/2}$. Thus the third integral is also $O(1)$.

If $\epsilon \geq t^{1/2}$,

$$(2.7.12) \sim -t\int_{\partial\Omega} \frac{1}{(|x|^2+\epsilon^2)^k}\frac{\epsilon}{\epsilon^2+|x|^2}e^{i|x|^2/t}dx$$

$$+t\int_{\Omega} O(|x|^{-2k-2})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx + t^{1/2}\int_{\Omega} O(|x|^{-2k-1})\phi'(t^{-1/2}x)e^{i|x|^2/t}dx.$$

For the first integral,

$$\frac{1}{\epsilon^2+|x|^2} \leq \frac{1}{t}$$

Thus the integral converges by using Lemma [2.7.2]. The other two integrals are the same as before.

Now for the case when $n = 2k+1$.

$$t^{-1/2}\int_{\Omega} O(|x|^{-2k})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx$$

If $\epsilon \le t^{1/2}$,

$$(2.7.12) \sim t^{1/2}\int_{\partial\Omega} \frac{1}{(\epsilon^2+|x|^2)^k}\frac{\epsilon}{\epsilon^2+|x|^2}(1-\phi(2t^{-1/2}x))e^{i|x|^2/t}dx$$

$$-t^{1/2}\int_{\Omega} O(|x|^{-2k-2})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx - \int_{\Omega} O(|x|^{-2k-1})\phi'(t^{-1/2}x))e^{i|x|^2/t}dx$$

The first integral is of the form

$$t^{1/2}\int_{t^{1/2}}^{\infty}\frac{r^{n-2}}{r^n}dr = O(1).$$

The second integral is

$$\le t^{1/2}\int_{t^{1/2}}^{\infty}\frac{r^{n-1}}{r^{n+1}}dr = O(1).$$

Finally the last integral is

$$\le t^{-n/2}\int_{\mathbf{R}^n:|x|\le t^{1/2}} 1dx = O(1),$$

and if $\epsilon \ge t^{1/2}$,

$$(2.7.12) \sim t^{1/2}\int_{\partial\Omega} \frac{1}{(\epsilon^2+|x|^2)^k}\frac{\epsilon}{\epsilon^2+|x|^2}e^{i|x|^2/t}dx$$

$$-t^{1/2}\int_{\Omega} O(|x|^{-2k-2})(1-\phi(t^{-1/2}x))e^{i|x|^2/t}dx - \int_{\Omega} O(|x|^{-2k-1})\phi'(t^{-1/2}x))e^{i|x|^2/t}dx.$$

In this case

$$\frac{1}{(\epsilon^2+|x|^2)^{1/2}} \leq t^{-1/2}.$$

This integral is $O(1)$ by Lemma [2.7.2].

Remark: This correspondence proved in Section 6 can carry over some of the calculations on the Gibbs phenomenon for the Schrödinger equation over to Fourier inversion.

THEOREM 2.7.4. *There exists a constant* $0 < C < \infty$ *such that*

$$|S_R(\eta(x-x_0)u_0)(x_0)| \leq C < \infty, \tag{2.7.13}$$

for any point $x_0 \in \mathbf{R}^n$, $R \in (0,\infty)$, $u_0 = \chi_\Omega$. Ω *is some manifold with corners.*

Proof: Take the cutoff $\phi(Rx)$.

$$\|\phi(R(x-x_0))u_0(x)\|_{L^1(\mathbf{R}^n)} \leq C(n)R^{-n}. \tag{2.7.14}$$

As before translate so that $x_0 = 0$. Since $\|\chi_{|\xi|\leq R}\|_{\mathbf{R}^n} \leq C(n)R^n$,

$$\|S_R\phi(R(x-x_0))u_0(x)\|_\infty \leq C < \infty. \tag{2.7.15}$$

So it remains to consider the part outside the cutoff. The presence of the cutoff requires the computation of the commutator,

$$\frac{\partial}{\partial x_j}(\frac{x_j}{|x|^2}\eta(Rx)f) - \eta(Rx)\frac{\partial}{\partial x_j}(\frac{x_j}{|x|^2}f) = R\eta'(Rx)\frac{x_j}{|x|^2}f. \tag{2.7.16}$$

First consider the case where n = 2k. After k regularizations $e^{it\Delta}\eta(x)(1-\phi(Rx))$ is a sum of terms of the form

$$\int_{\mathbf{R}^{n-l}} e^{-i|x|^2/t}\frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}}\cdots\frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}}|x|^{-2q+m}R^m\phi^{(m)}(Rx), \tag{2.7.17}$$

where $q = k - \alpha$, $\alpha = \alpha_1 + ... + \alpha_l$, and $m \leq k - l$. By the subordination identity it is necessary to consider terms of the form

$$\int_0^\infty\int_0^\infty \frac{\sin(Rcu)}{c}e^{iu^2}e^{ic^2\Delta/4}u_0(x)dcdu \tag{2.7.18}$$

$$\sim \int_0^\infty e^{-iR^2c^2/4}e^{ic^2\Delta/4}u_0(x)dc. \tag{2.7.19}$$

(2.7.19) is a sum of terms of the form

$$\begin{aligned}&\int_0^\infty\int_{\mathbf{R}^{n-l}}\frac{1}{c}e^{-iR^2c^2/4}e^{-i|x|^2/c^2}\frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}}\\&\cdots\frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}}|x|^{-2q+m}R^m\phi^{(m)}(Rx)dxdc\end{aligned} \tag{2.7.20}$$

$$\sim \int_{\mathbf{R}^{n-l}}\frac{1}{R^{1/2}|x|^{1/2}}\frac{\epsilon_1}{(\epsilon_1^2+|x|^2)^{\alpha_1}}\cdots\frac{\epsilon_l}{(\epsilon_l^2+|x|^2)^{\alpha_l}}|x|^{-2q+m}R^m\phi^{(m)}(Rx)dx. \tag{2.7.21}$$

When $|x| \sim \frac{1}{R}$, the integrand is

$$\lesssim \frac{|x|^l}{|x|^{n+1/2}}|x|^m R^m \sim R^{n-l+1/2}.$$

On the other hand $\phi^{(m)}(Rx)$ is supported on $|x| \leq \frac{2}{R}$ when $m > 0$ so for $m > 0$,

$$(2.7.21) \leq O(1)$$

Finally consider the integral when $m = 0$.

$$(2.7.21) \lesssim R^{1/2} \int_{\mathbf{R}^{n-l}:|x|\geq \frac{1}{R}} \frac{1}{|x|^{n-l+1/2}} dx = O(1)$$

which takes care of the $n = 2k$ case.

In the $n = 2k + 1$ case $e^{it\Delta}u_0$ is a sum of integrals of the form

$$t^{1/2} \int_{\mathbf{R}^{n-l}} e^{-i|x|^2/t} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q-2+m} R^m \phi^{(m)}(Rx). \tag{2.7.22}$$

Thus $S_R u_0(x)$ is a sum of integrals of the form

$$\frac{1}{R} \int_{\mathbf{R}^{n-l}} \frac{\epsilon_1}{(\epsilon_1^2 + |x|^2)^{\alpha_1}} \cdots \frac{\epsilon_l}{(\epsilon_l^2 + |x|^2)^{\alpha_l}} |x|^{-2q-2+m} R^m e^{-i|x|R} \phi^{(m)}(Rx) dx, \tag{2.7.23}$$

when $|x| \sim \frac{1}{R}$ the integrand is $\sim R^{n-l}$. Meanwhile for $m > 0$, $\phi^{(m)}(Rx)$ is supported on $|x| \leq \frac{2}{R}$. Thus $|(2.7.23)| \leq O(1)$ for $m > 0$. When $m = 0$,

$$\frac{1}{R} \int_{\mathbf{R}^{n-l}:|x|\geq \frac{1}{R}} (1 - \phi(Rx)) \frac{1}{|x|^{n-l+1}} dx = O(1).$$

The proof is complete $\square$.

CHAPTER 3

Indefinite Signature

3.1. Focusing in the wave and Schrödinger equations

The following heuristic is introduced to shed some light on the formation of caustics of $e^{itL}u_0$. As in the positive definite signature case, it is helpful to compare the indefinite signature Schrödinger equation,

$$(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2})u = 0,$$

(3.1.1)

$$u(0,x) = u_0(x),$$

with the indefinite signature wave equation,

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u = 0,$$

(3.1.2)

$$u(0,x) = u_0(x),$$

$$u_t(0,x) = 0.$$

The wave operator

$$\Box_L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

would be an ordinary wave operator if x_1 was the time variable and x_2, t were the space variables. So redefine this to be the ordinary wave equation.

$$\Box v = 0$$

(3.1.3)

$$\frac{\partial}{\partial t}v(x_1,x_2,0) = 0$$

$$v(x_1,x_2,0) = u_0(x_1,x_2)$$

Under the rotation of coordinates, this is a wave equation whose value on the $\{y = 0\}$ line is specified for all time, and is symmetric about that line.

Suppose $u_0(x) = \chi_\Omega(x)\beta(x)$, where $\beta \in C_0^\infty(\mathbf{R}^2)$ and $\Omega = \{(x_1, x_2) : x_1^2 - x_2^2 \geq 1, x_1 \leq 0\}$.

$$Sing(u_0) \subset \{(x_1, x_2) : x_1^2 - x_2^2 = 1\}.$$

Let $d((x_1, x_2, t), (y_1, y_2, \tau))$ denote the Minkowski distance

$$d((x_1, x_2, t), (y_1, y_2, \tau))^2 = (y_2 - x_2)^2 + (t - \tau)^2 - (x_1 - y_1)^2, \tag{3.1.4}$$

$$d((-\sqrt{1 + r^2}, r, 0), (0, 0, 1)) = 0. \tag{3.1.5}$$

The origin at time t = 1 and the hyperbola at time $t = 0$ lie on the same light cone, which gives the perfect focusing phenomenon for $\beta(x)\chi_\Omega(x)$. For a metric of signature $(1, n)$, focusing would arise from a jump across $x_1^2 - x_2^2 - ... - x_{n+1}^2 = 1$. If the metric has signature (p, q), it would arise from a jump across

$$x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 = 1.$$

The stationary phase estimate will be needed.

LEMMA 3.1.1.

$$\int_0^1 \frac{1}{x^{1/2}} e^{ix/t} dx = \frac{t^{1/2}}{2} e^{i\pi/4} + O(t). \tag{3.1.6}$$

Proof: Make a change of variables, $x = y^2$, $dx = 2ydy$.

$$\int_0^1 \frac{1}{x^{1/2}} e^{ix/t} dx = \frac{1}{2}\int_0^1 e^{iy^2/t} dy = \frac{t^{1/2}}{2} e^{i\pi/4} + O(t).$$

Let $L_{p,q}$ denote the Laplacian of signature (p,q),

$$\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2}.$$

Let e_k be the eigenvector corresponding to $\frac{\partial}{\partial x_k}$. Let $V_p = \text{span}\,\{e_1, ..., e_p\}$ and $V_q = \text{span}\,\{e_{p+1}, ..., e_{p+q}\}$. Then let $y = (y_p, y_q)$, $(y_p, 0) \in V_p$, $(0, y_q) \in V_q$. Let $r_p = |y_p|$ and $r_q = |y_q|$.

The focusing phenomenon for $e^{itL_{(p,q)}}\chi_{B(0;1)}$ at the origin is not nearly as bad at the origin as it is in the Δ case. Without loss of generality suppose $p \geq q$.

(3.1.7)
$$e^{itL_{(p,q)}}\chi_{B(0;1)}(0) = t^{-n/2}\int_0^1 r_p^{p-1} e^{ir_p^2/t} \int_0^{\sqrt{1-|r_q|^2}} r_q^{q-1} e^{-ir_q^2/t} dr_p dr_q.$$

$$t^{-p/2}\int_0^{\sqrt{1-|r_q|^2}} r_p^{p-1} e^{-ir_p^2/t} = \frac{t^{-(p-2)/2}}{2i}[r_p^{p-2} e^{-ir_p^2/t}|_0^{\sqrt{1-r_q^2}}]$$
$$-\frac{t^{-(p-2)/2}(p-2)}{2i}\int_0^{\sqrt{1-|r_q|^2}} r_p^{p-3} e^{-ir_p^2/t} dr_p].$$

Case 1, $p > 2$:

(3.1.8)
$$(3.1.7) = Ct^{-(n-2)/2}[\int_0^1 r_q^{q-1}(1-r_q^2)^{(q-2)/2} e^{i(2r_q^2-1)/t} dr_q$$
$$-(p-2)\int_0^1 r_q^{q-1} e^{ir_q^2/t} \int_0^{\sqrt{1-r_q^2}} r_p^{p-3} e^{-ir_p^2/t} dr_q dr_p]$$

For the first integral, make a change of variables, $u = r_q^2$. Combining integration by parts with (1.1.23),

$$t^{-(n-2)/2} \int_0^1 u^{(p-2)/2}(1-u)^{(q-2)/2} e^{iu/t} du \sim t^{-n/2+1+q/2}. \tag{3.1.9}$$

The second integral can be regularized again, and has faster convergence.

Case 2, $p = 2$:

$$\frac{1}{2t} \int_0^{\sqrt{1-r_q^2}} e^{ir_p^2/t} 2r_p dr_p = \frac{1}{2t} \int_0^{(1-r_q^2)} e^{iu/t} du = \frac{1}{2i} [e^{i(1-r_q^2)/t} - 1].$$

$$e^{itL} \chi_{B(0;1)} = \frac{t^{-q/2}}{2i} \int_0^1 e^{i(1-2r_q^2)/t} r_q^{q-1} dr_q - \frac{t^{-q/2}}{2i} \int_0^1 e^{-ir_q^2/t} r_q^{q-1} dr_q$$

When $q = 2$,

$$\frac{e^{i/t}}{2it} \int_0^1 e^{-2ir_q^2/t} r_q dr_q = \frac{e^{i/t}}{2it} \int_0^{1/2} e^{-4iu/t} du = \frac{e^{i/t}}{8} [e^{-2i/t} - 1].$$

$$\frac{1}{2it} \int_0^1 e^{-ir_q^2/t} r_q dr_q = \frac{1}{4it} \int_0^1 e^{-iu/t} du = \frac{1}{4} [e^{-i/t} - 1].$$

When $q = 1$,

$$\frac{e^{i/t}}{2it^{1/2}} \int_0^1 e^{-2ir_q^2/t} dr_q = Ce^{i/t}$$

$$\frac{1}{2it^{1/2}} \int_0^1 e^{-ir_q^2/t} = C'$$

In either case the Pinsky phenomenon at the origin is oscillatory. This proves the theorem,

THEOREM 3.1.2. *When $q > 2$ $e^{it\Delta_{(p,q)}}\chi_{B(0;1)}$ has divergence $O(t^{(2-q)/2})$ at the center. When $q = 2$,*

$$e^{it\Delta_{(p,2)}}\chi_{B(0;1)}(0) = 1 + C_1 e^{i/t} + C_2 e^{-i/t} + O(t). \tag{3.1.10}$$

This divergence is an improvement over the signature $(n, 0)$ case by $O(t^{p/2})$. However, there is a price to pay for this improved convergence at the center.

Suppose $x_p = 0$ and $x_q \neq 0$.

$$t^{-n/2}\int_{B(0;1)} e^{i|x_q-y_q|^2/t}\int_{B(0;\sqrt{1-|y_q|^2})} e^{-i|y_q|^2/t}dy_q dy_p$$

$$= t^{-n/2}\int_{B(0;1)} e^{i|x_q-y_q|^2/t}\int_0^{\sqrt{1-|y_q|^2}} r_q^{q-1}e^{-ir_q^2/t}dr_q dy_p$$

$$= Ct^{-n/2+1}\int_{B(0;1)} e^{i|x_q-y_q|^2/t}[r_p^{p-2}e^{-ir_p^2/t}|_0^{\sqrt{1-r_q^2}} - C_2\int_0^{\sqrt{1-r_q^2}} r_p^{p-3}e^{ir_p^2/t}dr_p]dy_q.$$

Once again, separate into two cases.

Case 1, $p > 2$: In this case the first integral is

$$Ct^{-n/2+1}\int_{B(0;1)} e^{i|x_q-y_q|^2/t}(1-r_q^2)^{(p-2)/2}e^{-i(1-r_q^2)/t}dy_p \tag{3.1.11}$$

$$|x_q - y_q|^2 - (1-|y_q|^2) = |x_q|^2 + |y_q|^2 - 2x_q\cdot y_q - 1 + |y_q|^2.$$

Completing the square,

$$= 2|y_q|^2 + |x_q|^2 - 2x_q \cdot y_q - 1 = |\tilde{y}_q - \tilde{x}_q|^2 - 1 + \frac{3}{4}|x_q|^2,$$

$$\tilde{y}_q = \sqrt{2}y_q, \tilde{x}_q = \frac{1}{\sqrt{2}}x_q.$$

If $\frac{1}{\sqrt{2}}|x_q| < \sqrt{2}$, $|x_q| < 2$, $|\tilde{x}_q - \tilde{y}_q|^2$ has a stationary point in $|y_q| \leq 1$, and (3.1.11) is $O(t^{(2-p)/2})$. The coefficient of the leading order term will decay to 0 as $|x_q| \to 2$.

$$(3.1.11) = Ct^{-n/2+1} \int_{B(0;\sqrt{2})} e^{i(|\tilde{y}_q - \tilde{x}_q|^2 - 1 + \frac{3}{4}|x_q|^2)/t}(1 - \frac{\tilde{r}_q^2}{2})dy_q,$$

$$= Ce^{i\pi/4}t^{-p/2+1}e^{i(3|x_q|^2/4-1)/t}(1 - \frac{|x_q|^2}{4})^{(p-2)/2}.$$

For $|x_q| > 2$, the decay of $(1 - \tilde{r}_q^2/2)^{(p-2)/2}$ at $\tilde{r}_q = \sqrt{2}$ of order $\frac{p-2}{2}$ allows additional regularization.

$$(\sqrt{2}y_q - \frac{1}{\sqrt{2}}x_q) = 0 \Leftrightarrow x_q = 2y_q.$$

So if $|x_q| > 2$, $|\tilde{y}_q - \tilde{x}_q|^2$ is not stationary on B(0;1). Use the regularization operator

$$\mathcal{L} = \frac{t}{2i}\frac{(x_q - y_q) \cdot \nabla}{|x_q - y_q|^2}. \tag{3.1.12}$$

$$\mathcal{L}e^{i|x_q - y_q|^2/t} = \frac{t}{2i}\frac{(x_q - y_q) \cdot \nabla}{|x_q - y_q|^2}e^{i|x_q - y_q|^2/t} = e^{i|x_q - y_q|^2/t}.$$

Suppose p is even. Then

$$e^{i(-1/t+3|x_p|^2/4)/t}Ct^{-n/2+1} \int \mathcal{L}^{(p-2)/2}e^{i|\tilde{y}_q - \tilde{x}_q|^2/t}(1 - \tilde{r}_q^2/2)^{(p-2)/2}\chi_{B(0;1)}dy_q$$

$$= e^{i(-1/t+3|x_p|^2/4)/t} C t^{-n/2+1} \int e^{i|\tilde{y}_q - \tilde{x}_q|^2/t} (\mathcal{L}^t)^{(p-2)/2} ((1-\tilde{r}_q^2/2)^{(p-2)/2} \chi_{B(0;1)}) dy_q.$$

This integration by parts is allowable since

$$(1-\tilde{r}_q^2/2)^{(p-2)/2} \chi_{B(0;1)} \in C^{(p-2)/2}(\mathbf{R}^n).$$

$$t^{-n/2+p/2+1} \int e^{i|\tilde{y}_q - \tilde{x}_q|^2/t} (\mathcal{L}^t)^{(p-2)/2} ((1-\tilde{r}_q^2/2)^{(p-2)/2} \chi_{B(0;1)}) dy_q$$

$$\sim t^{-q/2+1} \int_{S^q(0;\sqrt{2})} e^{i|\tilde{x}_q - \tilde{y}_q|^2/t} f(\tilde{y}_q) d\tilde{y}_q = O(t^{1/2}),$$

since $f(\tilde{y}_q)$ is a smooth function on $B(0;\sqrt{2})$. $S(0;\sqrt{2})$ is the sphere of radius $\sqrt{2}$ centered at 0. The same thing holds when p is odd, making use of (3.1.6).

Case 2, $p = 2$: In this case the Pinsky phenomenon is oscillatory when $|x_p| < 2$, and there is the usual convergence when $|x_p| > 2$. This result can be generalized in the following way.

THEOREM 3.1.3. *Let A be the differential operator*

(3.1.13) $$A = \frac{1}{a_1} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \frac{1}{a_2} \sum_{j=p+1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Make the decomposition $\mathbf{R}^n = \mathbf{R}^p \oplus \mathbf{R}^q$. There is focusing of type $C(t)O(t^{(2-p)/2})$ along the axis $x_p = 0$ when $|x_p| < \frac{|a_1-a_2|}{|a_1|}$ and convergence when $|x_p| > \frac{|a_1-a_2|}{|a_1|}$. In particular, as $a_1 \to a_2$ the focusing concentrates to the center. C(t) is a function of the form $Ce^{ia/t}$, $a \in \mathbf{R}$. When $|C(t)| = 1$, but when $a \neq 0$, C oscillates more and more rapidly as $t \to 0$.

Proof: Make a calculation analogous to the calculation in the first theorem. For the first integral the phase function is

$$a_1|x_p - y_p|^2 + a_2(1 - |y_p|^2) = (a_1 - a_2)|y_p|^2 + a_1|x_p|^2 - 2a_1 x_p \cdot y_p + a_2. \tag{3.1.14}$$

Without loss of generality suppose $a_1 > 0$ and $a_1 > a_2$. Make a change of variables.

$$\tilde{y}_p = \sqrt{(a_1 - a_2)}y_p,$$

$$\tilde{x}_p = \frac{a_1}{\sqrt{(a_1 - a_2)}}x_p,$$

$$(3.1.14) = |\tilde{y}_p - \tilde{x}_p|^2 + (a_1 - \frac{a_1^2}{a_1 - a_2})|x_p|^2 + a_2. \tag{3.1.15}$$

If $|x_p| < \frac{a_1 - a_2}{a_1}$ there is focusing along the $x_p = 0$ axis. If $|x_p| > \frac{a_1 - a_2}{a_1}$ then there is convergence. If $a_2 \to a_1$, then the focusing concentrates to the center $\square$.

THEOREM 3.1.4. *As usual assume $p \geq q$.*

$$e^{itA}\chi_{B(0;1)}(0) = \frac{Ct^{-q/2+1}}{a_2}\frac{1}{(a_1 - a_2)^{p/2}} + O(t^{-q/2+2}) \tag{3.1.16}$$

In particular as $a_1 \to a_2$ this converges to the usual Pinsky phenomenon.

Proof: Without loss of generality suppose $a_2 = 1$, and $a_1 > 1/2$.

$$t^{-n/2}\int_0^1 r_q^{q-1}e^{ia_2 r_q^2/t}\int_0^{(1-r_p^2)^{1/2}} e^{ia_1 r_p^2/t}r_p^{p-1}dr_p dr_q$$

$$\sim \frac{t^{-n/2+1}}{2ia_1}\int_0^1 r_q^{q-1}e^{ia_2 r_q^2/t}(1 - r_q^2)^{(p-2)/2}r_q^{q-1}e^{ia_1(1-r_q^2)/t}dr_q + l.o.t.$$

$$= \frac{t^{-n/2+1}}{2ia_1} \int_0^1 e^{i/t} u^{(q-2)/2} e^{i(a_2-a_1)u/t} (1-u)^{(p-2)/2} du + l.o.t.$$

$$\sim \frac{t^{-p/2+1}}{a_1(a_2-a_1)^{q/2}}.$$

Remark: The geometric picture of this is to take $B(0;1) \subset \mathbf{R}^2$. Now allow the flow velocity to $v(\theta)$, $v(0) = v(\pi) = a_1$ and $v(\pi/2) = v(3\pi/2) = a_2$. The caustic will not be a single point, but it will be in a small neighborhood of the origin for $\frac{|a_1-a_2|}{|a_1|}$ small.

3.2. Gibbs Phenomenon for the indefinite signature Schrodinger equation

The Gibbs phenomenon for the characteristic function of a set Ω with smooth boundary, $e^{itL}\chi_\Omega$, is controlled by a boundary integral.

THEOREM 3.2.1. *Take $\Omega \subset \mathbf{R}^2$ to be a region with C^∞ boundary. Let*

$$L = \frac{1}{a_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{a_2}\frac{\partial^2}{\partial x_2^2}.$$

Then take the phase function

$$\psi(x,y) = a_1(x_1-y_1)^2 + a_2(x_2-y_2)^2. \tag{3.2.1}$$

$$e^{itL}\chi_\Omega = \frac{1}{2i}\int_{\partial\Omega:|x-y|>t^{1/2}} e^{i\psi(x,y)/t}\frac{\langle a_1(y_1-x_1), a_2(y_2-x_2)\rangle\cdot\overrightarrow{n}}{a_1^2(y_1-x_1)^2+a_2^2(y_2-x_2)^2}$$

$$(3.2.2)\qquad [1-\frac{t}{2i}(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2})(\frac{a_1(y_1-x_1)+a_2(y_2-x_2)}{a_1^2(y_1-x_1)^2+a_2^2(y_2-x_2)^2})]d\sigma(y)$$

$$+R(t,x,y),$$

$$R(t,x,y)=o(1).$$

Here, $\overrightarrow{n}$ is the unit normal to the boundary, pointing outward.

Proof: Take a C^∞ cut-off

$$(3.2.3)\qquad \eta(t,y)=\begin{cases}1, & |y|>2t^{1/2};\\ 0, & |y|<t^{1/2}.\end{cases}$$

Define the operator

$$(3.2.4)\qquad \mathcal{L}=\frac{t}{i}\frac{a_1(y_1-x_1)\frac{\partial}{\partial y_1}+a_2(y_2-x_2)\frac{\partial}{\partial y_2}}{a_1^2(y_1-x_1)^2+a_2^2(y_2-x_2)^2},$$

$$\mathcal{L}e^{i\psi(x,y)/t}=e^{i\psi(x,y)/t}.$$

It is possible to regularize the integral.

$$\int_{\mathbf{R}^n}\mathcal{L}e^{i\psi(x,y)/t}\chi_\Omega(y)dy=\int_{\mathbf{R}^n}e^{i\psi(x,y)/t}\mathcal{L}^t\chi_\Omega(y)dy.$$

From integration by parts,

$$\mathcal{L}^t\chi_\Omega = -\mathcal{L}\chi_\Omega(y) + \frac{t}{2i}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2})[\frac{a_1(y_1-x_1)+a_2(y_2-x_2)}{a_1^2(y_1-x_1)^2+a_2^2(y_2-x_2)^2}]\chi_\Omega(y). \tag{3.2.5}$$

$\mathcal{L}\chi_\Omega$ a finite measure supported on $\partial\Omega$. The second term can be regularized again, and gives (3.2.2) with remainder term $R(t,x,y)$.

$$\begin{aligned} R(t,x,y) = Ct\int_\Omega e^{i\psi(x,y)/t}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2})(\frac{a_1(y_1-x_1)+a_2(y_2-x_2)}{a_1^2(y_1-x_1)^2+a_2^2(y_1-x_1)^2}\times \\ (\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2})(\frac{\eta(y)(a_1(y_1-x_1)+a_2(y_2-x_2))}{a_1^2(y_1-x_1)^2+a_2^2(y_1-x_1)^2}))dy + (3.2.7). \end{aligned} \tag{3.2.6}$$

This is a second order differential operator. If both derivatives miss $\eta(y)$, then the integrand is $O(\frac{1}{|x-y|^4})$.

$$t\int_{t^{1/2}}^{\infty}\frac{1}{r^3}dr = O(1)$$

$\chi'(y) = O(t^{-1/2})$, $\chi''(y) = O(t^{-1})$, and are supported on $supp(\chi)$.

$$t^{1/2}\int_{t^{1/2}}^{2t^{1/2}} r^{-2}dr = O(1),$$

$$\int_{t^{1/2}}^{2t^{1/2}} r^{-1}dr = O(1).$$

Moreover, since $\psi(x,y)$ is non-constant,

$$\frac{1}{t}\int(1-\eta(y))e^{i((y_1-x_1)^2-(y_2-x_2)^2)/t}dy = o(1). \tag{3.2.7}$$

The proof is complete $\square$.

Remark: The second integral term is bounded by

$$t\int_{\partial\Omega\cap|x-y|>t^{1/2}} O(\frac{1}{|x-y|^3})d\sigma(y).$$

Parameterize the boundary by the unit speed curve $\gamma(t)$ where $\gamma(0)$ is the y-value closest to x in $\partial\Omega \cap |x-y| > t^{1/2}$. Then as long as $d(x,\gamma(t)) > ct$, the second integral is $O(1)$. So it is the first term where the most trouble is caused.

This can be generalized to higher dimensions. Let Ω be a set in $\mathbf{R}^n$ and let $u_0 = \chi_\Omega$.

COROLLARY 3.2.2. *Let $N = [\frac{n}{2}]$. Then*

$$f(x,y) = \frac{a_1(y_1-x_1)+...+a_n(y_n-x_n)}{a_1^2(y_1-x_1)^2+...+a_n^2(y_n-x_n)^2}$$

$$e^{itL}u_0 = \sum_{j=0}^{N} t^{-n/2+1+j}(-1)^j$$

(3.2.8)

$$\int_{\partial\Omega:|x-y|>t^{1/2}} (f(x,y)(\frac{\partial}{\partial y_1}+...+\frac{\partial}{\partial y_n}))^j(\eta(y)f(x,y))e^{i\psi(x,y)/t}d\sigma(y) + R(t,x,y).$$

As in the previous calculation $R(t,x,y) = o(1)$.

Proof: This is also proved by integration by parts.

Theorem [3.2.1] implies that calculation of the Gibbs phenomenon for L having signature $(1,1)$ is reduced to the computation of a one dimensional boundary integral. Make a change of variables so that $a_1 = 1$ and $a_2 = -1$. Suppose x_0 is a point on the boundary where the boundary is tangent to a line of slope ± 1. Rotate the coordinate axes so that the x_1 and x_2 axes are the lines of slope ± 1. The phase function is

$$(y_1-x_1)(y_2-x_2).$$

Make a translation so that x_0 is the origin. In a neighborhood of x_0 the boundary has the representation

$$\{(x, f(x))\}.$$

Suppose that locally $f(x) = x^m$. The region $\mathbf{R}^2 \setminus \{(0,y) : y \neq 0\}$ can be foliated by the curves (x, cx^m), where $c \in (-\infty, \infty)$. When δ and $c\delta^m$ are small,

$$\begin{aligned} &\int_{-1}^{1} \frac{b_1(x+\delta) + b_2(c\delta^m + x^m)}{(x+\delta)^2 + (c\delta^m + x^m)^2} e^{i(x^m + c\delta^m)(x+\delta)/t} dx = \\ &\int_{-\infty}^{\infty} \frac{b_1(x+\delta) + b_2(c\delta^m + x^m)}{(x+\delta)^2 + (c\delta^m + x^m)^2} e^{i(x^m + c\delta^m)(x+\delta)/t} dx + O(t). \end{aligned} \tag{3.2.9}$$

Then make a change of variables. Let $\tau = \frac{t}{\delta^{m+1}}$.

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{b_1(x+\delta) + b_2(c\delta^m + x^m)}{(x+\delta)^2 + (c\delta^m + x^m)^2} e^{i(x^m + c\delta^m)(x+\delta)/t} dx = \\ &\int_{-\infty}^{\infty} \frac{b_1(x+1) + b_2(c + x^m)}{(x+1)^2 + (c + x^m)^2} e^{i(x^m + c)(x+1)/\tau} dx. \end{aligned} \tag{3.2.10}$$

Let

$$K(c,\tau) = \int_{-\infty}^{\infty} \frac{b_1(x+1) + b_2(c + x^m)}{(x+1)^2 + (c + x^m)^2} e^{i(x^m + c)(x+1)/\tau} dx. \tag{3.2.11}$$

Combining (3.2.2), (3.2.9), and (3.2.10) gives

$$e^{itL}\chi_\Omega((\alpha t)^{1/(m+1)}, c(\alpha t)^{m/(m+1)}) = K(c, \frac{1}{\alpha}) + R(t,x). \tag{3.2.12}$$

This gives a good description of the Gibbs phenomenon along the different leaves of the foliation. For a sequence $t_n \to 0$,

$$(e^{it_nL}\chi_\Omega)((\alpha t_n)^{1/(m+1)}, c(\alpha t_n)^{m/(m+1)}) = K(c, \frac{1}{\alpha}) + o(1)$$

Remark: One leaf is the boundary. Another leaf is the caustic, as will soon be demonstrated.

Now to understand how the Gibbs phenomenon depends on c. This is more difficult to calculate, so it will be done only for $m = 2$. Here,

$$\frac{\partial}{\partial x}(x^2 + c)(x + 1) = 3x^2 + 2x + c,$$
$$\frac{\partial^2}{\partial x^2}(x^2 + c)(x + 1) = 6x + 2.$$

Checking the equation, the phase function has two critical points when $c < \frac{1}{3}$, and zero critical points when $c > \frac{1}{3}$. It has a double root when $c = \frac{1}{3}$, which is the equation for the caustic.

$$F'(c) = \frac{\partial}{\partial c}\int_{-\infty}^{\infty} \frac{b_1(x-1) + b_2(c + x^m)}{(x-1)^2 + (c + x^m)^2} e^{i(x^m+c)(x-1)/t} dx$$
$$= \frac{1}{t}\int_{-\infty}^{\infty} \frac{b_1(x-1)^2 + b_2(c + x^m)(x-1)}{(x-1)^2 + (c + x^m)^2} e^{i(x^m+c)(x-1)/t} dx.$$

First consider $c < 1/3$.

(3.2.13) $$F'(c) = \frac{\kappa}{t}\int_{-\infty}^{\infty} e^{i(x^3 - \epsilon^2 x + (4/3)(1/9+c))/t} dx + l.o.t.$$

κ denotes a fixed constant and

$$\epsilon(c) = \frac{\sqrt{1-3c}}{3}.$$

By stationary phase calculations

$$F'(c) = \frac{1}{t^{1/2}}\frac{e^{-i\pi/4}}{(1-3c)^{1/4}}e^{i(-(8/3)(1-3c)^{3/2}-4/9(1-3c)+16/27)/t} \tag{3.2.14}$$
$$+\frac{1}{t^{1/2}}\frac{e^{i\pi/4}}{(1-3c)^{1/4}}e^{i((8/3)(1-3c)^{3/2}-4/9(1-3c)+16/27)/t} + l.o.t.$$

$\frac{-8}{3}(1-3c)^{3/2} - \frac{4}{9}(1-3c) + \frac{16}{27}$ is not stationary when $c < 1/3$, so the calculations will concentrate on the second term in (3.2.14). Make a change in variables, $z = (1-3c)$,

$$\frac{1}{t^{1/2}}\int_0^{\alpha} x^{-1/4}e^{i((8/3)x^{3/2}-4/9x)/t}dx = \frac{1}{t^{1/2}}\int_0^{\alpha^{4/3}} e^{i((8/3)x^2-(4/9)x^{4/3})/t}dx. \tag{3.2.15}$$

The phase function has a root at $x = \frac{1}{27}$. This integral is $O(1)$. This gives a uniform bound for $c_1, c_2 \leq \frac{1}{3}$.

$$|K(c_1,t) - K(c_2,t)| \leq C < \infty. \tag{3.2.16}$$

When $c > \frac{1}{3}$, the phase function does not have a stationary point. First consider $c > 1$.

$$F'(c) = \frac{\kappa}{t}\int_{-\infty}^{\infty} e^{i(x^3-x^2+cx-c)/t}dx + l.o.t.$$

After a change of variables the phase function $\sim x^3 + cx + d$. Integrate by parts twice.

$$\frac{1}{t}\int_{-\infty}^{\infty} e^{i(x^3+cx)/t}dx = t\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(\frac{1}{3x^2+c}(\frac{\partial}{\partial x}\frac{1}{3x^2+c}))e^{i(x^3+cx)/t}dx.$$

$$\int_{-\infty}^{\infty}|\frac{\partial}{\partial x}(\frac{1}{3x^2+c}(\frac{\partial}{\partial x}\frac{1}{3x^2+c}))|dx \leq \frac{1}{c^2}.$$

When $1/3 < c < 1$, use the estimate

$$\int_0^1 \frac{1}{\epsilon + x^2} dx \lesssim |\frac{\ln(\epsilon)}{\sqrt{\epsilon}}|$$

to prove

$$|F'(c)| \lesssim |\frac{\ln(c - 1/3)}{\sqrt{c - 1/3}}|. \tag{3.2.17}$$

So if $c_1, c_2 \geq \frac{1}{3}$,

$$|K(c_1, t) - K(c_2, t)| \leq C < \infty. \tag{3.2.18}$$

Lemma 3.2.3.

$$e^{itL} \chi_{B(0;1)}$$

is uniformly bounded on $[0, T] \times \mathbf{R}^2$.

Proof: By the previous analysis, it suffices to prove an L^∞ bound on the boundary. The boundary has contact of order 2 with a line of slope ± 1.

$$F(-1) = \int_{-\infty}^{\infty} \frac{1}{x + 1} e^{i(x-1)(x+1)^2/t} dx, \tag{3.2.19}$$

which is bounded uniformly for $t \in [0, 1]$ $\Box$.

Remark: For pieces of the boundary that are some distance away from a line of slope ± 1, the phase function has a representation $f(x)^2 + x^2$. Thus the analysis for the positive definite case can be applied.

Remark: The calculation of the Gibbs phenomenon slicing between the foliations reduces the L^∞ boundedness question to the L^∞ along the boundary of the region in the case of the manifold being smooth of finite type 2 where it has slope of ± 1.

LEMMA 3.2.4. *$e^{itL}u_0 \in L^\infty(Q)$ uniformly for $t \in [0,1]$ iff $e^{itL}u_0 \in L^\infty(Q \cap \partial\Omega)$ uniformly for $t \in [0,1]$, where Q is some open neighborhood of where the boundary has slope ± 1, if $\partial\Omega$ has contact of order two at these points.*

In fact, it is possible to make a reduction to the boundary for a positive definite metric, and this can be copied over to the non-positive definite case when the boundary does not have slope ± 1.

LEMMA 3.2.5. *Let $\phi(y)$ be a phase function. Then*

$$\int_{[-\delta,\delta]} \frac{y}{y^2} e^{i\phi(y)/t} dy \tag{3.2.20}$$

lies in L^∞ uniformly for all $0 < \delta < 1$ iff

$$\int_{[-\delta,\delta]} \frac{y}{y^2+\epsilon^2} e^{i\phi(y)/t} dy \tag{3.2.21}$$

lies in L^∞ uniformly for all $0 < \delta < 1$ for positive definite phase functions, for $\epsilon < 1/4$.

Proof: Note that

$$\frac{y}{y^2+\epsilon^2} - \frac{1}{y} = \frac{\epsilon^2}{y(y^2+\epsilon^2)}$$

On the interval $[-\epsilon/3, \epsilon/3]$,

$$\frac{\epsilon^2}{y(y^2+\epsilon^2)} \sim \frac{1}{y}.$$

This integral can be handled by (3.2.20). On the other hand,

$$\int_{\epsilon/3}^{\infty} \frac{\epsilon^2}{x(x^2+\epsilon^2)} dx = \int_{1/3}^{\infty} \frac{1}{x(x^2+1)} dx.$$

So the other piece has uniformly bounded integral also.

This permits a reduction to the boundary for the positive definite phase function. Take $x_0 = 0$, and the boundary of the form $f(y) = a_0 + a_1 x^2 + ...$

$$e^{it\Delta} u_0(x) \sim \int_{-\delta}^{\delta} \frac{y}{y^2 + f(y)^2} e^{i\phi(y)/t} dy \tag{3.2.22}$$

$$\phi(y) = y^2 + a_0^2 + 2a_0 y^2 + ...$$

Define the vector field that is normal to the boundary at every point $x \in \partial\Omega$. After a change of variables, for small a_0, uniform boundedness of $e^{it\Delta}\chi_\Omega$ is linked to uniform boundedness along $\partial\Omega$ $\Box$.

On the other hand, a rotation of the coordinate system does influence the (1,1) phase function. In this case the phase function is rotated to

$$\phi(y) = \alpha y^2 + 2\beta y f(y) - \alpha f(y)^2. \tag{3.2.23}$$

If $a_0 = 0$ the phase function is of the form $\alpha y^2 + ...$ The phase function can be rewritten

$$\phi(y) = \alpha(y - g(a_0))^2 + ..., \tag{3.2.24}$$

where $\frac{\partial g}{\partial a_0}|_0 = \frac{\beta}{\alpha}$. In other words, take the flow along the boundary defined by the vector field (β, α). This mapping is 1-1 near the boundary, except when α is close to zero. So take $|\alpha| \geq \epsilon > 0$. In this case it is possible to reduce to the boundary along these flows. This takes care of the L^∞ bounds near the boundary, away from where it has slope ± 1. The other part was already dealt with via the foliation. The next lemma will be needed later.

THEOREM 3.2.6. *Let $u_0 = \chi_{B(0;1)}$ where $B(0;1) \subset \mathbf{R}^3$ and let*

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}.$$

Let $x_0 \in S^1$ lie some distance away from where the caustic intersects the boundary. Then

$$e^{itL} u_0$$

is uniformly bounded on this part of the boundary.

Proof: Near the point x_0 the surface integral is of the form

$$t^{-1/2} \int_{|r| \leq 1} \frac{y_1 \pm y_2}{y_1^2 + y_2^2} e^{i(y_1^2 + a y_2^2)/t} dy_1 dy_2,$$

where $|a| \geq \epsilon > 0$. Rewrite the phase function in polar coordinates.

$$\frac{r^2}{2}(1-a)\cos(2\theta) + \frac{r^2}{2}(1+a)$$

Make a change of variables $r \mapsto r t^{1/2}$

$$t^{-1/2} \int_0^{2\pi} \int_{t^{1/2}}^{1} e^{ir^2((1-a)\cos(2\theta)+(1+a))/2t} dr d\theta = \int_0^{2\pi} \int_1^{t^{-1/2}} e^{ir^2((1-a)\cos(2\theta)+(1+a))/2} dr d\theta$$

when a is close to one, $(1-a)\cos(2\theta) + (1+a)$ will be bounded below, and thus the integral is O(1). Otherwise take the $d\theta$ integral. Integrate in the cone where $\cos(2\theta) = \pm 1$. Since a is bounded away from zero, $(1-a)\cos(2\theta)+(1+a)$ is bounded away from zero, and the integral in that cone is O(1). Outside the cone, a stationary phase integral introduces r^2 in the denominator. But this is integrable on $[1,\infty)$ and the integral is O(1) □.

Remark: When the operator is not positive definite it is possible to have local blowup for a characteristic function of a manifold with corners. Consider

$$L = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}. \tag{3.2.25}$$

Now make a rotation of coordinates so that the kernel is of the form

$$K(x,y) = \frac{C}{t} e^{-i(x_1-y_1)(x_2-y_2)/t}, \tag{3.2.26}$$

$$u_0(y) = \eta(Ry_1)\eta(Ry_2)\chi_{[0,1]}(y_1)\chi_{[0,1]}(y_2), \tag{3.2.27}$$

where $\eta \equiv 1$ on $|x| < 1/2$, $\eta \equiv 0$ for $|x| > 1$, and R is a large number, $\chi_{[0,1]}$ is the characteristic function of [0,1].

$$\int_0^1 \int_0^1 e^{-ixy/t} dy dx = \int_0^1 \frac{1}{x}[e^{-ix/t} - 1]dx = \int_0^{1/t} \frac{1}{x}[e^{-ix} - 1]dx \sim \ln(t)$$

$$\int_0^1 (1 - \chi(Rx))\frac{1}{x} dx = O(R)$$

$$\int_0^1 (1 - \chi(Rx))\frac{1}{x} e^{-ix/t} dx = O(R)$$

$$\int_0^1 (1-\chi(Rx))\frac{1}{x}\int_0^1 (1-\chi(Ry))e^{-ixy/t}dydx = O(R)$$

(3.2.28) $$e^{itL}u_0(0) \sim \ln(t),$$

which is not uniformly bounded in L^∞. When integrating by parts with a positive definite phase function, the boundary integral is something of the form

(3.2.29) $$\int_{-\delta}^{\tilde{\delta}} \frac{\epsilon}{\epsilon^2 + x^2} e^{i\phi(x)/t} dx.$$

On the other hand for the phase function of signature (1,1) the boundary integral will be of the form

(3.2.30) $$\int_{-\delta}^{\tilde{\delta}} \frac{a\epsilon + bx}{\epsilon^2 + x^2} e^{i\phi(x)/t} dx,$$

which is not uniformly bounded for an arbitrary $\delta > 0$, $\tilde{\delta} > 0$.

3.3. Caustics

Suppose Ω is a compact, smoothly bounded region in $\mathbf{R}^n$. Recall the formula in Corollary [3.2.2] for $e^{itL}\chi_\Omega$. Since $\partial\Omega$ is compact, $\psi(x,y)$ has a maximum and a minimum. $\partial\Omega$ is n - 1 dimensional so by stationary phase,

$$t^{-n/2+1}\int_{\partial\Omega}(f(x,y)(\eta(y)f(x,y))e^{i\psi(x,y)/t}d\sigma(y) \geq O(t^{1/2}).$$

$$f(x,y) = \frac{a_1(y_1 - x_1) + ... + a_n(y_n - x_n)}{a_1^2(y_1 - x_1)^2 + ... + a_n^2(y_n - x_n)^2}.$$

From [**29**], the points where $S_R f$ has convergence worse than $O(R^{-1})$ are given by the caustics of the flow

$$\cos(t\sqrt{-\Delta})\chi_\Omega.$$

Meanwhile, the points where pointwise Fourier inversion is weaker than $O(R^{-1})$ are points where the convergence of $e^{it\Delta}\chi_\Omega$ is weaker than $O(t^{1/2})$.

DEFINITION 3.3.1. Guided by this, those points where the convergence of $e^{itL}\chi_\Omega$ is worse than $O(t^{1/2})$ will be called the caustics of $e^{itL}\chi_\Omega$.

Example: In $\mathbf{R}^n$,

$$e^{it\Delta}\chi_{B(0;1)} = 1 + O(t^{(2-n)/2}). \tag{3.3.1}$$

For the wave equation, when $\partial\Omega \subset \mathbf{R}^n$ is smooth, then in some collar neighborhood of the region,

$$\Phi : [0,1] \times \partial\Omega \to \mathbf{R}^n \tag{3.3.2}$$

is a 1-1 mapping onto the collar neighborhood. Then the caustics must lie outside some collar neighborhood of $\partial\Omega$. This is not true when L has mixed signature. Choose a point $x_0 \notin \partial\Omega$. Let $\eta \in C_0^\infty$, $\eta \equiv 1$ in a neighborhood of x_0, such that $supp(\eta)$ does not intersect $\partial\Omega$.

$$(3.3.3) \qquad e^{it\Delta}(\eta u_0)(x) = u_0(x) + O(t^\infty).$$

Recall the operator

$$(3.3.4) \qquad \mathcal{L} = \frac{t}{i}\frac{(y_1 - x_1)\frac{\partial}{\partial y_1} + (y_2 - x_2)\frac{\partial}{\partial y_2}}{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

The denominator is only zero when $(y_1, y_2) = (x_1, x_2)$.

$$(3.3.5) \qquad \begin{aligned} \frac{1}{t}\int e^{i|x-y|^2/t}u_0(y)dy = \int \mathcal{L}(e^{i|x-y|^2/t})\chi_\Omega(y)dy &= \int e^{i|x-y|^2/t}\mathcal{L}^t(\chi_\Omega(y))dy \\ = \int_{\partial\Omega} e^{i|x-y|^2/t} &- \int_\Omega e^{i|x-y|^2/t}\tilde{f}(y), \\ \tilde{f}(y_1, y_2) = \frac{\partial}{\partial y_1}(\frac{(y_1 - x_1)}{(y_1 - x_1)^2 + (y_2 - x_2)^2}) &+ \frac{\partial}{\partial y_2}(\frac{(y_2 - x_2)}{(y_1 - x_1)^2 + (y_2 - x_2)^2}), \end{aligned}$$

$$\int_\Omega \mathcal{L}e^{i|x-y|^2/t}\tilde{f}(y)dy = \int_\Omega e^{i|x-y|^2/t}\mathcal{L}^t(\tilde{f}(y))dy = O(t).$$

The formation of the caustics also depends on the boundary integral. The level sets of $|x-y|^2$ are circles centered at x.

Lemma 3.3.2. *If a circle $|y-x|^2 = c$ is tangent to $\partial\Omega$ of order n at a point $y \in \partial\Omega$, with no higher order of tangency anywhere on $\partial\Omega$, the convergence is $O(t^{1/(n+1)})$.*

Proof: Make a change of coordinates so that x is the origin. Near y, the boundary can be expressed in polar coordinates, $r = c + f(\theta)$, $\theta \in [\tilde{\theta} - \epsilon, \tilde{\theta} + \epsilon]$, $\frac{d^k}{d\theta^k} f(\tilde{\theta}) = 0$ for $k \leq n$.

$$\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} e^{i(c+f(\theta))/t}d\theta \sim \int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} e^{i(c+c_1\theta^n)/t}d\theta \sim t^{1/(n+1)}.$$

The analysis in $\mathbf{R}^2$ for a metric of signature (1,1) also utilizes a regularizing operator, which yields (3.2.2). However, in the (1,1) case, the caustic may intersect the boundary.

The level sets of the phase function

$$(x_1 - y_1)^2 - (x_2 - y_2)^2 \tag{3.3.6}$$

are hyperbolas.

As in the case of the positive definite Schrödinger equation, the level sets of the phase function determine the order of the caustic. For an indefinite signature Schrödinger equation the level sets are the hyperbolas centered at $x_0 = (x_1, x_2)$. If (x_1, x_2) is the center of a hyperbola that is tangent to the boundary to order $n-1$, the pointwise convergence at (x_1, x_2) is $O(t^{1/n})$. So when Ω has a smooth boundary, the caustic can be calculated by travelling along the boundary. Travel along the boundary according to the curve $\gamma(t)$. At each point $\gamma(t)$, determine the hyperbolas that are tangent to $\partial\Omega$ to at least second order. The centers of these hyperbolas will be caustics. Then continue on to the next point. Thus travelling along the boundary traces out a caustic.

When the tangent line to $\partial\Omega$ has slope -1 or +1, the tangent hyperbola is the hyperbola $(y_1 - x_1)^2 - (y_2 - x_2)^2 = 0$. This is the only hyperbola that is tangent to the boundary. Now for the other points. First take $T(t) \neq (0,1)$. Make a change of variables, $t = y_1 - x_1$ and $z = y_2 - x_2$. Define a curve $\alpha(t) = (t, z(t))$.

$$\alpha(t) = (t, z(t)),$$

(3.3.7)
$$\alpha'(t) = (1, \frac{dz}{dt}),$$

$$\alpha''(t) = (0, \frac{d^2z}{dt^2}),$$

$$\kappa = \frac{z''}{|\alpha'|^3}.$$

What hyperbola does this represent?

$$2 - 2(z')^2 - 2zz'' = 0,$$

$$z = \frac{1 - (z')^2}{z''} = \frac{2 - \csc^2(\theta)}{z''} = \frac{2 - \csc^2(\theta)}{\kappa|\alpha'|^3}.$$

The hyperbola with second order tangency to $\partial\Omega$ at x_0 has origin

(3.3.8)
$$x_0 - \frac{2 - |\alpha'|^2}{\kappa|\alpha'|^3}(\frac{dy}{dx}, 1).$$

Let R define the reflection over the line $y = x$. Parameterize $\partial\Omega$ as a unit speed curve in $\mathbf{R}^2$. As one travels along $\partial\Omega$, it sweeps out an equation for the caustic. Let $\gamma(t)$ denote the curve $\partial\Omega$.

$$|\alpha'(t)| = \frac{1}{|\sin(\theta)|}.$$

When $T(t) = (\cos(\theta), \sin(\theta))$.

$$(3.3.9) \qquad \begin{aligned} &\gamma(t) - (\frac{2-|\alpha'|^2}{\kappa|\alpha'|^3})R(\alpha'(t)) = \\ &\gamma(t) - (\frac{2-\csc^2(\theta)}{\kappa\csc^2(\theta)})RT(t) = \\ &\gamma(t) - \frac{2\sin^2(\theta)-1}{\kappa}RT(t), \end{aligned}$$

$$R(0,1) = (1,0) = -\vec{N},$$

$$R(0,-1) = (-1,0) = -\vec{N}.$$

Therefore, (3.3.10) is valid for any T(t). This proves the theorem.

THEOREM 3.3.3. *Let $\gamma(t)$ be a closed, C^∞ curve with a unit speed parameterization. The caustic is the continuous curve swept out by*

$$(3.3.10) \qquad \gamma(t) + \frac{2\sin^2(\theta)-1}{\kappa}R(T(t)).$$

Remark: Observe that the caustic intersects $\gamma(t)$ iff $\sin^2(\theta) = \frac{1}{2}$. This agrees with the previous analysis on B(0;1).

Remark: If Ω is not convex, part of the caustic could lie inside Ω.

Remark: The convergence along this caustic is at best $O(t^{1/3})$. When $\sin^2(\theta) \neq 0$, a unique hyperbola has been found that is tangent to $\partial\Omega$ at x_0 of at least second order. Checking whether higher order tangencies exist will give the exact order of convergence.

When $\sin^2(\theta) = \frac{1}{2}$, the only hyperbolas of slope -1 or +1 are the straight lines $y = x$ and $y = -x$. These hyperbolas have second order tangency to $\partial\Omega$ at x_0 iff $\frac{d^2y}{dx^2} = 0$, and similarly for higher order tangency.

Remark: The caustic is tangent to the region at x_0 iff a line of slope ± 1 is tangent to Ω at x_0. As the order of tangency with a line of slope ± 1 increases, the order of tangency of the caustic with Ω increases.

Remark: If Ω has third order tangency with a line of slope ± 1 at a point, then one could choose any other point x on that line as the center of the hyperbola. In this case the line will also be added in to the caustic.

Remark: When $-a_2 < 0 < a_1$, if $a_2(y_2 - x_2)^2 = a_1(y_1 - x_1)^2$ lies tangent to $\partial\Omega$ at (x_1, x_2), then the caustic intersect (x_1, x_2). These hyperbolas are the lines

$$(y_2 - x_2) = \pm\frac{a_2}{a_1}(y_1 - x_1). \tag{3.3.11}$$

Corollary 3.3.4. *If $\partial\Omega$ is a smooth closed curve, there exist at least four points where a caustic intersects $\partial\Omega$.*

3.4. Value of the Gradient

As has already been shown, the Pinsky phenomenon for $e^{itL}\chi_{B(0;1)}$ when L has signature (2,1), is merely oscillatory. In particular, $\|e^{itL}\chi_\Omega\|_{L^\infty_{t,x}(\mathbf{R}\times\mathbf{R}^n)} \leq C$. Therefore, it is possible to apply a modification of the method of [**27**] which requires computing the gradient of $e^{itL}\chi_{B(0;1)}$.

Where Ω is a closed region in $\mathbf{R}^n$, with C^∞ boundary $\partial\Omega$ it is possible to compute the value of $\nabla e^{itL}\chi_\Omega$ quite explicitly.

Assign a smooth normal vector $\overrightarrow{n}_x$ at each point $x \in \partial M$ of unit length one.

$$\frac{\partial}{\partial x_i} e^{itL} u_0 = C t^{-n/2} \int_{\partial\Omega} (\overrightarrow{e}_i \cdot \overrightarrow{n}_x) e^{i\psi(x,y)/t} d\sigma(y), \tag{3.4.1}$$

$$\psi(x,y) = (x_1 - y_1)^2 + ... + (x_p - y_p)^2 - (x_q - y_q)^2 - ... - (x_n - y_n)^2, \tag{3.4.2}$$

for

$$L = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_n^2}. \tag{3.4.3}$$

Consider the case $u_0 = \chi_{B(0;1)}$ for $B(0;1) \subset \mathbf{R}^2$ and L has signature (1,1), in polar coordinates (R, φ),

Recall the shape of the caustic. There is $O(t^{1/3})$ convergence of $e^{itL}\chi_\Omega$ along this region, and $O(t^{1/4})$ convergence at the cusp points. Split the analysis into two separate pieces. $e^{itL}\chi_\Omega$ will be evaluated both near the cusp point, and also along the fold set of the caustic.

$$\begin{aligned} \frac{\partial}{\partial x_1} e^{itL} u_0(R,\varphi) &= \frac{C}{t} \int_{-\pi}^{\pi} \cos(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta+\varphi) + R^2\cos(2\varphi))/t} d\theta, \\ \frac{\partial}{\partial x_2} e^{itL} u_0(R,\varphi) &= \frac{C}{t} \int_{-\pi}^{\pi} \sin(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta+\varphi) + R^2\cos(2\varphi))/t} d\theta. \end{aligned} \tag{3.4.4}$$

The caustic has four cusp points, (2,0), (0,2), (-2,0), (0, -2). The convergence is weaker at these cusp points then along the fold set of the caustic. Without loss of generality take the point (2,0) and consider the analysis in a neighborhood of this point.

THEOREM 3.4.1. *Let $\chi \in C_0^\infty$ and $\chi \equiv 1$ in a neighborhood of the point (2,0).*

$$t^{1/2}\chi\nabla e^{itL}\chi_\Omega \in L^{4-\epsilon}(\mathbf{R}^2). \tag{3.4.5}$$

Proof: Take a foliation of the region near the cusp point. Let $a \in (-\infty, \infty)$. The region

$$(\mathbf{R} \setminus 0) \times \mathbf{R}$$

can be foliated by the curves

$$\begin{gathered}(-x, ax^{3/2}) \\ (x, ax^{3/2}),\end{gathered} \tag{3.4.6}$$

$$\frac{\partial}{\partial x_1} e^{itL} u_0(x) = \int_{-\pi}^{\pi} \cos(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta$$

$$= \int_{-\pi/8}^{\pi/8} \cos(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta$$

$$+ \int_{7\pi/8}^{9\pi/8} \cos(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta + O(t^{1/2}),$$

since the phase function does not have any double roots outside of $[-\pi/8, \pi/8] \cup [7\pi/8, 9\pi/8] = A$. In this region A approximate the boundary of the circle with a parabola. Choose a point $(x, f(x))$.

$$\int_{-\pi/8}^{\pi/8} \cos(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta$$

$$= \int_{-\infty}^{\infty} e^{i((1-2x+x^2-f(x)^2)+y^4/4-xy^2+2f(x)y)/t} dy + O(t^{1/2}).$$

Define a function.

$$G(a,t) = \frac{1}{t}\int_{-\infty}^{\infty} e^{i(y^4/4-y^2+ay)/t} dy. \tag{3.4.7}$$

$$\frac{1}{t}\int_{-\infty}^{\infty} e^{i(y^4/4-xy^2+2ax^{3/2}y)/t} dy = \frac{\sqrt{x}}{t}\int_{-\infty}^{\infty} e^{ix^2(y^4/4-y^2+2ay)/t} dy$$

(Now let $\frac{1}{\tau} = \frac{x^2}{t}$.)

$$= \frac{x^{-3/2}}{\tau}\int_{-\infty}^{\infty} e^{i(y^4/4-y^2+2ay)/\tau} dy = x^{-3/2} G(a,\tau).$$

For most values of a the derivative

$$y^3 - 2y + 2a$$

will not have any double roots. In this case $G(a,\tau) = O(\tau^{-1/2})$.

$$x^{-3/2} G(a,\tau) = O(x^{-1/2} t^{-1/2}).$$

There is a double root when

$$\begin{aligned} y^3 - 2y - a &= 0, \\ 3y^2 - 2 &= 0. \end{aligned} \tag{3.4.8}$$

When $a = \pm\frac{4}{3}(\frac{2}{3})^{1/2}$ the phase function has a double root at $y = \pm\sqrt{\frac{2}{3}}$. Call $\frac{4}{3}(\frac{2}{3})^{1/2} = \alpha$.

$$x^{-3/2}G(\alpha, \tau) = x^{-3/2}O(\tau^{-2/3}) = O(x^{-1/6}t^{-2/3}).$$

So $\nabla e^{itL}u_0$ is large when a is close to α. The points $(-x, \alpha x^{3/2})$ are a good approximation of the caustic.

Remark: When $x > 0$ the second derivative is $3y^2 + 2 \geq 2$.

As a changes, the z such that $\varphi'(z) = 0$ will also change. Call this value $\tilde{z}(a)$.

$$(3\tilde{z}^2 - 2)\frac{\partial \tilde{z}}{\partial a} = 1.$$

Let

$$r(z) = (3\tilde{z}^2 - 2),$$

$$\frac{\partial r}{\partial a} = 6z(a) \cdot \frac{1}{3\tilde{z}(a)^2 - 2} = 6\tilde{z}(a) \cdot \frac{1}{r(\tilde{z}(a))}.$$

For small perturbations of a, z will stay in the interval, $[\sqrt{\frac{1}{3}}, 1]$. The change of r can be approximated by an ordinary differential equation. The solution to

(3.4.9)
$$\frac{dr}{da} = \frac{c}{r},$$
$$r(\alpha) = 0,$$

is of the form

$$r(a)^2 = 2ca - 2c\alpha. \tag{3.4.10}$$

Notice there are two roots that bifurcate as a moves away from α. These roots do not exist when $a < \alpha$. Now by stationary phase techniques,

$$\int_{-\infty}^{\infty} e^{ir^2/t} dr = O(t^{1/2}).$$

Therefore for any $p < 4$,

$$t^{1/2}\chi\nabla e^{itL}u_0 \in L^p(\mathbf{R}^n), \tag{3.4.11}$$

$$(\int_1^3 \int_{-1}^1 |\nabla e^{itL}u_0(x,y)|^p dxdy)^{1/p}$$

$$\leq t^{-1/2}(\int_{-2}^2 \int_{-2}^2 x^{3/2}|x^{-1/2}|\alpha - a|^{-1/4}|^p dadx)^{1/p}$$
$$+t^{-1/2}(\int_{|a|>2} \int_{-|a|^{-3/2}}^{|a|^{-3/2}} x^{3/2}|x|^{-p/2}|\alpha - a|^{-p/4} dxda)^{1/p}.$$

This integral converges. The derivative in the x_2 direction is much simpler.

$$\frac{\partial}{\partial x_2} e^{itL}u_0(x) = \frac{1}{t}\int_{-\pi}^{\pi} \sin(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta$$

$$= \frac{1}{t}\int_{-\pi/8}^{\pi/8} \sin(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta$$
$$+\frac{1}{t}\int_{7\pi/8}^{9\pi/8} \sin(\theta) e^{i(\cos(2\theta) - 2R\cos(\theta) + R^2\cos(2\varphi))/t} d\theta + O(t^{-1/2})$$

$$\frac{1}{t}\int_{-\pi/8}^{\pi/8}\sin(\theta)e^{i(\cos(2\theta)-2R\cos(\theta)+R^2\cos(2\varphi))/t}d\theta$$

$$=\frac{1}{t}\int_{-\infty}^{\infty}ye^{i((1-2x+x^2-f(x)^2)+y^4/4-xy^2+2f(x)y)/t}dy+O(t^{-1/2}).$$

Define the function

$$H(a,t)=\frac{1}{t}\int_{-\infty}^{\infty}ye^{i(y^4/4-y^2+ay)/t}dy. \tag{3.4.12}$$

$$\frac{1}{t}\int_{-\infty}^{\infty}ye^{i(y^4/4-xy^2+2ax^{3/2}y)/t}dy=\frac{x^{3/2}}{t}\int_{-\infty}^{\infty}e^{ix^2(y^4/4-y^2+2ay)/t}dy$$

(Now let $\frac{1}{\tau}=\frac{x^2}{t}$.)

$$=\frac{x^{-1/2}}{\tau}\int_{-\infty}^{\infty}e^{i(y^4/4-y^2+2ay)/\tau}dy=x^{-1/2}G(a,\tau).$$

This completes the proof $\square$.

Metric of Signature (2,1): For a metric of signature (2,1), $\chi_{B(0;1)}$, a weaker result holds. In the (1,1) signature case, there were caustic points at $(2,0)$, $(0,2)$, $(0,-2)$, and $(-2,0)$. If $(-2,0)$ is renamed the origin, the circle near $(-1,0)$ has the form $(1+\frac{y^2}{2},y)$. In signature $(2,1)$, the cusp will form at the points where $|y_p|=2$, $|y_q|=0$ and $|y_p|=0$, $|y_q|=2$. Without loss of generality, suppose x_0 has the coordinates $(x,0,f(x))$ and let $(-2,0,0)$ be the origin.

$$|(x,0)-(1+\frac{y^2}{2})(\cos\theta,\sin\theta)|^2=x^2+(1+\frac{y^2}{2})^2-2x(1+\frac{y^2}{2})\cos\theta,$$

$$\nabla e^{itL}u_0(x) = \frac{1}{t^{3/2}} \int_{-1}^{1} e^{i(x^2+(1+\frac{y^2}{2})^2-(y-f(x))^2)/t} \int_0^{2\pi} e^{-2ix(1+\frac{y^2}{2})\cos\theta/t} d\theta dy + O(t^{-1/2}).$$

Then apply stationary phase.

$$t^{-1/2} \int_0^{2\pi} e^{-2ix(1+y^2/2)\cos\theta/t} d\theta = c_1 x^{-1/2}(1+\frac{y^2}{2})^{-1/2} e^{-2ix(1+y^2/2)/t}$$

$$+c_2 x^{-1/2}(1+\frac{y^2}{2})^{-1/2} e^{2ix(1+y^2/2)/t} + O(t^{1/2}).$$

The $O(t^{-1/2})$ term is fine to integrate over a bounded region.

$$t^{-1} \int_{-1}^{1} e^{i(x^2+(1+\frac{y^2}{2})^2-(y-f(x))^2)/t} e^{\pm 2ix(1+\frac{y^2}{2})} dy = t^{-1} \int_{-1}^{1} e^{i((x \pm (1+\frac{y^2}{2}))^2-(y-f(x))^2)/t} dy.$$

These integrals are the integrals in the case (1,1). The L^p integral is

$$\leq Ct^{-1/2}(\int_{-2}^{2}\int_{-2}^{2} x^{5/2}|x^{-1}|\alpha - a|^{-1/4}|^p da dx)^{1/p}$$

$$+Ct^{-1/2}(\int_{|a|>2}\int_{-|a|^{-3/2}}^{|a|^{-3/2}} x^{5/2}|x|^{-p}|\alpha - a|^{-p/4} dx da)^{1/p}$$

(3.4.13) $$t^{1/2}\|\chi \nabla e^{itL}u_0\|_p < \infty,$$

for $p < 7/2$.

Fold: Next, evaluate the derivative along the fold part of the caustic, the part away from the cusps. First, a lemma is needed.

Lemma 3.4.2. *Suppose (R, φ) lies along the caustic, away from the cusp points. For the θ such that $\varphi(\theta) = \cos(2\theta) - 2R\cos(\theta + \varphi) + R^2\cos(2\varphi)$ has a double root,*

$$\begin{aligned} |\sin(2\theta)| &\geq \epsilon \\ |\sin(\theta + \varphi)| &\geq \epsilon \end{aligned} \tag{3.4.14}$$

where $\epsilon > 0$ depends on the size of the forbidden neighborhood around the cusp points.

Proof: For such a double root,

$$2\sin(2\theta) = 2R\sin(\theta + \varphi),$$

$$4\cos(2\theta) = 2R\cos(\theta + \varphi),$$

$$16\sin^2(2\theta) + 16\cos^2(2\theta) = 16 = 12R^2\sin^2(\theta + \varphi) + 4R^2.$$

The points on the caustic outside the forbidden regions are given by (R, φ) where $R \leq 2 - \delta$. Thus

$$\sin(\theta + \varphi)^2 \geq \delta' > 0.$$

Convert to rectangular coordinates. Let (x, y) be a point near the caustic. Without loss of generality suppose that $y > 0$.

$$\begin{aligned} \frac{\partial}{\partial x_1} e^{itL} u_0(x, y) = \frac{C}{t} \int_{-1}^{1} e^{i((z-x)^2 - (y - \sqrt{1-z^2})^2)/t} \frac{z}{(1-z^2)^{1/2}} dz \\ + \frac{C}{t} \int_{-1}^{1} e^{i((z-x)^2 - (y + \sqrt{1-z^2})^2)/t} \frac{z}{(1-z^2)^{1/2}} dz, \end{aligned} \tag{3.4.15}$$

$$\frac{d^2}{dy^2}(z-x)^2-(y+\sqrt{1-z^2})^2=4+\frac{y}{(1-z^2)^{3/2}}\geq 4.$$

Also, a root near $z^2=1$ corresponds to a root near $\theta=0$ or $\theta=\pi$, which is excluded by Lemma [3.4.2]. The second term is $O(t^{-1/2})$ uniformly. So the analysis depends on the first term. Next, define the ϵ-sets.

DEFINITION 3.4.3. The ϵ-sets are varieties in $\mathbf{R}^2$ where there exists $z_0\in[-1,1]$ such that

$$\varphi'(z_0)=0,$$

(3.4.16)

$$\varphi''(z_0)=\epsilon.$$

The caustic corresponds to the $\epsilon=0$ set.

LEMMA 3.4.4. *The ϵ-sets are given by the equation (x,y) such that*

$$x=\pm(1-(\frac{2y}{4-\epsilon})^{2/3})^{1/2}(4-(4-\epsilon)(\frac{2y}{4-\epsilon})^{2/3}).$$

Proof: Fix y.

$$\varphi''(z)=4-2y(1-z^2)^{3/2}=\epsilon.$$

This gives an equation for $|z|$. Plug this into

$$\varphi'(z)=4z-2x+\frac{2yz}{\sqrt{1-z^2}}=0,$$

which gives the appropriate equation for x $\square$.

For ϵ small, since

$$\frac{\partial}{\partial\epsilon}[(1-(\frac{2y}{4-\epsilon})^{2/3})^{1/2}(4-(4-\epsilon)(\frac{2y}{4-\epsilon})^{2/3})]|_{\epsilon=0}=0,$$

$$\frac{\partial^2}{\partial\epsilon^2}[(1-(\frac{2y}{4-\epsilon})^{2/3})^{1/2}(4-(4-\epsilon)(\frac{2y}{4-\epsilon})^{2/3})]|_{\epsilon=0}=C,$$

for small ϵ there is a Taylor expansion

$$[(1-(\frac{2y}{4-\epsilon})^{2/3})^{1/2}(4-(4-\epsilon)(\frac{2y}{4-\epsilon})^{2/3})]=[(1-(\frac{2y}{4})^{2/3})^{1/2}(4-4(\frac{2y}{4})^{2/3})]+C(y)\epsilon^2+O(\epsilon^3),$$

$$\int_{1/4}^{7/4}[(1-(\frac{2y}{4-\epsilon})^{2/3})^{1/2}(4-(4-\epsilon)(\frac{2y}{4-\epsilon})^{2/3})]-[(1-(\frac{2y}{4})^{2/3})^{1/2}(4-4(\frac{2y}{4})^{2/3})]dy=O(\epsilon^2),$$

$$t^{1/2}\|\frac{\partial}{\partial x_1}e^{itL}u_0\|_{4-\delta}\leq\int_{-1}^{1}\epsilon^2(\frac{C}{\sqrt{\epsilon}})^{4-\delta}d\epsilon<\infty,$$

(3.4.17)
$$\begin{aligned}\frac{\partial}{\partial x_2}e^{itL}u_0(x,y)&=\frac{C}{t}\int_{-1}^{1}e^{i((z-x)^2-(y-\sqrt{1-z^2})^2)/t}dz\\&+\frac{C}{t}\int_{-1}^{1}e^{i((z-x)^2-(y+\sqrt{1-z^2})^2)/t}dz,\end{aligned}$$

so by the same analysis,

$$t^{1/2}\|\frac{\partial}{\partial x_1}e^{itL}u_0\|_{4-\delta}<\infty.$$

Now take the same reduction in $\mathbf{R}^3$ with signature (2,1). Due to rotational symmetry let $x=(x_1,0,x_2)$. Let S(0;1) denote the sphere of radius one centered at zero.

$$t^{-3/2}\int_{S(0;1)} e^{i\psi(x,y)/t}dy$$

$$= t^{-3/2}\int_{-\pi/2}^{\pi/2} |x_1| e^{i(|x_1|^2+\cos^2(\theta)-(|x_2|-\sin^2(\theta))^2)/t}\int_0^{2\pi} e^{-2i|x_1||\cos\theta|\cos(\varphi)/t}d\varphi d\theta$$

$$= c_1 t^{-1}\int_{-\pi/2}^{\pi/2} |x_1|^{1/2}|\cos(\theta)|^{-1/2}e^{i((x_1-\cos(\theta))^2-(x_2-\sin(\theta))^2)/t}d\theta$$

$$+c_2 t^{-1}\int_{-\pi/2}^{\pi/2} |x_1|^{1/2}|\cos(\theta)|^{-1/2}e^{i((x_1+\cos(\theta))^2-(x_2-\sin(\theta))^2)/t}d\theta + O(t^{-1/2}),$$

thus reducing to the (1,1) signature metric. Let η be a smooth cutoff, $\eta(x) \equiv 1$ for $|x| \leq 2$, and $\eta \equiv 0$ for $|x| > 3$.

$$t^{1/2}(\eta-\chi)\nabla e^{itL}u_0 \in L^{4-\epsilon}(\mathbf{R}^3). \tag{3.4.18}$$

Lemma 3.4.5. *Away from the caustics,*

$$|\nabla e^{itL}\chi_{B(0;1)}| \lesssim t^{-1/2} \tag{3.4.19}$$

and for $|x| > 10$*,*

$$|\nabla e^{itL}\chi_{B(0;1)}| \lesssim t^{-1/2}|x|^{-1} \tag{3.4.20}$$

Proof: For a point x away from the caustic sets,

$$t^{-3/2}\int_{|y|=1} e^{i|x_p-y_p|^2/t-(x_q-y_q)^2/t}d\sigma(y) \lesssim t^{-1/2},$$

since all the stationary points have Hessians that are bounded below. Now for $|x|$ large, say, $|x| > 10$,

$$t^{-3/2} \int_{|y|=1} e^{i|x_p - y_p|^2/t - (x_q - y_q)^2/t} d\sigma(y),$$

The Hessians of the stationary points are $\sim |x - y| \sim |x|$, which proves (3.4.20) $\square$.

3.5. Local Well-posedness for a metric of signature (2,1)

Let $u_0 = \chi_{B(0;1)}$ in $\mathbf{R}^3$ with

$$L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}.$$

Since $e^{itL}u_0$ is uniformly bounded in L^∞, this suggests the possibility of using the iteration scheme found in [T]. However, there is an additional obstacle. In [T] to iterate in $L^\infty$the Sobolev embedding theorem

$$H^{\sigma,p} \subset L^\infty$$

when $p\sigma > n$ was used. The dispersive estimates in the Duhamel term require $(t - s)^{3(1/p-1/2)}$ to be integrable, forcing $1/2 - 1/p < 1/3$, and $p < 6$. However, a characteristic function lies in $H^{1/2-\epsilon,2}$, which falls just short of the Sobolev embedding needed. So a smoothing estimate is needed.

Theorem 3.5.1. *Let $F \in C^\infty$, $F : \mathbf{R}^2 \to \mathbf{R}^2$, $F(0) = F'(0) = 0$ be the nonlinearity. Then the equation*

$$iu_t + Lu = F(u),$$

(3.5.1)

$$u(0) = u_0 = \chi_{B(0;1)}$$

is locally well-posed on some time interval $[0, T_0)$.

Proof: Start with the function space

(3.5.2)
$$X = \{w(t,x) : \|w(t,\cdot)\|_{H^{1,2}} \leq Ct^{1/2}, \sup_{t\in[0,T]} \|w(t,\cdot)\|_\infty \leq C\},$$

(3.5.3)
$$\Phi(w(t,x)) = \int_0^t e^{i(t-s)L} F(e^{isL}u_0 + w(s,x))ds,$$

for a sufficiently small T, $\Phi : X \to X$. For now, assume three estimates. First,

(3.5.4)
$$\sup_{[0,T]} \|e^{itL}u_0\|_\infty \leq C < \infty.$$

Let χ be a smooth cutoff with compact support and make the decomposition $u_1 = \chi u$ and $u_2 = (1-\chi)u$. The other estimates to arrange for the present are

$$\nabla e^{itL}u_0 = u_1(t,x) + u_2(t,x)$$

(3.5.5)

$$\|u_1(t,x)\|_{L^{7/2-\epsilon}} \leq Ct^{-1/2}, \|u_2(t,x)\|_{L^{3+}\cap L^\infty} \leq Ct^{-1/2},$$

(3.5.6)
$$\sup_{[0,T]} \| \int_0^t e^{i(t-s)L} F(e^{isL}u_0)ds\|_\infty \leq C.$$

(3.5.4) has already been proved via the Gibbs phenomenon. (3.5.5) and (3.5.6) will be proved later.

$$\| \int_0^t e^{i(t-s)L} F(e^{isL}u_0 + w(s,x))ds\|_\infty$$

$$\leq C + \| \int_0^t e^{i(t-s)L}[F(e^{isL}u_0 + w(s,x)) - F(e^{isL}u_0)]ds\|_\infty,$$

$$F(e^{isL}u_0 + w(s,x)) - F(e^{isL}u_0) = w(s,x)\int_0^1 F'(e^{isL}u_0 + \tau w(s,x))d\tau$$

$$= w(s,x)G(e^{isL}u_0, w(s,x)).$$

Now apply the Sobolev embedding theorem $H^{1,3+} \subset L^\infty$.

$$\| \int_0^t e^{i(t-s)L}[F(e^{isL}u_0 + w(s,x)) - F(e^{isL}u_0)]ds\|_{\dot{H}^{1,3+}} \leq$$

$$\| \int_0^t e^{i(t-s)L} w(s,x)\nabla G(e^{isL}u_0, w(s,x))ds\|_{3+}$$

$$+\| \int_0^t e^{i(t-s)L} (\nabla w(s,x))G(e^{isL}u_0, w(s,x))ds\|_{3+}$$

$$\nabla \int_0^1 F'(e^{isL}u_0 + \tau w(s,x))d\tau = \int_0^1 F''(e^{isL}u_0 + \tau w(s,x))(\nabla e^{isL}u_0 + \tau \nabla w(s,x))d\tau.$$

$e^{isL}u_0 + \tau w(s,x)$ uniformly bounded implies

$$F''(e^{isL}u_0 + \tau w(s,x)) \in L^\infty,$$

$$\nabla w(s,x) \in L^2,$$

$$\|\nabla e^{isL} u_0\|_{L^{7/2-\epsilon}+L^{3+}\cap L^\infty} \leq C s^{-1/2}.$$

By the Sobolev embedding theorem, $\|w(s,x)\|_{3-} \leq C s^{1/2}$. This gives the estimate.

$$\|w(s,x)\nabla G(e^{isL}u_0, w(s,x))\|_{3/2-} \leq C.$$

Similarly, $\nabla w(s,x) \in L^2$ and $G(e^{isL}u_0, w(s,x)) \in L^2 \cap L^\infty$, so

$$\|(\nabla w(s,x))G(e^{isL}u_0, w(s,x))\|_{3/2-} \leq C.$$

Combine these results with the dispersive estimates

$$\begin{aligned} &\| \int_0^t e^{i(t-s)L}[F(e^{isL}u_0 + w(s,x)) - F(e^{isL}u_0)]ds\|_{\dot{H}^{1,3+}} \\ &\leq \int_0^t \frac{1}{(t-s)^{3(1/2-1/3+)}} C' ds \leq C' t^{1/2-}, \end{aligned} \tag{3.5.7}$$

taking T sufficiently small proves the L^∞ bounds.

Now for the $\dot{H}^{1,2}$ bounds. $\nabla w \in L^2$ and $G(e^{isL}u_0, w(s,x)) \in L^\infty$, so

$$\nabla w(s,x) G(e^{isL}u_0, w(s,x)) \in L^2.$$

On the other hand, $\nabla \int_0^1 F'(e^{isL}u_0 + \tau w(s,x))d\tau \in L^2 + s^{-1/2}L^{7/2-\epsilon} + s^{-1/2}L^\infty$. The L^2 term and $s^{-1/2}L^\infty$ terms are fine since $w(s,x) \in s^{1/2}L^2 \cap L^\infty$. By the Sobolev embedding $w(s,x) \in s^{1/2}L^6(\mathbf{R}^3) \cap s^{1/2}L^2(\mathbf{R}^3)$. Combine the estimates

$$\| \int_0^t e^{i(t-s)L}[F(e^{isL}u_0 + w(s,x)) - F(e^{isL}u_0)]ds\|_{\dot{H}^{1,2}} \leq \int_0^t C' ds \leq C't, \tag{3.5.8}$$

which satisfies the $\dot{H}^{1,2}$ estimate.

Finally, since $F'(0) = 0$ and $F \in C^\infty$, $G(e^{isL}u_0, w(s,x)) \in L^2 \cap L^\infty$, $w(s,x)G(e^{isL}u_0, w(s,x)) \in L^1 \cap L^2$,

$$\begin{aligned} \| \int_0^t e^{i(t-s)L} w(s,x) G(e^{isL}u_0, w(s,x)) ds\|_2 &\leq \int_0^t C s^{1/2} = C' t^{3/2} \\ \| \int_0^t e^{i(t-s)L} w(s,x) G(e^{isL}u_0, w(s,x)) ds\|_{3+} &\leq \int_0^t C' \frac{s^{1/2}}{(t-s)^{1/2+}} ds = C' t^{1-}, \end{aligned} \tag{3.5.9}$$

which proves $\Phi : X \to X$ for $t \in [0,T]$, T sufficiently small.

A similar argument proves that Φ is a contraction. Define the function.

$$\begin{aligned} G(e^{isL}u_0 + &w_1(s,x), w_2(s,x) - w_1(s,x)) = \\ (w_2(s,x) - w_1(s,x)) &\int_0^1 F'(e^{isL}u_0 + w_1(s,x) + \tau(w_2(s,x) - w_1(s,x)) d\tau. \end{aligned} \tag{3.5.10}$$

Define the norm

(3.5.11)

$$\|w_1(t,x) - w_2(t,x)\|_Y = \|w_1(t,x) - w_2(t,x)\|_\infty + \sup_{t \in [0,T]} t^{-1/2} \|w_1(t,x) - w_2(t,x)\|_{H^{1,2}}.$$

Let $\|w_1 - w_2\|_Y = A$. Then plugging in to (3.5.8) - (3.5.11) gives a contraction, possibly after shrinking T.

$$\|\nabla[(w_1(s,x)-w_2(s,x))G(e^{isL}u_0+w_1(s,x),w_2(s,x)-w_1(s,x))]\|_{3/2-} \leq AC,$$

$$\|\nabla(w_1(s,x)-w_2(s,x))G(e^{isL}u_0+w_1(s,x),w_2(s,x)-w_1(s,x))\|_2 \leq AC,$$

$$\|(w_1(s,x)-w_2(s,x))G(e^{isL}u_0+w_1(s,x),w_2(s,x)-w_1(s,x))\|_{3/2-} \leq AC,$$

$$\|(w_1(s,x)-w_2(s,x))G(e^{isL}u_0+w_1(s,x),w_2(s,x)-w_1(s,x))\|_2 \leq ACs^{1/2},$$

$$t^{-1/2}\int_0^t AC ds \leq ACt^{1/2},$$

$$t^{-1/2}\int_0^t ACs^{1/2} ds \leq ACt,$$

$$\int_0^t \frac{1}{(t-s)^{1/2+}} AC ds \leq ACt^{1/2-}.$$

This gives a contraction for $t \in [0, T_0]$, T_0 sufficiently small $\square$.

The proof will be complete once the estimates (3.5.5) and (3.5.6) are established. By the previous section if η is the cutoff $\eta \equiv 1$ on B(0;3) and $\eta = 0$ for $|x| > 4$, then $\nabla \eta e^{itL} u_0 \in L^{7/2-\epsilon}(\mathbf{R}^3)$.

$$e^{itL}u_0 = t^{-3/2}\int_{B(0;1)} e^{i\psi(x,y)/t}dy.$$

By a rotation of coordinates let $x_2 = 0$. Then $x = (R\cos\varphi, 0, R\sin\varphi)$.

$$\int_{S^1} e^{i\psi(x,y)/t}d\sigma(y)$$

$$= \int_{-\pi/2}^{\pi/2} \cos(\phi)e^{i\sin(\phi)^2/t}\int_0^{2\pi} e^{i(\cos(\phi)^2\cos(2\theta)-2R\cos(\phi)\cos(\theta+\varphi)+R^2\cos(2\varphi))/t}d\varphi d\theta.$$

When $R > 3$, the inner phase function has no double roots. Moreover,

$$\frac{\partial}{\partial\theta}(\sin(\phi)^2\cos(2\theta) - 2R|\sin(\phi)|\cos(\theta+\varphi)) = -2\cos(\phi)^2\sin(2\theta) + 2R\cos(\phi)\sin(\theta+\varphi)$$

$$\frac{\partial^2}{\partial\theta^2}(\sin(\phi)^2\cos(2\theta) - 2R|\sin(\phi)|\cos(\theta+\varphi))$$

$$= -4\cos(\phi)^2\cos(2\theta) + 2R\cos(\phi)\cos(\theta+\varphi) = O(R).$$

For R large, $\sin(\theta+\varphi)$ must be close to zero at a stationary point, so $\cos(\theta+\varphi)$ will be close to one.

$$\int_0^{2\pi} e^{i(\cos(\phi)^2\cos(2\theta)-2R\cos(\phi)\cos(\theta+\varphi)+R^2\cos(2\varphi))/t}d\varphi d\theta$$

$$= t^{1/2}R^{-1/2}\cos(\phi)^{-1/2}e^{i\chi(\tilde{\theta})/t} + tO(R^{-1}).$$

$\tilde{\theta}$ is a stationary point of the phase function

$$\chi(\theta) = \cos(\phi)^2\cos(2\theta) - 2R\cos(\phi)\cos(\theta+\varphi) + R^2\cos(2\varphi).$$

$$\int_{-\pi/2}^{\pi/2} tO(R^{-1}) = tO(R^{-1}).$$

So concentrate on the first term

$$R^{-1/2}t^{1/2}\int_{-\pi/2}^{\pi/2}\cos(\phi)^{1/2}e^{i\sin^2(\phi)/t}e^{i(cos^2(\phi)\cos(2\tilde{\theta})-2R\cos(\phi)\cos(\tilde{\theta}+\varphi)+R^2\cos(2\varphi))/t}d\phi.$$

Since $\cos(\tilde{\theta}+\varphi)$ is close to one,

$$\frac{\partial}{\partial\phi}[\sin^2(\phi)+(cos^2(\phi)\cos(2\tilde{\theta})-2R\cos(\phi)\cos(\tilde{\theta}+\varphi)+R^2\cos(2\varphi))]=0$$

iff $\sin(\phi)$ is close to zero. But in that case

$$\frac{\partial^2}{\partial\phi^2}[\sin^2(\phi)+(cos^2(\phi)\cos(2\tilde{\theta})-2R\cos(\phi)\cos(\tilde{\theta}+\varphi)+R^2\cos(2\varphi))]\approx R,$$

which proves

$$=t^{1/2}R^{-1/2}\int_{-\pi/2}^{\pi/2}\cos(\phi)^{1/2}e^{i\chi(\tilde{\theta})/t}=tO(R^{-1}).$$

Thus outside $B(0;3)$ the pointwise convergence of $e^{itL}u_0$ is $O(\frac{t^{1/2}}{|x|^2})$. The pointwise convergence of $e^{itL}\nabla u_0$ is $O(\frac{1}{|x|t^{1/2}})$. Thus (3.5.5) is proved.

Lemma 3.5.2.

$$\nabla e^{itL}u_0\in L^{7/2-\epsilon}+L^{\infty}\cap L^{3+}. \tag{3.5.12}$$

There is one final piece to prove.

Lemma 3.5.3.

$$\int_0^t e^{i(t-s)L}F(e^{isL}u_0)ds \in L^\infty. \tag{3.5.13}$$

Proof: Use the fact that $e^{isL}u_0 \in C^\infty \cap H^{1/2-\epsilon,2}$ for any $s > 0$. Let x_0 be a point in $\mathbf{R}^3$. Suppose that χ is a smooth cutoff function, $\chi \equiv 1$ on $B(0;7) \cup B(x_0;7)$, and $\chi \equiv 0$ on $(\mathbf{R}^3 \setminus B(0;8)) \cap (\mathbf{R}^3 \setminus B(x_0,8))$.

Make a change of coordinates so that $x_0 = 0$, and then regularize the integral

$$\frac{1}{(t-s)^{3/2}}\int e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}F(e^{isL}u_0)(y)dy$$

$$= \frac{i}{(t-s)^{1/2}}\int F(e^{isL}u_0)(y)\frac{2(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}-y_3\frac{\partial}{\partial y_3})}{y_1^2+y_2^2+y_3^2}e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$= \frac{2i}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_1}(\frac{y_1F(e^{isL}u_0)(y)\chi(y)}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$+\frac{2i}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_2}(\frac{y_2F(e^{isL}u_0)(y)\chi(y)}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$-\frac{2i}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_3}(\frac{y_3F(e^{isL}u_0)(y)\chi(y)}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$+\frac{2i}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_1}(\frac{y_1F(e^{isL}u_0)(y)(1-\chi(y))}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$+\frac{2i}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_2}(\frac{y_2F(e^{isL}u_0)(y)(1-\chi(y))}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy$$

$$-\frac{2i}{(t-s)^{1/2}}\int\frac{\partial}{\partial y_3}(\frac{y_3F(e^{isL}u_0)(y)(1-\chi(y))}{y_1^2+y_2^2+y_3^2})e^{i(y_1^2+y_2^2-y_3^2)/(t-s)}dy,$$

by the product rule, the estimates on $\nabla F(e^{isL}u_0)$, that χ has compact support, and $\frac{1}{|y|^2}$ is integrable in $\mathbf{R}^3$. $F(e^{isL}u_0)\in L^\infty$ and $\frac{1}{|y|^2}\in L^{3/2-}$ locally, so

$$F(e^{isL}u_0)\chi(y)\frac{\partial}{\partial y_i}(\frac{y_i}{|y|^2})\in L^1.$$

If the derivative hits $F(e^{isL}u_0)$ apply the chain rule and the fact that $F'(e^{isL}u_0)\in L^\infty$, to get

$$\|\chi(y)F'(e^{isL}u_0)\frac{\partial}{\partial y_i}e^{isL}u_0\|_{L^{4-\epsilon}}\leq Cs^{-1/2}.$$

On the other hand, $\frac{1}{|y|}\in L^{3-}(supp(\chi))$, so by Holder's inequality,

$$\|\chi(y)\frac{y_i}{|y|^2}F'(e^{isL}u_0)\frac{\partial}{\partial y_i}e^{isL}u_0\|_{L^1}\leq Cs^{-1/2}.$$

Finally, $\nabla\chi\in L^\infty$ and is compactly supported on the support of χ, $F(e^{isL}u_0)\in L^\infty$ and $\frac{1}{|y|}\in L^{3-}(supp(\chi))$ so

$$\frac{\partial}{\partial y_i}(\frac{y_i}{|y|^2}F(e^{isL}u_0)(y)\chi(y))\in s^{-1/2}L^1 \tag{3.5.14}$$

$$\frac{1}{(t-s)^{1/2}}\int\frac{\partial}{\partial y_i}(\frac{y_iF(e^{isL}u_0)(y)\chi(y)}{y_1^2+y_2^2+y_3^2})dy\leq\frac{C}{(t-s)^{1/2}s^{1/2}}.$$

Now for the integral on $1-\chi(y)$.

Outside B(0;7), $e^{isL}u_0\sim s^{1/2}r^{-2}$, where r is the distance from $\partial\Omega$ to the point. Since $F'(0)=0$ and $F\in C^\infty$,

$$|\frac{\partial}{\partial y_i}F(e^{isL}u_0)| = |F'(e^{isL}u_0)\frac{\partial}{\partial y_i}(e^{isL}u_0)| \leq C\frac{s^{1/2}}{r^2}\frac{1}{s^{1/2}r} = \frac{C}{r^3}$$

$$\Rightarrow |\frac{y_i}{|y|^2}\nabla F(e^{isL}u_0)(1-\chi(y))| \in L^1,$$

$$\frac{1}{|y|^2} \in L^2(\mathbf{R}^3 \setminus B(0;7)),$$

$$\frac{1}{(t-s)^{1/2}}\int \frac{\partial}{\partial y_i}(\frac{y_i F(e^{isL}u_0)(y)(1-\chi(y))}{y_1^2+y_2^2+y_3^2})dy \leq \frac{C}{(t-s)^{1/2}}.$$

Since

$$\int_0^t \frac{C}{(t-s)^{1/2}s^{1/2}} + \frac{C}{(t-s)^{1/2}}ds \leq C',$$

the proof is complete □.

CHAPTER 4

Strichartz Estimates, power-type nonlinearities

4.1. Strichartz Estimates

Define the space of functions, $L_t^p L_x^q$.

$$\|f(t,x)\|_{L_t^p L_x^q(\mathbf{R}\times\mathbf{R}^n)} = [\int_{\mathbf{R}} \{\int_{\mathbf{R}^n} |f(t,x)|^q dx\}^{p/q} dt]^{1/p}. \tag{4.1.1}$$

DEFINITION 4.1.1. (p,q) is an admissible pair for n if $p > 2$, $q > 2$, and

$$\frac{2}{q} = n(\frac{1}{2} - \frac{1}{p}). \tag{4.1.2}$$

For such admissible pairs (p,q).

$$\|e^{it\Delta}\phi\|_{L_t^p L_x^q(\mathbf{R}\times\mathbf{R}^n)} \leq C(n,p,q)\|f\|_{L^2(\mathbf{R}^n)}. \tag{4.1.3}$$

This estimate is proved in [**2**], [**24**]. This estimate is also proved in [**23**] for any family of operators $U(t)$ where

$$\begin{aligned} \|U(t)(U(s))^* g\|_{L^\infty(\mathbf{R}^n)} &\lesssim |t-s|^{-n/2}\|g\|_{L^1(\mathbf{R}^n)} \\ \|U(t)(U(s))^* g\|_{L^2(\mathbf{R}^n)} &\lesssim \|g\|_{L^2(\mathbf{R}^n)} \end{aligned} \tag{4.1.4}$$

The operator e^{itL},

$$L = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}, \tag{4.1.5}$$

also satisfies the dispersive estimates in (4.1.4). The kernel of the operator is

$$K'(x,y) = \frac{1}{4\pi t} e^{-i\frac{(x_1-y_1)^2}{4t}} e^{i\frac{(x_2-y_2)^2}{4t}}. \tag{4.1.6}$$

Remark: For an operator of signature (p,q) take the decomposition $\mathbf{R}^n = \mathbf{R}^p \oplus \mathbf{R}^q$, x_p is the projection onto $\mathbf{R}^p$ and x_q is the projection onto $\mathbf{R}^q$, and make a similar decomposition for $y = (y_p, y_q)$. The kernel is

$$K''(x,y) = \frac{e^{i\pi(p-q)/4}}{(4\pi t)^{n/2}} e^{-i\frac{|x_p-y_p|^2}{4t}} e^{i\frac{|x_q-y_q|^2}{4t}}. \tag{4.1.7}$$

For the convenience of the reader, the non-endpoint Strichartz estimates will be proved. An identical proof can be found in [**23**].

Theorem 4.1.2. *Suppose (p,q) are admissible pairs. Then*

$$\| \int_{\mathbf{R}} e^{-isL} F(s,\cdot) ds \|_2 \leq C(n,q,p) \|F\|_{L_t^{q'} L_x^{p'}(\mathbf{R}\times\mathbf{R}^n)}. \tag{4.1.8}$$

Proof:

$$\| \int_{\mathbf{R}} e^{i(t-s)L} F(s,\cdot) ds \|_{L_t^q L_x^p(\mathbf{R}\times\mathbf{R}^n)} \leq \| \int_{\mathbf{R}} \| e^{i(t-s)L} F(s,\cdot) \|_{L_x^p(\mathbf{R}^{\mathbf{n}})} ds \|_{L_t^q(\mathbf{R})} \tag{4.1.9}$$

$$\leq C(n,p) \| \int_{\mathbf{R}} \|F(s,\cdot)\|_{L_x^{p'}(\mathbf{R}^d)} \frac{1}{|t-s|^{n(1/2-1/p)}} ds \|_{L_t^q(\mathbf{R})}.$$

Since (p,q) is an admissible pair, $n(\frac{1}{2} - \frac{1}{p}) = \frac{2}{q}$. Thus

(4.1.10) $$(4.1.9) \le C(n,q) \| \int \|F(s,\cdot)\|_{L_x^{p'}(\mathbf{R}^n)} \frac{1}{|t-s|^{2/q}} ds \|_{L_t^q(\mathbf{R})}.$$

Now apply the Hardy - Littlewood - Sobolev theorem

THEOREM 4.1.3.

(4.1.11) $$\|f * \frac{1}{|x|^\alpha}\|_{L_x^q(\mathbf{R}^m)} \le C(p,r,m)\|f\|_{L_x^p(\mathbf{R}^m)}$$

for $1 < p < q < \infty$, $0 < \alpha < m$, *and* $\frac{1}{p} = \frac{1}{q} + \frac{m-\alpha}{m}$.

Proof: See [**13**].

Apply the theorem to (4.1.10) when m = 1, $\frac{1}{q} + (1 - \frac{2}{q}) = 1 - \frac{1}{q} = \frac{1}{q'}$.

(4.1.12) $$(4.1.9) \le C(m,p,q)\|F\|_{L_t^{q'} L_x^{p'}}$$

Now choose f(t,x) such that $\|f\|_{L_t^{q'} L_x^{p'}(\mathbf{R}\times\mathbf{R}^n)} = 1$. Let $F(s) = f(s,\cdot)$.

$$\| \int_{\mathbf{R}} e^{-isL} F(s,\cdot) ds\|_2^2 = \int_{\mathbf{R}} \int_{\mathbf{R}} \langle e^{-isL} F(s,\cdot), e^{-itL} F(t,\cdot)\rangle ds dt$$

(4.1.13) $$= \int_{\mathbf{R}} \int_{\mathbf{R}} \langle e^{i(t-s)L} F(s,\cdot), F(t,\cdot)\rangle ds dt \le \| \int_{\mathbf{R}} e^{i(t-s)L} F(s,\cdot) ds\|_{L_t^q L_x^p} \|F(t,\cdot)\|_{L_t^{q'} L_x^{p'}}.$$

By (4.1.12),

(4.1.14) $$\| \int_{\mathbf{R}} e^{i(t-s)L} F(s,\cdot) ds\|_{L_t^q L_x^p(\mathbf{R}\times\mathbf{R}^n)} \le C(n,p,q)\|F\|_{L_t^{q'} L_x^{p'}(\mathbf{R}\times\mathbf{R}^n)},$$

$$\| \int e^{-isL} F(s,\cdot)ds\|^2_{L^2(\mathbf{R}^n)} \leq C(n,p,q)\|F\|^2_{L_t^{q'}L_x^{p'}(\mathbf{R}\times\mathbf{R}^n)}. \tag{4.1.15}$$

The proof is complete □.

Now apply a duality argument. Take $G(t,x) \in L_t^{q'}L_x^{p'}$.

$$\int_{\mathbf{R}} \langle e^{itL}u_0, G(t,x)\rangle dt = \langle u_0, \int_{\mathbf{R}} e^{-itL}G(t,x)dt\rangle \leq \|u_0\|_2 \| \int_{\mathbf{R}} e^{-itL}G(t,x)dt\|_2$$

By Theorem [4.1.2]

$$\| \int_{\mathbf{R}} e^{-itL}G(t,\cdot)dt\|_2 \leq C(n,p,q)\|G\|_{L_t^{q'}L_x^{p}},$$

This proves the estimate,

$$\|e^{it\Delta}u_0\|_{L_t^pL_x^q(\mathbf{R}\times\mathbf{R}^n)} \leq C\|u_0\|_{L^2(\mathbf{R}^n)}. \tag{4.1.16}$$

Remark: For the Hardy - Littlewood - Sobolev theorem, $0 < \alpha < n$. This requires $\frac{2}{q} > 0$, so $q < \infty$, and $\frac{2}{q} < 1$, so $q > 2$.

Remark: The same estimates also hold for $e^{t\Delta/(a+ib)}$, where $a > 0$.

4.2. Local Well-posedness for L^2 critical

It will first be necessary to prove a local well-posedness result in L^2. This proof can be found in [**24**], the original proof is given by [**31**].

Theorem 4.2.1. *Suppose there is the equation*

$$iu_t + \Delta u = |u|^{4/d}u, \quad u(0,x) = u_0(x) \in L^2. \tag{4.2.1}$$

This equation is locally wellposed on some $[0,T]$, *for* $T(u_0)$.

Proof: Define the Strichartz space. Let I be some interval. Recall that (p,q) is an admissible pair if $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$. Set

$$\|u(t,x)\|_{S^0(I\times\mathbf{R}^n)} = \sup_{(p,q)} \|u(t,x)\|_{L_t^p L_x^q(I\times\mathbf{R}^n)}. \tag{4.2.2}$$

By the dominated convergence theorem, given u_0, for I sufficiently small

$$\|e^{it\Delta}u_0\|_{S^0(I\times\mathbf{R}^n)} < \epsilon. \tag{4.2.3}$$

Let $N^0(I\times\mathbf{R}^n)$ be the dual Strichartz space to $S^0(I\times\mathbf{R}^n)$.

$$\|u\|_{N^0(I\times\mathbf{R}^n)} \leq \|u\|_{L_t^{p'} L_x^{q'}(I\times\mathbf{R}^n)}, \tag{4.2.4}$$

where p', q' are the dual exponents of p and q.

Suppose $u(t,x)$ solves Duhamel's formula,

$$u(t,x) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}F(s)ds. \tag{4.2.5}$$

$$\|u(t,x)\|_{S^0(I\times\mathbf{R}^n)} \leq C\|u_0\|_{L^2(\mathbf{R}^n)} + C\|F\|_{N^0(I\times\mathbf{R}^n)}, \tag{4.2.6}$$

$$F(s) = |u_{j-1}(s)|^{4/d}u_{j-1}(s),$$

$$\||u_{j-1}(s)|^{4/n}u_{j-1}\|_{L_t^2L_x^{2n/(n+2)}} \leq \|u_{j-1}\|_{L_t^2L_x^{2n/(n-2)}(I\times\mathbf{R}^n)}\||u_{j-1}|^{4/n}\|_{L_t^\infty L_x^{n/2}(I\times\mathbf{R}^n)}$$

$$\leq C\|u_{j-1}\|_{S^0}^{1+4/n},$$

so for $\|u_{j-1}\|_{S^0(I\times\mathbf{R}^n)} \leq \epsilon$ for ϵ sufficiently small, the Duhamel map,

$$\Phi(u_j(s)) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}|u_{j-1}(s)|^{4/n}u_{j-1}(s)ds, \tag{4.2.7}$$

satisfies

$$\begin{gathered}\Phi : X \to X,\\ X = \{u : \|u\|_{S^0(I\times\mathbf{R}^n)} \leq 2\epsilon\}.\end{gathered} \tag{4.2.8}$$

Moreover, the map is a contraction.

$$\|u_j - u_{j-1}\|_{S^0} \leq C\|u_j - u_{j-1}\|_{S^0(I\times\mathbf{R}^n)}\||u_j|^{4/d} + |u_{j-1}|^{4/d}\|_{S^0(I\times\mathbf{R}^n)}$$

$$\leq C\epsilon^{4/d}\|u_j - u_{j-1}\|_{S^0(I\times\mathbf{R}^n)}.$$

Thus the map is a contraction for ϵ sufficiently small.

Take $u_1(t,x) = e^{it\Delta}u_0$. As $\delta \to 0$,

$$\|e^{it\Delta}u_0\|_{S^0([t,t+\delta)\times \mathbf{R}^n)} \to 0, \tag{4.2.9}$$

so for δ sufficiently small, $\|e^{it\Delta}u_0\|_{S^0([t,t+\delta)\times \mathbf{R}^n)} \leq \epsilon$. This gives local well-posedness $\square$.

Remark: The δ depends on the profile of the initial data, not just the L^2 norm.

Remark: If $u_0 \in H^s(\mathbf{R}^n)$ for some $s > 0$ then $I = [0,T]$, $T(\|u_0\|_{H^s}) > 0$, where T depends only on the size of $\|u_0\|_{H^s(\mathbf{R}^n)}$.

LEMMA 4.2.2. *If $u_0 \in \dot{H}^\rho(\mathbf{R}^n)$ for some $0 \leq \rho < \frac{n}{2}$, $n \geq 3$, then*

$$\begin{aligned} iu_t + \Delta u &= |u|^\alpha u, \\ u(0,x) &= u_0(x), \\ \alpha &= \frac{4}{n-2\rho}, \end{aligned} \tag{4.2.10}$$

has a solution for some interval $[0,T]$, where $T(u_0) > 0$.

Proof: By the Sobolev embedding theorem, when $\frac{1}{p} = \frac{1}{2} - \frac{\rho}{n}$ and $\rho < \frac{n}{2}$,

$$\|e^{it\Delta}u_0\|_{L_t^\infty L_x^p} \leq C\|u_0\|_{\dot{H}^\rho(\mathbf{R}^n)}.$$

Consider three cases separately. Let $\alpha = \frac{4}{n-2\rho}$. For any u_0, (4.2.8) and the dominated convergence theorem implies that for any $\epsilon > 0$ there is an interval I such that $\||\nabla|^\rho e^{it\Delta}u_0\|_{L_t^2 L_x^{2n/(n-2)}(I\times\mathbf{R}^n)} \leq \epsilon$.

Case 1, $\alpha \geq 1$ and $\rho < \frac{n}{2} - 1$:

$$\||\nabla|^{\rho}|u|^{\alpha}u\|_{L^1_tL^2_x} \leq C\|u\|_{L^2_tL^q_x}\||\nabla|^{\rho}u\|_{L^2_tL^q_x}\||u|^{\alpha-1}\|_{L^\infty_tL^r_x},$$

when $\frac{1}{q} = \frac{1}{2} - \frac{\rho+1}{n}$, $\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$, $\frac{1}{r} = \frac{1}{2} - \frac{\rho}{n}$.

$$\||u|^{\alpha-1}\|_{L^\infty_tL^r_x(I\times\mathbf{R}^n)} \leq \||\nabla|^{\rho}u\|^{\alpha-1}_{L^\infty_tL^2_x(I\times\mathbf{R}^n)},$$

by the Sobolev embedding theorem. So iterate in the space

$$\{u : \|u\|_{L^\infty_t\dot{H}^{\rho}(I\times\mathbf{R}^n)} < 2\|u_0\|_{\dot{H}^{\rho}(\mathbf{R}^n)} = A, \|u\|_{L^2_tL^{2n/(n-2)}_x(I\times\mathbf{R}^n)} < 2\epsilon\}. \tag{4.2.11}$$

Define a sequence with $v_0 \equiv 0$, and

$$v_n(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}|v_{n-1}(s)|^{\alpha}v_{n-1}(s)ds. \tag{4.2.12}$$

This sequence has the estimates

$$\|v_n(t)\|_{L^2_tL^{2n/(n-2)}_x(I\times\mathbf{R}^n)} \leq \|e^{it\Delta}u_0\|_{L^2_tL^{2n/(n-2)}_x(I\times\mathbf{R}^n)} + (2\epsilon)^2A^{\alpha-1},$$

$$\|v_n(t)\|_{L^\infty_t\dot{H}^{\rho}(I\times\mathbf{R}^n)} \leq \|e^{it\Delta}u_0\|_{L^\infty_t\dot{H}^{\rho}(I\times\mathbf{R}^n)} + (2\epsilon)^2A^{\alpha-1}.$$

So for ϵ sufficiently small the iteration stays in (4.2.8). A contraction is obtained by a similar method.

Case 2, $\alpha < 1$*:*

$$\||\nabla||u|^{\alpha}u\|_{L^{p'}_tL^{q'}_x(I\times\mathbf{R}^n)} \leq \||\nabla|^{\rho}u\|_{L^2_tL^{2n/(n-2)}_x(I\times\mathbf{R}^n)}\|u\|^{\alpha}_{L^{2/\alpha}_tL^{r/\alpha}_x(I\times\mathbf{R}^n)},$$

where $\frac{1}{p'} = \frac{1}{2} + \frac{2}{\alpha}$ and $\frac{1}{q'} = \frac{1}{2} + \frac{1}{n} - \frac{\alpha}{n}$. (p', q') are again the dual exponents to an admissible pair, and $\frac{1}{r} = \frac{1}{2} - \frac{1}{n} - \frac{\rho}{n}$. Once again apply the Duhamel estimates

$$\|v_n(t)\|_{L_t^2 L_x^{2n/(n-2)}(I \times \mathbf{R}^n)} \leq \|e^{it\Delta} u_0\|_{L_t^2 L_x^{2n/(n-2)}(I \times \mathbf{R}^n)} + (2\epsilon)^{\alpha+1} A^{\alpha-1},$$

$$\|v_n(t)\|_{L_t^\infty \dot{H}^\rho(I \times \mathbf{R}^n)} \leq \|e^{it\Delta} u_0\|_{L_t^\infty \dot{H}^\rho(I \times \mathbf{R}^n)} + (2\epsilon)^{\alpha+1} A^{\alpha-1}.$$

There is also local existence for ϵ sufficiently small.

Case 3, $\rho \geq \frac{n}{2} - 1$:

$$\||\nabla|^\rho |u|^\alpha u\|_{L_t^2 L_x^{2n/(n+2)}(I \times \mathbf{R}^n)} \leq C \|u\|_{L_t^\infty \dot{H}^\rho(I \times \mathbf{R}^n)} \|u\|^\alpha_{L_t^{2\alpha} L_x^q(I \times \mathbf{R}^n)},$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{n\alpha}$. Again apply the Duhamel estimates $\Box$.

LEMMA 4.2.3. *There exists a $T(\|u_0\|_{H^{\rho+\epsilon}(\mathbf{R}^n)}) > 0$ such that (**??**) has a local solution for $[0, T]$.*

Proof: By the Sobolev embedding theorem,

$$\||\nabla|^\rho e^{it\Delta} u_0\|_{L_t^2 L_x^{2n/(n-2)}([0,T] \times \mathbf{R}^n)} \leq T^{\delta(\epsilon,n)} \||\nabla|^\rho e^{it\Delta} u_0\|_{L_t^p L_x^q([0,T] \times \mathbf{R}^n)}, \tag{4.2.13}$$

for some admissible pair (p, q) with $\frac{1}{q} = \frac{n-2}{2n} + \frac{\epsilon}{n}$. Then Holder's inequality and the Sobolev embedding theorem imply (4.2.13). Another linear estimate will be needed later in the paper.

LEMMA 4.2.4. *For u radial,*

$$\||x|^{(n-1)/2}u\|_{L^\infty(\mathbf{R}^n)} \lesssim \|u\|_{H^{1/2+\epsilon,2}(\mathbf{R}^n)}. \tag{4.2.14}$$

Proof: By stationary phase,

$$\int_{|\xi|\geq 1/|x|} e^{ix\cdot\xi}\hat{f}(|\xi|)d\xi,$$

$$= C_1\int_{1/|x|}^{\infty}\frac{r^{(n-1)/2}}{|x|^{(n-1)/2}}\hat{f}(r)e^{ir|x|}dr + C_2\int_{1/|x|}^{\infty}\frac{r^{(n-1)/2}}{|x|^{(n-1)/2}}\hat{f}(r)e^{-ir|x|}dr$$

$$+ \text{Lower order terms}$$

Since $|\xi|^{1/2+\epsilon}|\xi|^{(n-1)/2}\hat{f}(|\xi|) \in L^2(\mathbf{R})$, the integral converges.

$$\int_{|\xi|\leq\frac{1}{|x|}} |\xi|^{(n-1)}\hat{f}(\xi)d\xi \leq |x|^{(n/2-1-\epsilon)}\|f\|_{H^{1/2+\epsilon}(\mathbf{R}^n)}.$$

This proves the lemma $\square$.

4.3. Global well-posedness of the Schrödinger equation

For the defocusing nonlinear Schrödinger equation, it is possible to prove global well-posedness for initial data $u_0 \in H^1$. This can be found in [**38**].

THEOREM 4.3.1. *The de-focusing Schödinger equation*

$$\begin{aligned} iu_t + \Delta u &= |u|^\sigma u, \\ u(0,x) &= u_0(x) \in H^1(\mathbf{R}^n), \end{aligned} \tag{4.3.1}$$

is globally well-posed for $\sigma < \frac{4}{n-2}$.

Proof: The length of the interval where the solution to Lemma [4.2.2] is controlled by $\|u_0\|_{H^1(\mathbf{R}^n)}$. Therefore, global well-posedness can only fail if the H^1 norm of $u(t)$ explodes to infinity sufficiently rapidly. This cannot happen, due to the following calculation.

$$\partial_t \langle \nabla u, \nabla u \rangle = -\langle u_t, \Delta u \rangle - \langle \Delta u, u_t \rangle$$

$$= -\langle u_t, |u|^\sigma u - iu_t \rangle - \langle |u|^\sigma u - iu_t, u_t \rangle.$$

$$= -\frac{1}{2}\partial_t \int |u|^{\sigma+2} dx$$

Therefore there is conservation of the energy

$$E(u)(t) = \|u\|_{\dot{H}^1} + \frac{1}{2}\|u\|_{\sigma+2}^{\sigma+2}. \tag{4.3.2}$$

The Sobolev embedding theorem gives $H^1(\mathbf{R}^n) \subset L^{\sigma+2}$ for $\sigma \leq \frac{4}{n-2}$, $n \geq 3$. There is also conservation of mass.

$$M(u)(t) = \|u(t, \cdot)\|_{L^2(\mathbf{R}^n)}^2 \tag{4.3.3}$$

$$\frac{d}{dt} \int |u(t,x)|^2 dx = \int (i\Delta u)\bar{u} dx - \int (i|u|^\alpha u)\bar{u} dx - \int (i\Delta \bar{u}) u dx + \int u(i|u|^\alpha \bar{u}) dx = 0.$$

Therefore mass is conserved as well. □

4.4. Some Remarks about previous results of local and global well-posedness

In a recent paper [**27**], local existence for $n = 1, 2$, $u_0 = \chi_{B(0;1)}$ was proved.

THEOREM 4.4.1. *There exists a $T_* > 0$ such that*

$$iu_t + \Delta u = F(u) \tag{4.4.1}$$

has a solution in the function space

$$\|u\|_X = \sup_{0 \le t \le T_*} \{\|u(t,\cdot)\|_{H^{\sigma,2}(\mathbf{R}^n)} + \|u(t,\cdot)\|_{L^\infty(\mathbf{R}^n)}\} \tag{4.4.2}$$

when $u_0 = \chi_{B(0;1)}$, where the nonlinearity $F : \mathbf{C} \to \mathbf{C}$ obeys $F(0) = 0$ and $DF(0) = 0$.

The proof relies heavily on the uniform bound $\|e^{it\Delta}u_0\|_{L^\infty(\mathbf{R}^2)} \le C$. This crucial fact is not true for higher dimensions due to the focusing phenomenon. When $n = 1$

$$e^{it\Delta}u_0(x) \to u_0(x) \tag{4.4.3}$$

pointwise as $t \to 0$. When $n = 2$ there is pointwise convergence everywhere except the center of $B(0; 1)$, at the origin the convergence has the form

$$e^{it\Delta}u_0(0) = 1 + Ce^{-i\alpha/t} + o(1) \tag{4.4.4}$$

For some constant C, and $\alpha \in \mathbf{R}$, $\alpha \neq 0$. In higher dimensions for small t the Pinsky phenomenon for small t is of the form

$$|e^{it\Delta}u_0(0)| \sim t^{-(n-2)/2} \tag{4.4.5}$$

So instead it is necessary to restrict the type of nonlinearities F to power-type nonlinearities that are $\dot{H}^1$ - subcritical.

Some of these local solutions can be extended to global solutions. Progress of extending the results of the previous section began with [**1**], where the equation

$$iu_t + \Delta u = |u|^2 u,$$

was proved to have a global solution in $\mathbf{R}^2$ for $u_0 \in H^s(\mathbf{R}^2)$, $s > 3/5$. These results lead [**20**], [**18**], [**19**], and [**32**] to prove to extend global well-posedness results in $H^1(\mathbf{R}^n)$ for a $\dot{H}^1$ subcritical equation (usually L^2 critical or $\dot{H}^{1/2}$ critical in $\mathbf{R}^3$) for $s < 1$. This was accomplished by solving the equation

$$iIu_t + I\Delta u = I(|u|^{\alpha}u), \tag{4.4.6}$$

where

$$If = m(|\xi|)\hat{f}(\xi), \tag{4.4.7}$$

and $m(|\xi|)$ decays like $|\xi|^{s-1}$ for large $|\xi|$. Then $Iu_0 \in H^1$, and for certain values of s depending on α and n, $\|Iu(t)\|_{\dot{H}^1}$ is bounded by some function of time for all t.

The I - method relies heavily on scaling. $\|Iu_0\|_{\dot{H}^1(\mathbf{R}^n)} \sim N^{1-s}\|u_0\|_{H^s(\mathbf{R}^n)}$. This is rescaled so that $\|Iu_0\|_{\dot{H}^1(\mathbf{R}^n)} \sim 1$. If $N = \infty$,

$$\frac{d}{dt}E(I_N u(t)) = 0.$$

This would suggest that taking N large would lead to a small change in energy, with $E(I_N u_0) \sim 1$. Using bilinear estimates and careful harmonic analysis, estimates like

$$\frac{d}{dt}E(I_N u(t)) \sim N^{-c},$$

for some $c > 0$ on the interval of local well-posedness. Then, for s sufficiently close to 1, the method can be iterated many times, and eventually the interval $[0, T]$ is covered for any $T > 0$. More recently, in [**11**] the results have been extended using a Morawetz estimate.

Here a different approach is used. Recall that for the local solution of

(4.4.8)
$$iu_t + Lu = F(u),$$
$$u(0) = \chi_{B(0;1)}$$

in $\mathbf{R}^3$, the solution was of the form $e^{itL}u_0 + w(t,x)$, where $w(t,x) \in H^1(\mathbf{R}^3)$. In other words the Duhamel term is smoother than u_0. For certain initial data, the linear term $e^{it\Delta}u_0$ can be treated as a perturbation of the solution of

(4.4.9)
$$iv_t + \Delta v = |v|^{\alpha}v,$$
$$v(T,\cdot) = w(T,\cdot),$$

proving in fact the solution is of the form $e^{it\Delta}u_0 + w(t,x)$, $w(t,x) \in H^1(\mathbf{R}^n)$ for all time.

This method can be generalized. There are three principal advantages to this method.

1. The method can extend certain supercritical local existence results.

2. The method does not rely on rescaling, and therefore can be extended to combined power-type nonlinearities.

3. The method also proves an estimate of the form

$$\|u(t,\cdot) - e^{it\Delta}u_0\|_{H^{\sigma,2}(\mathbf{R}^n)} \leq C < \infty,$$

for a higher order regularity $\sigma > 0$, $u_0 \notin H^{\sigma}(\mathbf{R}^n)$. The disadvantage of the method is it requires some additional structure on u_0.

4.5. Supercritical Local Existence

Let Ω be a compact, smoothly bounded region in $\mathbf{R}^n$. By [**27**], when $n = 1, 2$, the equation

$$iu_t + \Delta u = |u|^{2k}u, \qquad (4.5.1)$$
$$u(0,x) = \chi_{\Omega},$$

has a local solution on $[0,T]$ for some $T(\Omega) > 0$, for any k. When $n = 2$, this means (4.5.1) has a local solution as long as the exponent is $\dot{H}^{\rho}$ - critical for $\rho < 1$, despite the fact that $\chi_{\Omega} \in H^{1/2-}(\mathbf{R}^n)$. In higher dimensions, if (4.5.1) is $\dot{H}^1$ - subcritical, then the solution exists locally.

Theorem 4.5.1. *The initial value problem*

$$iu_t + \Delta u = |u|^\alpha u, \tag{4.5.2}$$
$$u_0 = \chi_{B(0;1)},$$

has a local solution as long as $\alpha < \frac{4}{n-2}$.

Combining the estimate (4.6.16) with $e^{it\Delta}u_0 \in L_t^\infty L_x^2([0,\infty) \times \mathbf{R}^n)$ proves

$$e^{it\Delta}u_0 \in L_t^\infty L_x^p([0,\infty) \times \mathbf{R}^n), \tag{4.5.3}$$

as long as $2 \leq p < \frac{2n}{n-2}$. This is in fact true for $u_0 = \chi_\Omega$ where Ω is any compact set with smooth boundary. Set $p = 2 + \alpha$,

$$p < 2 + \frac{4}{n-2} = \frac{2n-4}{n-2} + \frac{4}{n-2} = \frac{2n}{n-2}.$$

Define the Duhamel operator

$$\Phi(u(t)) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}u(s)ds \tag{4.5.4}$$

Then

$$\|\Phi u(t)\|_p \leq \|e^{it\Delta}u_0\|_p + \int_0^t \frac{1}{(t-s)^{n(1/2-1/p)}}\|u(s)\|_p^{1+\alpha}ds, \tag{4.5.5}$$

$$\frac{1}{2+\alpha} = 1 - \frac{1+\alpha}{2+\alpha},$$

$$\| |u(s)|^{\alpha} u(s) \|_{L^{p'}(\mathbf{R}^n)} = \|u\|^{1+\alpha}_{L^p(\mathbf{R}^n)}, \tag{4.5.6}$$

$$\frac{1}{2} - \frac{1}{p} < \frac{1}{2} - \frac{n-2}{2n} = \frac{1}{n}.$$

Thus the Duhamel term is integrable, and

$$\Phi : L_t^{\infty} L_x^p \to L_t^{\infty} L_x^p \tag{4.5.7}$$

$$\|\Phi u(t)\|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)} \leq \|e^{it\Delta} u_0\|_{L_t^{\infty} L_x^p(\mathbf{R}\times\mathbf{R}^n)} + T^{\epsilon} \|u\|^{1+\alpha}_{L_t^{\infty} L_x^p}. \tag{4.5.8}$$

Suppose $\|e^{it\Delta} u_0\|_{L_t^{\infty} L_x^p(\mathbf{R}\times\mathbf{R}^n)} \leq C$. For T sufficiently small

$$\|\Phi(u(t))\|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)} \leq 2C, \tag{4.5.9}$$

$$\|\Phi(u(t)) - \Phi(v(t))\|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)} =$$

$$\| \int_0^t e^{i(t-s)\Delta} [|u(s)|^{\alpha} u(s) - |v(s)|^{\alpha} v(s)] ds \|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)}$$

$$\leq \|u - v\|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)} (2C)^{\alpha} \int_0^T \frac{1}{(t-s)^{1/2-1/p}} dt$$

$$\leq \|u - v\|_{L_t^{\infty} L_x^p([0,T]\times\mathbf{R}^n)} (2C)^{\alpha} T^{\epsilon}.$$

When T is sufficiently small there is a contraction $\square$.

Remark: These carry over for any $u_0 = \chi_\Omega$, Ω a compact subset of $\mathbf{R}^n$ for smooth boundary.

4.6. A Local Result in $\mathbf{R}^3$

THEOREM 4.6.1. *The nonlinear Schrodinger equation for* $u : \mathbf{R}^3 \to \mathbf{C}$*,*

$$iu_t + \Delta u = |u|^2 u, \tag{4.6.1}$$

has a local solution when $u(0) = \chi_{B(0;1)}$ *of the form*

$$e^{it\Delta} u_0 + w(t,x), \tag{4.6.2}$$

where $w(t,x) \in L_t^\infty H_x^1([0,T_*] \times \mathbf{R}^3)$.

Proof: $e^{it\Delta}u_0$ can be calculated explicitly.

$$e^{it\Delta}u_0(x) = \frac{C}{t^{3/2}} \int_0^1 r^2 e^{i(r^2+|x|^2)/4t} \int_{-\pi/2}^{\pi/2} e^{-2i|x|r\sin(\theta)/4t} \cos(\theta) d\theta dr$$

$$= Ct^{-3/2} \int_0^1 r^2 e^{i(r^2+|x|^2)/4t} \int_{-1}^{1} e^{-2i|x|ru/4t} du dr$$

$$= Ct^{-1/2} \int_0^1 \frac{r}{|x|} [e^{i(r-|x|)^2/4t} - e^{i(r+|x|)^2/4t}] dr$$

Now make a stationary phase estimate.

$$Ct^{-1/2} \int_0^1 e^{i(r\pm|x|)^2/4t} dr = C \int_{\pm|x|t^{-1/2}}^{(1\pm|x|)t^{-1/2}} e^{ir^2} dr = O(1).$$

For large $|x|$, say $|x| > 10$,

$$t^{-1/2}\int_0^1 e^{i(r\pm|x|)^2/4t}dr = \int_0^1 \frac{2t^{1/2}}{i(r\pm|x|)}\frac{d}{dr}e^{i(r\pm|x|)^2/4t}dr$$

$$= \frac{2t^{1/2}}{i(r\pm|x|)}e^{i(r\pm|x|)^2/4t}|_0^1 + \frac{2t^{1/2}}{i}\int_0^1 \frac{1}{(r\pm|x|)^2}e^{i(r\pm|x|)^2/4t}dr = O(\frac{t^{1/2}}{|x|}).$$

Also, by a change of variables,

$$Ct^{-1/2}\int_0^1 \frac{(r-|x|)}{|x|}e^{i(r-|x|)^2/4t}dr \leq O(\frac{t^{1/2}}{|x|}),$$

$$Ct^{-1/2}\int_0^1 \frac{(r+|x|)}{|x|}e^{i(r+|x|)^2/4t}dr \leq O(\frac{t^{1/2}}{|x|}).$$

Since the radial derivative of $\chi_{B(0;1)}$ is a Dirac measure supported on $|x| = 1$,

$$\frac{d}{dr}e^{it\Delta}\chi_{B(0;1)} = Ct^{-3/2}e^{i|x|^2/4t}\int_{-1}^1 e^{-2i|x|u/4t}du = Ct^{-1/2}|x|^{-1}e^{i|x|^2/4t}[e^{-2i|x|/4t} - e^{2i|x|/4t}].$$

More generally,

$$|\nabla e^{it\Delta}\chi_\Omega| \lesssim |x|^{-1}t^{-1/2}.$$

Next, set up a Duhamel iteration. Let $w_0(t,x) = 0$ and

$$w_n(t,x) = \int_0^t e^{i(t-s)\Delta}|e^{is\Delta}u_0 + w_{n-1}(s,x)|^2(e^{is\Delta}u_0 + w_{n-1}(s,x))ds. \tag{4.6.3}$$

If $w(t,x)$ is a fixed point in $L_t^\infty H_x^1([0,T]\times\mathbf{R}^n)$ then $e^{it\Delta}u_0 + w(t,x)$ is a solution of (4.6.1). The Strichartz estimates give

$$
\begin{aligned}
&\|w_n(t,x)\|_{S^1([0,T]\times\mathbf{R}^3)} = \\
&\|\int_0^t e^{i(t-s)\Delta}|e^{is\Delta}u_0 + w_{n-1}(s,x)|^2(e^{is\Delta}u_0 + w_{n-1}(s,x))ds\|_{S^1([0,T]\times\mathbf{R}^3)} \\
&\lesssim \||\nabla(e^{is\Delta}u_0 + w_{n-1}(s,x))||e^{is\Delta}u_0 + w_{n-1}(s,x)|^2\|_{L_s^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)} \\
&+\||(e^{is\Delta}u_0 + w_{n-1}(s,x))||e^{is\Delta}u_0 + w_{n-1}(s,x)|^2\|_{L_s^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)},
\end{aligned} \tag{4.6.4}
$$

where p', q' are some the duals of an admissible pair (p, q) (The pair in the first case does not have to be the same as the pair in the second case). The first term is more difficult to estimate, so that is the one that will be estimated here by a sum of six terms. In the ensuing calculations the admissible pair (p, q) may change from term to term.

$$\||\nabla(e^{is\Delta}u_0 + w_{n-1}(s,x))||e^{is\Delta}u_0 + w_{n-1}(s,x)|^2\|_{L_s^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)} \lesssim$$

$$\||\nabla e^{is\Delta}u_0||e^{is\Delta}u_0|^2\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$+\||\nabla e^{is\Delta}u_0||e^{is\Delta}u_0||w_{n-1}(s,x)|\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$+\||\nabla e^{is\Delta}u_0||w_{n-1}(s,x)|^2\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$+\||e^{is\Delta}u_0|^2|\nabla w_{n-1}(s,x)|\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$+\||e^{is\Delta}u_0||w_{n-1}(s,x)||\nabla w_{n-1}(s,x)\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$+||w_{n-1}(s,x)|^2|\nabla w_{n-1}(s,x)\|_{L_t^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}.$$

If (p,q) is an admissible pair, and (p',q') are the dual exponents,

$$\frac{2}{p'} = 2 - 3(\frac{1}{q'} - \frac{1}{2}).$$

Let χ be a C^∞ cutoff,

$$\chi = \begin{cases} 1, & |x| \leq 2; \\ 0, & |x| > 3. \end{cases} \tag{4.6.5}$$

Term 1:

$$|e^{is\Delta}u_0|^2(\nabla e^{is\Delta}u_0)| \sim s^{-1/2}|x|^{-2}$$

$$\|\chi(x)e^{is\Delta}u_0|^2(\nabla e^{is\Delta}u_0)\|_{L_s^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)} < \infty,$$

provided $q' < 3/2$. If (p,q) is an admissible pair and $q' = \frac{3}{2} - \epsilon$ then

$$\frac{1}{p'} = 1 - 3(\frac{1}{3-2\epsilon} - \frac{1}{4}),$$

so $p' = 3/4 - \delta(\epsilon)$.

$$\|\chi(x)|e^{is\Delta}u_0|^2(\nabla e^{is\Delta}u_0)\|_{L_s^{p'}L_x^{q'}([0,T]\times\mathbf{R}^3)}$$

$$\leq C(\epsilon)\int_0^T (t^{-1/2})^{4/3+\delta'} = C(\epsilon)T^{1/3-\delta''}.$$

On $|x| > 2$,

$$(1-\chi)e^{it\Delta}u_0 \in L^2 \cap L^\infty,$$

$$\|(1-\chi(x))|e^{is\Delta}u_0|^2(\nabla e^{is\Delta}u_0)\|_{L_s^1 L_x^2([0,T]\times\mathbf{R}^3)} \leq CT^{1/2}.$$

Term 2: In this term use the Sobolev embedding $H^1(\mathbf{R}^3) \subset L^6(\mathbf{R}^n)$.

$$\|\chi(x)|e^{is\Delta}u_0||w_{n-1}(s,x)|(\nabla e^{is\Delta}u_0)\|_{L_s^{p'} L_x^{q'}([0,T]\times\mathbf{R}^3)}$$
$$\leq C(\epsilon)\|w_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)} \int_0^T (t^{-1/2})^{4/3+\delta'} = C(\epsilon)\|w_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)} T^{1/3-\delta''},$$

$$\|(1-\chi(x))|e^{is\Delta}u_0||w_{n-1}(s,x)|(\nabla e^{is\Delta}u_0)\|_{L_s^1 L_x^2([0,T]\times\mathbf{R}^3)} \leq CT^{1/2}\|w_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)}.$$

Term 3:

$$\|\chi(x)|e^{is\Delta}u_0||w_{n-1}(s,x)|^2\|_{L_s^{p'} L_x^{q'}([0,T]\times\mathbf{R}^3)}$$
$$\leq C(\epsilon)\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)} \int_0^T (t^{-1/2})^{4/3+\delta'} = C(\epsilon)\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)} T^{1/3-\delta''},$$

$$\|(1-\chi(x))|e^{is\Delta}u_0||w_{n-1}(s,x)|^2\|_{L_s^1 L_x^2([0,T]\times\mathbf{R}^3)} \leq CT^{1/2}\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)}.$$

Term 4:

$$\|\chi(x)|\nabla w_{n-1}(s,x)||e^{is\Delta}u_0|^2\|_{L_s^{1+} L_x^{2-}([0,T]\times\mathbf{R}^3)} \leq C\|w_{n-1}\|_{S^1([0,T]\times\mathbf{R}^3)} T^{1/2-\delta''}.$$

Since $|x|^{-1/2}$ does not quite live in $L^6(B(0;1))$, set $q' = 2-\epsilon$ and then set $\frac{1}{p'} = 1 - 3(\frac{1}{2-2\epsilon} - \frac{1}{2})$.

$$\|(1-\chi(x))|\nabla w_{n-1}(s,x)||e^{is\Delta}u_0|^2\|_{L^1_x L^2_x([0,T]\times\mathbf{R}^3)} \leq CT\|w_{n-1}\|_{S^1([0,T]\times\mathbf{R}^3)}.$$

Term 5:

$$\|\chi(x)|\nabla w_{n-1}(s,x)||e^{is\Delta}u_0||w_{n-1}(s,x)|\|_{L^{1+}_s L^{2-}_x([0,T]\times\mathbf{R}^3)} \leq C\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)}T^{1/2-\delta''}$$

$$\|(1-\chi(x))|\nabla w_{n-1}(s,x)||e^{is\Delta}u_0|^2\|_{L^1_x L^2_x([0,T]\times\mathbf{R}^3)} \leq CT^{3/4}\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)}.$$

Term 6:

$$\||\nabla w_{n-1}(s,x)||w_{n-1}(s,x)|^2\|_{L^1_s L^2_x([0,T]\times\mathbf{R}^3)} \leq CT^{1/2}\|w_{n-1}\|^3_{S^1([0,T]\times\mathbf{R}^3)}.$$

Adding all this together gives the estimate,

$$\begin{aligned}\|w_n(t,x)\|_{S^1([0,T]\times\mathbf{R}^3)} &\lesssim \\ T^{1/3-\delta''} + T^{1/3-\delta''}\|w_{n-1}\|^2_{S^1([0,T]\times\mathbf{R}^3)} &+ T^{1/2}\|w_{n-1}\|^3_{S^1([0,T]\times\mathbf{R}^3)},\end{aligned} \tag{4.6.6}$$

so if $\|w_1\|_{S^1([0,T]\times\mathbf{R}^3)} \leq CT^{1/3-\delta''}$, $\|w_n\|_{S^1([0,T]\times\mathbf{R}^3)}$ will stay $\leq 2CT^{1/3-\delta''}$ for T sufficiently small for all n.

Contraction: Using the same calculations suppose there are two sequences starting with w_0 and $\tilde{w}_0$,

$$\|w_n(t,x) - \tilde{w}_n(t,x)\|_{S^1([0,T]\times\mathbf{R}^3)}$$

(4.6.7)
$$\leq CT^{1/3-\delta''}(1 + \|w_{n-1}(t,x) + \tilde{w}_{n-1}(t,x)\|^2_{S^1([0,T]\times\mathbf{R}^3)})\times$$

$$\|w_{n-1}(t,x) - \tilde{w}_{n-1}(t,x)\|_{S^1([0,T]\times\mathbf{R}^3)}$$

which gives a contraction for T sufficiently small. This proves the existence of a solution of the form (4.6.2) $\square$.

The Gibbs Phenomenon:

$$e^{it\Delta}\chi_{B(0;1)} = Ct^{-3/2}e^{i|x|^2/4t}\int_0^1 r^2 e^{ir^2/4t}\int_{-1}^1 e^{-i|x|ru/2t}dudr,$$

$$= Ct^{-1/2}\frac{1}{|x|}\int_0^1 r[e^{i(r-|x|)^2/4t} - e^{i(r+|x|)^2/4t}]dr,$$

$$= Ct^{-1/2}\frac{1}{|x|}\int_0^1 re^{i(r-|x|)^2/4t}dr + O(t^{1/2}),$$

$$= Ct^{-1/2}\int_0^1 e^{i(r-|x|)^2/4t}dr + \frac{Ct^{-1/2}}{|x|}\int_0^1 (r-|x|)e^{i(r-|x|)^2/4t}dr + O(t^{1/2}),$$

$$= Ct^{-1/2}\int_0^1 e^{i(r-|x|)^2/4t}dr + O(t^{1/2}).$$

So the Gibbs phenomenon near the edges is the same as the Gibbs phenomenon in $\mathbf{R}$ for $\chi_{[-1,1]}$. Let

(4.6.8)
$$\chi = \begin{cases} 0, & |x| \leq \frac{1}{2}; \\ 1, & |x| > 3/4., \end{cases}$$

(4.6.9)
$$\|\chi(x)u(x)\|_{L^\infty_x} \leq C\|u\|_{H^1},$$

when u is a radial function. This gives a uniform estimate for the Gibbs phenomenon near the boundary.

$$\lim_{t \to 0} \|\chi(x) w(t,x)\|_{L^\infty_x} = 0 \tag{4.6.10}$$

So the Gibbs phenomenon near the boundary for the solution to (4.6.1) is controlled by the Gibbs phenomenon for $e^{it\Delta}\chi_{B(0;1)}$. More generally,

Lemma 4.6.2.

$$iu_t + \Delta u = |u|^\alpha u \tag{4.6.11}$$

has a local solution of the form $e^{it\Delta}u_0 + w(t,x)$ *on some time interval [0, T],* $T > 0$, *where*

$$\|w(t,x)\|_{S^1([0,T]\times\mathbf{R}^3)} \leq C(\delta) T^{1-\frac{2}{5-\alpha}-\delta}, \tag{4.6.12}$$

where $C(\delta) \nearrow \infty$ *as* $\delta \to 0$ *for* $2 \leq \alpha < 3$.

Thus for $|x| > 1/2$, let $u_\alpha(t)$ be the solution to (4.6.11) with initial condition $u_0 = \chi_{B(0;1)}$,

$$|u(t,x) - u_0(x)| = |e^{it\Delta}u_0(x) - u_0(x)| + O(t^{1-\frac{2}{5-\alpha}-\delta}). \tag{4.6.13}$$

The decay of the last term is uniform with respect to x.

Remark: The calculations of $e^{it\Delta}u_0$ can be carried over to $u_0 = \chi_\Omega$, where $\Omega \subset \mathbf{R}^3$ is a compact region with smooth boundary. If the function is no longer radial, then we do not have the Sobolev embedding $H^1(\mathbf{R}^3) \subset L^\infty(|x| > 1/2)$, however.

Higher Dimensions: Now consider $u_0 = \chi_{B(0;1)}$ for higher dimensions. As the dimensions increase the Pinsky phenomenon becomes worse and worse.

$$\frac{d}{dr} e^{it\Delta}u_0 = t^{-n/2} e^{i|x|^2/4t} \int_{-1}^{1} e^{-2i|x|u/4t}(1-u^2)^{(n-3)/2} du,$$

$$|\nabla e^{it\Delta}u_0| \lesssim t^{-1/2}|x|^{-(n-1)/2}. \tag{4.6.14}$$

Placing the remainder term in $L_t^\infty H_x^1([0,T_*] \times \mathbf{R}^n)$ would require

$$e^{it\Delta}u_0 \in L_t^\infty L_x^{\frac{2n\alpha}{3}+}([0,T_*] \times \mathbf{R}^n), \tag{4.6.15}$$

for some $T_* > 0$. This is severely restricted in higher dimensions due to the blowup near $|x| = 0$. When $|x|$ is large $e^{it\Delta}u_0 \in L^2 \cap L^\infty(|x| > 1)$. By stationary phase calculations, for $|x|$ close to 0 the solution has the form

$$t^{-n/2} \int_0^1 r^{n-1} e^{i(|x|^2+r^2)/t} \int_{-\pi/2}^{\pi/2} e^{-i|x|r\sin(\theta)/t} \cos(\theta)^{n-2} d\theta dr$$

$$= t^{-n/2} \int_0^1 r^{n-1} e^{i(|x|^2+r^2)/t} \int_{-1}^{1} e^{-i|x|ru/t}(1-u^2)^{(n-3)/2} du dr$$

$$\sim t^{-1/2} \int_0^1 r^{(n-1)/2} \frac{1}{|x|^{(n-1)/2}} e^{i(|x| \pm r)^2/t} dr + ...$$

$$t^{-1/2}\int_0^1 \frac{(r^{(n-1)/2} \pm |x|^{(n-1)/2})}{|x|^{(n-1)/2}} e^{i(|x|\pm r)^2/t} dr \leq O(\frac{t^{1/2}}{|x|^{(n-1)/2}})$$

$$t^{-1/2}\int_0^1 e^{i(|x|\pm r)^2/t} dr \leq O(1),$$

so when $t < |x|$ we have $|e^{it\Delta}u_0| \lesssim |x|^{-(n-2)/2}$. When $|x| < t$ using the Pinsky estimate we also have $|e^{it\Delta}u_0| \lesssim t^{-(n-2)/2} \leq |x|^{-(n-2)/2}$.

Therefore, close to 0,

$$|e^{it\Delta}u_0(x)| \lesssim |x|^{-(n-2)/2}. \tag{4.6.16}$$

This gives a fairly strong restriction on how large α can be. (4.6.15) is satisfied for $\alpha < \frac{3}{n-2}$. Once $n \geq 8$, α will in fact be L^2 - subcritical.

Remark: This calculation can be carried over to a more general nonlinearity of the form

$$\begin{aligned} &g(|u|^2)u, \\ 0 <&g(x) < \sup(C_2|x|^{\frac{3}{2(n-2)}-\delta}, C_1|x|^{1/2n+\epsilon}), \\ 0 <&g'(x) < \sup(C_2|x|^{\frac{3}{2(n-2)}-\delta-1}, C_1|x|^{1/2n+\epsilon-1}), \end{aligned} \tag{4.6.17}$$

for $\epsilon, \delta > 0$.

4.7. Global Continuation

From [**18**], if $u_0 \in H^s(\mathbf{R}^3)$, $s > 4/5$, the solution to (4.6.1) is global. It is conjectured that if $u_0 \in H^{1/2+}(\mathbf{R}^3)$, (4.6.1) is globally well-posed. If the solution is of the form

$e^{it\Delta}\chi_{B(0;1)} + w(t,x)$, $\|w(t,x)\|_{H^1_x(\mathbf{R}^n)} \leq C(t)$ on $[0,\infty)$, then a global solution exists. The solution to (4.6.1) has conserved mass.

LEMMA 4.7.1. *If $u(t)$ solves (4.6.17), the L^2 norm is conserved.*

$$\|u(t,\cdot)\|_{L^2(\mathbf{R}^n)} = \|u(0,\cdot)\|_{L^2(\mathbf{R}^n)}. \tag{4.7.1}$$

Proof:

$$\frac{d}{dt}\Re\langle u,u\rangle = 2\Re\langle u_t,u\rangle$$

$$= 2\Re\langle i\Delta u,u\rangle - 2\Re\langle ig(|u|^2)u,u\rangle = 0$$

THEOREM 4.7.2. *The partial differential equation*

$$iu_t + \Delta u = |u|^\alpha u \tag{4.7.2}$$

has a global solution $\frac{2}{n} < \alpha < \frac{3}{n-2}$, $\alpha \leq 2$. The solution is of the form

$$e^{it\Delta}u_0 + w(t,x), \tag{4.7.3}$$

where $w(t,x) \in L^\infty_t H^1_x([0,T]\times\mathbf{R}^n)$.

Proof: The theorem has already been proved for the time interval $[0,T]$. Moreover, $u(T,x)$ is of the form

$$e^{iT\Delta}u_0 + w(T,x), \tag{4.7.4}$$

where $\|w(T,x)\|_{H^1(\mathbf{R}^n)} \leq C$. The solution will be continued past T, and a Gronwall-type inequality will give and estimate on $\|w(t,x)\|_{H^1_x(\mathbf{R}^n)}$. If u is a solution to (4.7.2) then $u = v + w$, where v and w solve the system of equations

$$iv_t + \Delta v = 0,$$

$$i\tilde{w}_t + \Delta \tilde{w} = |u|^\alpha u, \tag{4.7.5}$$

$$v(T) = e^{iT\Delta}u_0; w(T,x) = w(T,x).$$

In order to prove $w(t,x) \in L^\infty_t H^1_x([0,\infty) \times \mathbf{R}^n)$, it suffices to obtain a bound on the energy

$$E(w(t)) = \int |\nabla w|^2 dx + \frac{2}{\alpha+2}\int |u|^{\alpha+2} dx. \tag{4.7.6}$$

$$\frac{d}{dt}\Re\langle \nabla\tilde{w}, \nabla\tilde{w}\rangle = -2\Re\langle \tilde{w}_t, \Delta\tilde{w}\rangle$$

$$= -2\Re\langle \tilde{w}_t, |u|^2 u\rangle - \Re\langle \tilde{w}_t, i\tilde{w}_t\rangle = -2\Re\langle (u_t - v_t), |u|^2 u\rangle$$

$$= -\frac{1}{2+\alpha}\frac{d}{dt}|u|^{2+\alpha}dx + 2\langle v_t, |u|^\alpha u\rangle.$$

$$2\Re\int v_t |u|^\alpha \bar{u} dx = 2\Re\int i\Delta v|u|^\alpha \bar{u} dx = -2i\int \nabla v \nabla(|u|^\alpha \bar{u})dx.$$

$$\frac{d}{dt}E(u(t)) \leq C\int |\nabla v||\nabla u||u|^\alpha dx. \tag{4.7.7}$$

Consider two cases separately.

Term 1:

Case 1: Assume $1 \leq \alpha \leq 2$.

$$\int |\nabla v||\nabla w||u|^{\alpha}dx \leq \|\nabla v\|_{L^{\infty}(\mathbf{R}^n)}\|\nabla w\|_{L^2(\mathbf{R}^n)}\||u|^{\alpha}\|_{L^2(\mathbf{R}^n)},$$

$$\||u|^{\alpha}\|_{L^2(\mathbf{R}^n)} \leq \|u\|^{\alpha}_{L^{2\alpha}(\mathbf{R}^n)}.$$

Let $\theta = \frac{2-\alpha}{\alpha^2},$.

$$\int |u|^{2\alpha}dx \leq \|u\|_2^{2\theta\alpha}\|u\|_{2+\alpha}^{2\alpha(1-\theta)}$$

when $\alpha \leq 2$, $2\alpha(1-\theta) \leq 2+\alpha$.

$$\|u\|_{2\alpha}^{\alpha} \leq E(w(t))^{\gamma(\alpha)}, \tag{4.7.8}$$

where $\gamma(\alpha) = \frac{2\alpha(1-\theta)}{2(2+\alpha)} \leq \frac{1}{2}$.

Case 2: Assume $\frac{1}{n} < \alpha \leq 1$. From previous calculations

$$\|\nabla v\|_{L^{2n/(n-1)+\epsilon}(\mathbf{R}^n)} \leq C_1(\epsilon)t^{-1/2} + C_2 t^{-n/2}$$

Interpolate this with

$$\|\nabla v\|_{L^{\infty}(\mathbf{R}^n)} \leq C_2 t^{-n/2}.$$

$$\||u|^\alpha\|_{L^{2/\alpha}(\mathbf{R}^n)} \leq \|u\|_2^{1/\alpha}.$$

As $\alpha \to \frac{1}{n}$, $\epsilon \to 0$ and $C_1(\epsilon) \to \infty$.

Remark: When $\alpha > \frac{2}{n}$ make the estimate

$$\|\nabla v\|_{L^{2n/(n-1)+\epsilon}(\mathbf{R}^n)} \leq C_3(\epsilon)t^{-1-\epsilon} + C_4 t^{-n/2}.$$

Term 2:

$$\int |\nabla v|^2 |u|^\alpha dx \leq \|u\|^\alpha_{L^2(\mathbf{R}^n)} \|\nabla v\|^2_{L^p(\mathbf{R}^n)}. \tag{4.7.9}$$

In this case take $\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{4}$. When $\alpha > \frac{2}{n}$, $|x|^{-(n-1)/2} \subset L^p(\{x \in \mathbf{R}^n : |x| > 1\})$.

$$\|u\|^\alpha_{L^2(\mathbf{R}^n)} \|\nabla v\|^2_{L^p(\mathbf{R}^n)} \leq C(\epsilon)t^{-1-\delta}, \tag{4.7.10}$$

where $\delta \to 0$ as $\alpha \to \frac{2}{n}$.

Now apply Gronwall's inequality.

$$\frac{d}{dt}E(w(t)) \leq C_1 t^{-1-\delta} + C_2 t^{-1-\epsilon} E(w(t))^{\rho(\alpha)} \tag{4.7.11}$$

Since $\rho(\alpha) \leq 1$ and $E(w(T)) < \infty$, $E(w(t)) < \infty$ on $[T, \infty)$ $\Box$.

4.8. More general continuation

Suppose the local solution to (4.5.1) is defined on $[0, 2T_*]$,

$$u(t,x) \in L_t^\infty L_x^p([0,2T_*] \times \mathbf{R}^n).$$

The solution on $[0, 2T_*]$ is of the form

$$e^{it\Delta}\Omega + w_1(t,x). \tag{4.8.1}$$

When $\frac{2}{n} < \alpha < \frac{3}{n-2}$, $w_1(t,x) \in L_t^\infty H_x^1([0,T_*] \times \mathbf{R}^n)$, which was then extended to a global solution. However, this still leaves open the possibility of global existence for $\frac{3}{n-2} \leq \alpha < \frac{4}{n-2}$. Although $w_1(t,x)$ may not lie in $H_x^1(\mathbf{R}^n)$, $w_1(t,x) \in L_t^\infty H_x^s([0,T_*] \times \mathbf{R}^n)$ for some $s > 1/2$. Continuing, the solution on $[T_*, \frac{3T_*}{2}]$ will look like

$$e^{it\Delta}u_0 + e^{i(t-T_*)\Delta}w_1(T,x) + w_2(t,x). \tag{4.8.2}$$

Using the smoothing properties of the Duhamel integral, $w_2(t,x)$ lies in an even higher order Sobolev space, and so forth. After a finite number of iterations, $w_n(t,x) \in L_t^\infty H^1([\frac{2^n-1}{2^n}T_*, T_*) \times \mathbf{R}^n)$. Thus by induction, the solution on $[\frac{2^n-1}{2^n}T_*, T_*)$ can be expressed in the form

$$e^{it\Delta}u(0) + e^{i(t-T_*/2)\Delta}u(\frac{T_*}{2}) + ... + e^{i(t-\frac{2^n-1}{2^n}T_*)\Delta}u(\frac{2^n-1}{2^n}T_*) + w(t), \tag{4.8.3}$$

where $w(t) \in H^1(\mathbf{R}^n)$ and the linear solutions have "nice" asymptotics.

Now to make this rigorous. There is a sequence $u_n(s)$ that converges to a solution in $L^\infty_t L^p_x([0, 2T_* - \delta] \times \mathbf{R}^n)$ for any $\delta > 0$. What happens to the derivative? Since $\nabla \chi_\Omega$ is an L^1 measure,

$$|\nabla e^{it\Delta} \chi_\Omega| \lesssim t^{-n/2}. \tag{4.8.4}$$

When $\Omega = B(0; 1)$,

$$|\nabla e^{it\Delta} \chi_\Omega| \lesssim t^{-1/2} |x|^{-(n-1)/2}. \tag{4.8.5}$$

This is also true for any Ω with smooth boundary. By interpolation,

$$|\nabla e^{it\Delta} \chi_\Omega(x)| \in t^{-1+\epsilon} L^p(\{|x| \leq 2\}) + t^{-1/2} L^p(\{|x| \geq 2\}). \tag{4.8.6}$$

Define a function space

$$\|u(t)\|_{X([0,T] \times \mathbf{R}^n)} = \sup_{[0,T]} \|t^{1-\epsilon} \nabla u(t)\|_p. \tag{4.8.7}$$

$$|\nabla u_n(t)| \leq |\nabla e^{it\Delta} u_0| + |\nabla \int_0^t e^{i(t-s)\Delta} |u_{n-1}(s)|^\alpha u_{n-1}(s) ds|,$$

$$\|\nabla u_n(t)\|_X \leq \|e^{it\Delta} u_0\|_X + t^{1-\epsilon} \int_0^t \frac{1}{(t-s)^{1-\epsilon}} \|u(s)\|_p^\alpha \|\nabla u_{n-1}(s)\|_X \frac{1}{s^{1-\epsilon}} ds$$

$$\leq \|e^{it\Delta} u_0\|_X + 2C' T^\epsilon \|u_{n-1}(s)\|_X.$$

If $\|e^{it\Delta}u_0\|_X \leq C$, then for T sufficiently small,

$$\|u_{n-1}(t)\|_X \leq 2C \Rightarrow \|u_n(t)\|_X \leq 2C. \tag{4.8.8}$$

Now look at the free evolution of the solution $u(T)$, where T is the time length where there is a local solution and $\|u(t)\|_X \leq 2C$.

$$\begin{aligned} \|\nabla e^{i(t-T)\Delta}u(T)\|_p &\leq \|\nabla e^{it\Delta}u_0\|_p + \int_0^T \frac{1}{(t-s)^{1-\epsilon}}\|u(s)\|_p^\alpha \|\nabla u(s)\|_p ds \\ &\leq \|\nabla e^{it\Delta}u_0\|_p + \frac{1}{(t-T)^{1-2\epsilon}}2C' \leq C_1 + \frac{1}{(t-T)^{1-2\epsilon}}2C'. \end{aligned} \tag{4.8.9}$$

Continue the iteration. Although the solution will only be defined on $[T, \frac{3T}{2}]$, notice that the regularity of the solution is improved slightly. $\|\nabla e^{i(t-T)\Delta}u(T)\|_p \sim \frac{1}{\frac{1}{t-T}^{1-2\epsilon}}$. This process can be terminated after a finite number of iterations.

Lemma 4.8.1. *Suppose the solution on some* $[0,T)$ *is of the form*

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}|u(s)|^\alpha u(s) ds, \tag{4.8.10}$$

where $\|u(t)\|_{L_t^\infty L_x^p([0,T)\times \mathbf{R}^n)} < \infty$ *and* $\|\nabla u(t)\|_{L_x^p(\mathbf{R}^n)} \lesssim t^{-1/2+\epsilon}$. *Then*

$$u(t) = e^{it\Delta}u_0 + w(t), \tag{4.8.11}$$

where $w(t) \in L_t^\infty H^1([0,T_*) \times \mathbf{R}^n)$. *Recall* $p = 2+\alpha$.

Proof: First,

$$\|\nabla u\|_{L_t^2 L_x^p([0,T_*)\times \mathbf{R}^n)} \lesssim \int_0^{T_*} t^{-1+2\epsilon} dt < T_*^{2\epsilon} C(\epsilon).$$

The conservation of the L^2 norm is well known, so

$$\| |\nabla u| |u|^{\alpha} \|_{L_t^2 L_x^q([0,T_*)\times \mathbf{R}^n)} < C(\epsilon) T_*^{2\epsilon},$$

for some $q < p'$. On the other hand, since $u \in L_t^{\infty} L^p([0,T_*) \times \mathbf{R}^n)$,

$$\| |\nabla u| |u|^{\alpha} \|_{L_t^2 L_x^r([0,T_*)\times \mathbf{R}^n)} < C(\epsilon) T_*^{2\epsilon},$$

for $\frac{1}{r} = \frac{1+\alpha}{2+\alpha} = \frac{1}{2} + \frac{\alpha/2}{2+\alpha} < \frac{1}{2} + \frac{2/(n-2)}{2+4/(n-2)} = \frac{1}{2} + \frac{1}{n}$. So by interpolation,

$$\| |\nabla u| |u|^{\alpha} \|_{L_t^2 L_x^{2n/(n+2)}([0,T_*)\times \mathbf{R}^n)} < C(\epsilon) T_*^{2\epsilon}. \tag{4.8.12}$$

$$\| e^{i(t-s)\Delta} |u(s)|^{\alpha} u(s) ds \|_{\dot{S}^1([0,T]\times \mathbf{R}^n)} \lesssim \| |\nabla u| |u|^{\alpha} \|_{L_t^2 L_x^{2n/(n+2)}([0,T]\times \mathbf{R}^n)}, \tag{4.8.13}$$

so by Strichartz estimates we are done $\Box$.

Eventually the solution on some interval $[\frac{2^{n-1}-1}{2^{n-2}} T_*, \frac{2^n-1}{2^{n-1}} T_*)$ will be of the form

$$u(t) = e^{it\Delta} u_0 + e^{i(t-T)\Delta} u(T) + ... + e^{i(t - \frac{(2^k-1)T}{2^{k-1}})\Delta} u(\frac{T(2^k-1)}{2^{k-1}}) + w(t), \tag{4.8.14}$$

where $w(T_*) \in H_x^1(\mathbf{R}^n)$. Then set up a system of equations as before,

$$
\begin{aligned}
&iv_t + \Delta v = 0,\\
&i\tilde{w}_t + \Delta \tilde{w} = |v + \tilde{w}|^{\alpha}(v + \tilde{w}),\\
&v(t_0) = e^{it_0\Delta}u_0 + ... + e^{i(t_0 - \frac{(2^k-1)T}{2^{k-1}})\Delta}u(\frac{T(2^k-1)}{2^{k-1}}),\\
&\tilde{w}(t_0) = w(t_0),\\
&t_0 = T_*.
\end{aligned} \tag{4.8.15}
$$

$$
\frac{d}{dt}(\|\nabla w\|_2^2 + \frac{2}{2+\alpha}\|v + w\|_{2+\alpha}^{2+\alpha}) \lesssim \langle |\nabla v|, |v + w|^{\alpha}(|\nabla v| + |\nabla w|)\rangle,
$$

so define the energy E(w).

$$
E(w) = \|\nabla w\|_2^2 + \frac{2}{2+\alpha}\|v + w\|_{2+\alpha}^{2+\alpha}. \tag{4.8.16}
$$

Again, estimate the terms separately.

Term 1:

$$
\int |\nabla v(x,t)|^2 |v(x,t) + w(x,t)|^{\alpha} dx \leq \|\nabla v(x,\cdot)\|_{L^{2+\alpha}(\mathbf{R}^n)}^2 \|v(x,t) + w(x,t)\|_{L^{2+\alpha}(\mathbf{R}^n)}^{\alpha}, \tag{4.8.17}
$$

$$
\|\nabla v(x,t)\|_{L^{2+\alpha}(\mathbf{R}^n)} \lesssim [t - \frac{(2^k-1)T}{2^{k-1}}]^{-1/2-\epsilon'}, \tag{4.8.18}
$$

$$
\|v(t,x) + w(t,x)\|_{L^{2+\alpha}(\mathbf{R}^n)}^{\alpha} \leq E(w(t))^{\alpha/2}, \tag{4.8.19}
$$

so this term does not cause any problems.

Term 2:

$$\int |\nabla v(x,t)||\nabla w(x,t)||v(x,t)+w(x,t)|^{\alpha}dx \leq \tag{4.8.20}$$

$$\|\nabla v(\cdot,t)\|_{L^p(\mathbf{R}^n)}\|\nabla w(\cdot,t)\|_{L^2(\mathbf{R}^n)}\|v(\cdot,t)+w(\cdot,t)\|^{\alpha}_{L^{2+\alpha}(\mathbf{R}^n)},$$

where $p = \frac{2-\alpha}{2(2+\alpha)}$. To obtain such a p it is necessary to restrict $1 \leq \alpha \leq 2$.

$$\|\nabla v(\cdot,t)\|_{L^r} \leq \sum_{i=0}^{n} \int_{\frac{2^i-1}{2^{i-1}}T}^{\frac{2^{i+1}-1}{2^i}T} \frac{1}{(t-s)^{n(1/2-1/r)}} \|u(\cdot,s)\|^{\alpha}_{L^2(\mathbf{R}^n)}\|\nabla u(\cdot,s)\|_{L^{2+\alpha}(\mathbf{R}^n)}ds$$

$$+\|e^{it\Delta}u_0\|_{L^r(\mathbf{R}^n)},$$

where $\frac{1}{r} = 1 - \frac{\alpha}{2} - \frac{1}{2+\alpha} = 1 - \frac{2\alpha+\alpha^2+2}{2(2+\alpha)} = \frac{2-\alpha^2}{4+2\alpha}$. $2-\alpha^2 < 2-\alpha$ when $\alpha \geq 1$.

Thus (4.7.2) has a global solution when $1 \leq \alpha \leq 2$, where $\frac{2}{n} < \alpha < \frac{4}{n-2}$.

Remark: This is a somewhat unsatisfying result, since what about $\frac{2}{n} < \alpha < 1$? When $u_0 = \chi_{B(0;1)}$, the fact that u_0 is a radial function will localize the problem to the origin and extend the results further. This will be explored in the next section.

A digression to the heat equation: This method can also be applied to a Schrödinger equation that has a damping term. Consider,

$$(a+ib)u_t = \Delta u, \tag{4.8.21}$$

where $a \geq 0$ and $|a+ib| = 1$. The solution is the Fourier multiplier

$$e^{t\Delta/(a+ib)}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^2/(a+ib)}\hat{u}_0(\xi)). \tag{4.8.22}$$

$$u(x,t) = \int K(x,y,t)u_0(y)dy, \tag{4.8.23}$$

$$K(x,y,t) = \frac{(-a-ib)^{n/2}}{(4\pi t)^{n/2}} e^{(a+ib)\frac{|x-y|^2}{4t}}. \tag{4.8.24}$$

This operator obeys the operator bounds

$$\begin{aligned} &e^{t\Delta(a-ib)} : t^{n/2}L^1 \to L^\infty, \\ &e^{t\Delta(a-ib)} : L^2 \to L^2. \end{aligned} \tag{4.8.25}$$

Therefore it obeys the same Strichartz estimates as the Schrodinger equation.

Now consider the nonlinear equation with power-type nonlinearity.

$$(a+ib)u_t + \Delta u = |u|^\alpha u, \tag{4.8.26}$$

with $|a+ib| = 1$, is $\dot{H}^\rho(\mathbf{R}^n)$ - critical for $\alpha = \frac{4}{n-2\rho}$. When $u_0 \in H^{\rho+\epsilon}(\mathbf{R}^n)$ for $\alpha = \frac{4}{n-2\rho}$ there exists $T(\|u_0\|_{H^{\rho+\epsilon}(\mathbf{R}^n)} > 0$ such that a solution to (4.8.26) exists on $[0,T)$.

Theorem 4.8.2. *When $a < 0$, (4.8.26) has a global solution $u(t)$, and*

$$\|u(t)\|_{H^{\rho+\epsilon}(\mathbf{R}^n)} \leq F(t, \|u_0\|_{H^{\rho+\epsilon}(\mathbf{R}^n)}, a). \tag{4.8.27}$$

Proof: Apply the same expansion method as was used for the Schrodinger equation $(a = 0)$. For some uniform constant C,

$$a^{\sigma/2}t^{\sigma/2}|\xi|^{\sigma}e^{-at|\xi|^2} \leq C(\sigma). \tag{4.8.28}$$

Thus the Strichartz estimates hold

$$\begin{aligned} \|\nabla^{\rho+\epsilon}e^{-t\Delta/(a+ib)}u_0\|_{L_t^2 L_x^{2n/(n-2)}} &\lesssim C\|u_0\|_{\dot{H}^{\rho+\epsilon}}, \\ \|\nabla^{\rho+\epsilon+\delta}e^{-t\Delta/(a+ib)}u_0\|_2 &\lesssim |a|^{-\delta/2}t^{-\delta/2}\|u_0\|_{\dot{H}^{\rho+\epsilon}}. \end{aligned} \tag{4.8.29}$$

Meanwhile by the Sobolev embedding

$$\||e^{-t\Delta/(a+ib)}u_0|^{4/(n-2\rho)}\|_{L_t^\infty L_x^q} \lesssim \|u_0\|_{H^{\rho+\epsilon}}, \tag{4.8.30}$$

where $\frac{1}{q} = \frac{4}{2n} - \frac{4\epsilon}{n(n-2\rho)}$. Let $\frac{1}{p} = \frac{2n}{n-2} + \frac{4\epsilon}{n(n-2\rho)}$. If

$$\|\nabla^{\sigma}e^{-t\Delta/(a+ib)}u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{\dot{H}^{\rho+\epsilon}} \tag{4.8.31}$$

for $\frac{1}{q} = \frac{1}{2} - \frac{2\epsilon}{n-2\rho}$, then the Duhamel term lies in $H^{\sigma}(\mathbf{R}^n)$. By interpolation,

$$\|\nabla^{\rho+\epsilon+\delta}e^{-t\Delta/(a+ib)}u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{\dot{H}^{\rho+\epsilon}} \tag{4.8.32}$$

for some $\delta(\epsilon) > 0$, $\delta \geq c\epsilon$ for some $c > 0$. Expressing the solution in Duhamel's equation

$$u(t) = e^{-t\Delta/(a+ib)}u_0 + \int_0^t e^{-(t-s)\Delta/(a+ib)}|u(s)|^{\alpha}u(s)ds. \tag{4.8.33}$$

The second term will be smoothed, belonging to $H^{\rho+\epsilon+\delta}(\mathbf{R}^n)$ for some $\delta > c\epsilon$. But the first term also belongs to $H^{\rho+\epsilon+\delta}(\mathbf{R}^n)$ by (4.8.29) on $[T/2, T)$. Thus $u(T/2) \in H^{\rho+\epsilon+\delta}(\mathbf{R}^n)$. We can iterate this procedure, obtaining a smoother and smoother solution after each step. Eventually, the solution will lie in $H^1(\mathbf{R}^n)$, at which point we can use the conservation of H^1 norm $\Box$.

THEOREM 4.8.3. *The energy does not increase*

$$E(u(t)) = \langle \nabla u, \nabla u \rangle + \frac{2}{2+\alpha} \int |u|^{\alpha+2} dx. \tag{4.8.34}$$

Proof:

$$\frac{d}{dt} \int |\nabla u|^2 dx = -\langle u_t, \Delta u \rangle = -\int u_t |u|^\alpha \bar{u} dx - \int a |u_t|^2 dx,$$

so

$$\frac{d}{dt}[E(u(t))] \leq 0. \tag{4.8.35}$$

So once the solution $u(T') \in H^1(\mathbf{R}^n)$, the solution can be continued to a global solution.

4.9. Proof of Theorem [1.1.15]:

For the reader's convenience, (1.1.30) will be rewritten here.

$$\begin{gathered} iu_t + \Delta u = \pm |u|^{\frac{4}{n-2\rho}} u, \\ u(0, x) = u_0(x), \end{gathered} \tag{4.9.1}$$

For radial data in high dimensions, the linear operator $e^{it\Delta}u_0$ has solutions which are asymptotically very nice, and can be treated very effectively in the same manner as was used to prove Theorem [1.1.14].

LEMMA 4.9.1. *Let u_0 be a radial function.*

$$|\nabla e^{it\Delta}u_0(x)| \leq t^{-3/2}(|x|^{-(n-1)/2} + |x|^{-(n-3)/2})(\|u_0\|_{L^2(\mathbf{R}^n)} + \|u_0\|_{L^1(\mathbf{R}^n)}), \tag{4.9.2}$$

$$|\nabla e^{it\Delta}u_0(x)| \leq (t^{-n/2} + t^{-n/2+1})|x|^{-1}(\|u_0\|_{L^2(\mathbf{R}^n)} + \|u_0\|_{L^1(\mathbf{R}^n)}), \tag{4.9.3}$$

$$|e^{it\Delta}u_0(x)| \leq t^{-n/2}(\|u_0\|_{L^1(\mathbf{R}^n)}). \tag{4.9.4}$$

Proof: (4.9.4) is just the dispersive estimate.

$$\frac{\partial}{\partial x_i}\frac{1}{t^{n/2}}\int_{\mathbf{R}^n} e^{i|x-y|^2/4t}f(y) = \frac{1}{2it^{n/2+1}}\int_{\mathbf{R}^n}(x_i - y_i)e^{i|x-y|^2/4t}f(y)dy.$$

It suffices to bound two terms with stationary phase calculations. The first term is

$$t^{-n/2-1}|x|\int_0^\infty f(r)r^{n-1}e^{ir^2/4t}\int_{S_r^{n-1}} e^{-2ix\cdot\xi/4t}d\sigma_r(\xi)dr$$

$$\sim C_1 t^{-3/2}|x|^{-(n-3)/2}\int_0^\infty f(r)r^{(n-1)/2}e^{i(r+|x|)^2/4t}dr$$

$$+C_2 t^{-3/2}|x|^{-(n-3)/2}\int_0^\infty f(r)r^{(n-1)/2}e^{i(r-|x|)^2/4t}dr.$$

$$|t^{-3/2}|x|^{-(n-3)/2}\int_0^1 f(r)r^{(n-1)/2}e^{i(r\pm|x|)^2/4t}| \leq Ct^{-3/2}|x|^{-(n-3)/2}\|f\|_{L^2(\mathbf{R}^n)}.$$

$$|t^{-3/2}|x|^{-(n-3)/2}\int_1^\infty f(r)r^{(n-1)/2}e^{i(r\pm|x|)^2/4t}| \leq Ct^{-3/2}|x|^{-(n-3)/2}\|f\|_{L^1(\mathbf{R}^n)}.$$

Also, for the estimate for (4.9.3),

$$|t^{-n/2-1}|x|\int_0^\infty f(r)r^{n-1}e^{ir^2/4t}\int_{S_r^{n-1}} e^{-2ix\cdot\xi/4t}d\sigma_r(\xi)dr|$$

$$\leq t^{-n/2+1}|x|^{-1}\int_0^\infty f(r)r^{n-3}dr \leq t^{-n/2}|x|^{-1}\int_0^1 f(r)r^{n-3}dr + t^{-n/2+1}|x|^{-1}\int_1^\infty f(r)r^{n-3}dr$$

$$\leq t^{-n/2}(\|f\|_{L^{(n-1)/(n-3)}(\mathbf{R}^n)} + \|f\|_{L^1(\mathbf{R}^n)}).$$

The second term is

$$t^{-n/2-1}\int_0^\infty f(r)r^n e^{ir^2/4t}\int_{S_r^{n-1}} e^{-2ix\cdot\xi/4t}d\sigma_r(\xi)dr,$$

$$\leq t^{-3/2}|x|^{-(n-1)/2}\int_0^\infty f(r)r^{(n+1)/2}dr.$$

$$\int_0^1 f(r)r^{(n+1)/2}dr \leq C\|f\|_{L^{2(n-1)/(n+1)}(\mathbf{R}^n)}.$$

$$\int_1^\infty f(r)r^{(n+1)/2}dr \leq C\|f\|_{L^1(\mathbf{R}^n)}$$

Also for (4.9.3),

$$t^{-n/2-1}\int_0^\infty f(r)r^n e^{ir^2/4t}\int_{S_r^{n-1}} e^{-2ix\cdot\xi/4t}d\sigma_r(\xi)dr$$

$$\leq Ct^{-n/2}|x|^{-1}\int_0^{|x|} f(r)r^{n-1} \leq Ct^{-n/2}\|f\|_{L^1(\mathbf{R}^n)}.$$

This proves (4.9.2) and (4.9.3) □.

Now to prove Theorem [1.1.15].

THEOREM 4.9.2. *(1.1.30) has a global solution for u_0 radial,*

$$u_0 \in H^{\rho+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,2}(\mathbf{R}^n) \cap H^{1/2+\epsilon,1}(\mathbf{R}^n). \tag{4.9.5}$$

The idea of this proof is to obtain an expansion of the local solution into a sum of the form (4.8.14). Then, once the solution is on $[\tau, T]$ is of the form

$$e^{i(t-\tau)\Delta}u(\tau,x) + w(t,x),$$

where $w(t,x) \in L_t^\infty H_x^1([\tau,T]\times\mathbf{R}^n)$, the solution to (1.1.30) with initial data $w(T,x) + e^{i(T-\tau)\Delta}u(\tau,x)$ can be treated as a perturbation of the solution with initial data $w(T,x)$. Let $\alpha = \frac{4}{n-2\rho}$. When $n \geq 6$ and $0 < \rho < 1$, which forces $\alpha \leq 1$.

Define a set $X_{\rho,n}$ and let $\alpha = \frac{4}{n-2\rho}$.

$$\begin{aligned} X_{\rho,n} = \{\sigma : (1.1.30) \text{ exists globally}, u_0 = u_1 + u_2; u_1 \in H^{\sigma,2}(\mathbf{R}^n), \\ u_1 \text{ radial }, u_1 \in H^{1/2+\epsilon,1}(\mathbf{R}^n); u_2 \in H^{1,2}(\mathbf{R}^n)\}. \end{aligned} \tag{4.9.6}$$

THEOREM 4.9.3.

$$(1/2, 1] \cap (\rho, 1] \subset X_{\rho,n}, \tag{4.9.7}$$

for any $0 < \rho < 1$.

Theorem [4.9.3] implies Theorem [1.1.15].

Method of Proof: For any $\rho < 1$, the conservation of the energy

$$E(u(t)) = \|\nabla u(t)\|^2_{L^2(\mathbf{R}^n)} + \frac{2}{2+\alpha}\|u(t)\|^{2+\alpha}_{L^{2+\alpha}(\mathbf{R}^n)} \tag{4.9.8}$$

ensures that $X_{\rho,n}$ is a nonempty set for every $\rho < 1$, n. So $(1/2, 1] \cap (\rho, 1] \subset X_{\rho,n}$ will be proved by induction on σ. First prove a simpler result.

LEMMA 4.9.4. *Let* $u_0 = u_1 + u_2$*, where* $u_1 \in H^{1,2}(\mathbf{R}^n)$ *and* $u_2 \in H^{\rho+\epsilon,2}(\mathbf{R}^n)$ *has asymptotics of the form*

$$|e^{it\Delta}u_2(x)| \leq C(1+t)^{-n/2}, \tag{4.9.9}$$

$$\begin{aligned} |\nabla e^{it\Delta}u_2(x)| &\leq C(1+t)^{-1}(|x|^{-(n-2)/2}), \\ |\nabla e^{it\Delta}u_2(x)| &\leq C(1+t)^{-n/2-1}. \end{aligned} \tag{4.9.10}$$

Then the solution exists globally and is of the form

$$e^{it\Delta}u_2 + u(t,x), \tag{4.9.11}$$

$$u(t,x) \in C_t^0 H_x^{1,2}(\mathbf{R}^n) \cap L_t^\infty H_x^{1,2}(\mathbf{R}^n). \tag{4.9.12}$$

Proof: For $u_0 = u_1 + u_2$, where $\|u_1\|_{H_x^{1,2}(\mathbf{R}^n)} \leq C$, then (1.1.30) exists locally on some time interval [0, T) for $T(C) > 0$. Using Duhamel's principle define a sequence of functions, $v_0 = e^{is\Delta} u_1$,

$$v_n = \int_0^t e^{i(t-s)\Delta} |e^{is\Delta} u_2 + v_{n-1}(s)|^\alpha (e^{is\Delta} u_2 + v_{n-1}(s)) ds. \tag{4.9.13}$$

Use the Strichartz estimate

$$\|v_n\|_{S^1([0,T]\times\mathbf{R}^n)} \leq C \|F(s)\|_{N^1([0,T]\times\mathbf{R}^n)},$$

$$F(s) = |e^{is\Delta} u_2 + v_{n-1}(s)|^\alpha (e^{is\Delta} u_2 + v_{n-1}(s)),$$

$$\|F(s)\|_{N^1([0,T]\times\mathbf{R}^n)} \leq \|\nabla F(s)\|_{L_t^2 L_x^{2n/(n-2)}([0,T]\times\mathbf{R}^n)} + \|F(s)\|_{L_t^2 L_x^{2n/(n-2)}([0,T]\times\mathbf{R}^n)}.$$

From the asymptotic estimates.

$$\|\nabla e^{it\Delta} u_2\|_{L_t^\infty L_x^{2n/(n-2)+}(\mathbf{R}^n)} \leq C,$$

$$\|\nabla v_{n-1}(t,x)\|_{L_t^2 L_x^{2n/(n-2)}} \leq \|v_{n-1}(t,x)\|_{S^1},$$

$$\||e^{it\Delta} u_2 + v_{n-1}(t,x)|^\alpha\|_{L_t^\infty L_x^2} \leq C + \|v_{n-1}\|_{S^1}^\alpha.$$

This gives the estimate

$$\|\nabla F(s)\|_{N^1([0,T]\times\mathbf{R}^n)} \leq CT^{\delta}\|v_{n-1}\|^{1+\alpha}_{S^1([0,T]\times\mathbf{R}^n)} + T^{\delta}\|v_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)}$$

$$+T^{\delta}\|v_{n-1}\|^{\alpha}_{S^1([0,T])} + CT^{1/2},$$

for some $T > 0$, $\delta > 0$. This map is a bounded map for T sufficiently small, since $\|v_0\|_{S^1} < \infty$. Let

$$G(s) = |e^{is\Delta}u_2 + v_n(s)|^{\alpha}(e^{is\Delta}u_2 + v_n(s)) - |e^{is\Delta}u_2 + v_{n-1}(s)|^{\alpha}(e^{is\Delta}u_2 + v_{n-1}(s)),$$

$$|G(s)| \leq C(|e^{is\Delta}u_2 + v_n(s)|^{\alpha} + |e^{is\Delta}u_2 + v_{n-1}(s)|^{\alpha})(|v_n(s) - v_{n-1}(s)|),$$

$$\|\nabla G(s)\|_{N^1([0,T]\times\mathbf{R}^n)}$$

$$\leq CT^{\delta}\|v_{n-1}\|^{\alpha}_{S^1([0,T]\times\mathbf{R}^n)}\|v_n - v_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)} + T^{\delta}\|v_n - v_{n-1}\|_{S^1([0,T]\times\mathbf{R}^n)}.$$

This yields a contraction, and local well-posedness is established, and the solution is of the form $e^{it\Delta}u_2 + u_1(t,x)$ with $u_1 \in H^{1,2}(\mathbf{R}^n)$ $\Box$.

THEOREM 4.9.5. *Let $u_0 = u_1 + u_2 + u_3$, where $u_3 \in H^{1,2}(\mathbf{R}^n)$, $u_1 \in H^{\sigma,2}(\mathbf{R}^n)$, $u_2 \in H^{\rho+,2}(\mathbf{R}^n)$ and u_1, u_2 obey the asymptotic estimates*

$$|\nabla e^{it\Delta}u_1| \leq \|u_1\|_{H^{\sigma,2}(\mathbf{R}^n)}t^{-n/2-1},$$

(4.9.14)

$$|\nabla e^{it\Delta}u_1| \leq \|u_1\|_{H^{\sigma,2}(\mathbf{R}^n)}t^{-1+\epsilon}|x|^{-(n-2)/2-\epsilon},$$

$$|e^{it\Delta}u_2(x)| \le C(1+t)^{-n/2},$$

$$|\nabla e^{it\Delta}u_2(x)| \le C(1+t)^{-1}|x|^{-(n-2)/2}, \tag{4.9.15}$$

$$|\nabla e^{it\Delta}u_2(x)| \le C(1+t)^{-n/2}.$$

Then there exists $T(\|u_1\|_{H^{\sigma,2}}, C, \|u_3\|_{H^{1,2}(\mathbf{R}^n)}) > 0$ *such that (1.1.30) has a local solution of the form*

$$u(x,t) = e^{it\Delta}u_0 + g(x,t), \tag{4.9.16}$$

where $g(x,t) \in H^{\sigma+2\epsilon/n}(\mathbf{R}^n)$ *with bounded norm on [0, T], where* $\sigma - \rho > \epsilon$, $\sigma - 1/2 > \epsilon$. *Furthermore, g(x,t) has a decomposition,*

$$g(x,t) = v_1(x,t) + v_3(x,t), \tag{4.9.17}$$

where v_3 *has bounded* $H^{1,2}(\mathbf{R}^n)$ *norm and*

$$v_1(x,t) = \int_0^t e^{i(t-s)\Delta}\chi \cdot |u(x,s)|^\alpha u(x,s)ds. \tag{4.9.18}$$

Here $\chi \in C_0^\infty(\mathbf{R}^n)$ *is a radial cutoff function,* $\chi \equiv 1$ *on B(0;1) and* $\chi(y) \equiv 0$ *for* $|y| > 2$. v_1 *obeys (4.9.14)-type estimates on* $[T, \infty)$.

Proof: As before take the Duhamel expansion,

$$g_n(t) = \int_0^t e^{i(t-s)\Delta}|e^{is\Delta}u_0 + g_{n-1}(s)|^\alpha(e^{is\Delta}u_0 + g_{n-1}(s))ds, \tag{4.9.19}$$

$$F(s) = |e^{is\Delta}u_0 + g_{n-1}(s)|^\alpha (e^{is\Delta}u_0 + g_{n-1}(s)).$$

By the Sobolev embedding theorem

$$|e^{is\Delta}u_0|^\alpha \in L^{\frac{n}{2}\cdot\frac{n-2\rho}{n-2\rho-8\epsilon}}(\mathbf{R}^n) \cap L^{n/2-}(\mathbf{R}^n). \tag{4.9.20}$$

Now take the local smoothing estimate found in [RV] and interpolate with the Strichartz estimate

$$\begin{aligned} \|e^{it\Delta}u_0\|_{L_t^2 L_x^{2n/(n-2)}(\mathbf{R}\times\mathbf{R}^n)} &\leq C\|u_0\|_2, \\ \|\nabla^{1/2}\chi e^{it\Delta}u_0\|_{L_t^2 L_x^2(\mathbf{R}\times\mathbf{R}^n)} &\leq C\|u_0\|_2, \end{aligned} \tag{4.9.21}$$

$$\|\nabla^{2\epsilon/n}\chi e^{it\Delta}u_0\|_{L_t^2 L_x^p(\mathbf{R}\times\mathbf{R}^n)} \leq C\|u_0\|_2, \tag{4.9.22}$$

where $\frac{1}{p} = \frac{n-2}{2n} + \frac{4\epsilon}{n^2}$.

By the asymptotic estimates (4.9.15)

$$\nabla e^{it\Delta}u_2 \in L^{\frac{2n}{n-2}+}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n),$$

and by the Strichartz estimates

$$\nabla e^{it\Delta}u_3 \in L_t^2 L_x^{2n/(n-2)}(\mathbf{R} \times \mathbf{R}^n).$$

Finally use the estimate on u_1.

$$|\nabla(1-\chi)e^{it\Delta}u_1| \leq Ct^{-3/2}|x|^{-(n-3)/2}\{\|u_1\|_{L^1(\mathbf{R}^n)} + \|u_1\|_{L^2(\mathbf{R}^n)}\}. \tag{4.9.23}$$

Interpolate this with the trivial estimate.

$$\|(1-\chi)e^{it\Delta}u_1\|_{L^2(\mathbf{R}^n)} \leq \|u_1\|_{L^2(\mathbf{R}^n)}, \tag{4.9.24}$$

$$\|\nabla^{1/3-}(1-\chi)e^{it\Delta}u_1\|_{L^2_t L^{3-}_x([0,T]\times\mathbf{R}^n)} \leq CT^{0+}, \tag{4.9.25}$$

$$\|\nabla^{\sigma+2\epsilon/n}e^{it\Delta}u_0\|_{L^2_t L^p_x} \leq C\|u_1\|_{H^{\sigma,2}} + C, \tag{4.9.26}$$

where $\frac{1}{p} = \frac{n-2}{2n} + \frac{4\epsilon}{n^2}$. This gives the Strichartz estimate.

$$\|\nabla^{\sigma+4\epsilon/n}|e^{it\Delta}u_0 + g_{n-1}(t)|^\alpha(e^{it\Delta}u_0 + g_{n-1}(t))\|_{L^{2-}_t L^{2n/(n+2)+}_x([0,T]\times\mathbf{R}^n)}$$

$$\leq \|u(t)\|^\alpha_{S^\rho([0,T])}(C + \|g_{n-1}\|_{S^1([0,T])})T^\delta,$$

for some $\delta > 0$. Taking T small enough gives a bounded mapping. A contraction can also be obtained.

Then for the solution at $t = T$, $e^{iT\Delta}(u_1 + u_2)$ will become the new u_2, since there is a uniform bound on the free evolution of u_1 when $T \geq \tau > 0$. Also one can put $e^{iT\Delta}u_3$ for the new u_3 without problem. Thus, all that remains is to split $g(x,t)$ in the proper way. Let $v_0 = w_0 = 0$.

(4.9.27) $$v_n(t) = \int_0^t e^{i(t-s)\Delta}\chi\cdot|e^{is\Delta}u_0+v_{n-1}(s)+w_{n-1}(s)|^{\alpha}(e^{is\Delta}u_0+v_{n-1}(s)+w_{n-1}(s))ds,$$

(4.9.28)
$$w_n(t) = \int_0^t e^{i(t-s)\Delta}(1-\chi)\cdot|e^{is\Delta}u_0 + v_{n-1}(s) + w_{n-1}(s)|^{\alpha}(e^{is\Delta}u_0 + v_{n-1}(s) + w_{n-1}(s))ds.$$

Since $\sigma > \rho$, the solution has an appropriate S^{σ} bound. For now assume the following estimates.

(4.9.29)
$$|\nabla e^{it\Delta}u_0(x)| \leq |x|^{-(n-3)/2}t^{-3/2}\|u_0\|_1,$$
$$|\nabla e^{it\Delta}u_0(x)| \leq |x|^{-(n-1)/2}t^{-1/2}\|\nabla u_0\|_1.$$

Thus by interpolation,

(4.9.30) $$|\nabla e^{it\Delta}u_0(x)| \leq |x|^{-(n-2)/2-\epsilon}t^{-1+\epsilon}\|u_0\|_{H^{1/2+\epsilon,1}}.$$

This implies,

$$|\int_0^t e^{i(t-s)\Delta}\nabla\chi|u(s)|^{\alpha}u(s)(x)ds| \leq \int_0^t \|u(s)\|_{H^{1/2+\epsilon,2}}^{1+\alpha}(t-s)^{-1+\epsilon}|x|^{-(n-2)/2-\epsilon}ds$$

$$\leq \|u_0\|_{H^{1/2+\epsilon,2}}^{1+\alpha}t^{\epsilon}|x|^{-(n-2)/2-\epsilon}.$$

This gives the estimates

$$\||u(t)|^{\alpha}\|_{L_t^{\infty}L_x^{n/2-}(\mathbf{R}\times\mathbf{R}^n)} \leq C\|u(t)\|_{S^0}^{\alpha},$$

$$\|\nabla(1-\chi)e^{is\Delta}(u_1+u_3)\|_{L_t^1L_x^{2n/(n-4)}(\mathbf{R}\times\mathbf{R}^n)} \leq CT^{\epsilon}\|u_1\|_{H^{1/2+\epsilon,2}(\mathbf{R}^n)} + C'T,$$

$$\|\nabla e^{it\Delta}u_2\|_{L_t^2L_x^{2n/(n-2)}([0,T]\times\mathbf{R}^n)} \leq C\|u_2\|_{H^{1,2}(\mathbf{R}^n)}.$$

Now apply the Strichartz estimates to the other term.

$$\|w_n(t)\|_{\dot{S}^1} \leq \|\nabla(1-\chi)|e^{is\Delta}u_0 + v_n(s) + w_n(s)|^{\alpha}(e^{is\Delta}u_0 + v_n(s) + w_n(s))\|_{N^0}$$

$$\leq T^{\delta'}\|u(t)\|_{S^{\rho+\epsilon}(\mathbf{R}\times\mathbf{R}^n)}^{\alpha}(C'T^{\epsilon} + CT + C\|u_2\|_{H^{1,2}(\mathbf{R}^n)} + C\|u_0\|_{H^{1/2+\epsilon,2}}^{1+\alpha}T^{\epsilon+1} + T^{\delta}\|w_{n-1}\|_{S^1}).$$

Set T sufficiently small, this map keeps the $\dot{S}^1$ norm bounded. Finally,

$$e^{iT\Delta}\int_0^t e^{i(t-s)\Delta}\chi|u(s)|^{\alpha}u(s)ds$$

obeys the appropriate bounds, by (4.9.1) $\square$.

After some point the solution will almost be in the form of Lemma [4.9.4], with initial data $u_1 \in H^{1,2}(\mathbf{R}^n)$ and

$$\|\nabla e^{it\Delta}u_2\|_{L^{n-\epsilon}(\mathbf{R}^n)} \leq C(1+t)^{-n/2}, \tag{4.9.31}$$

$$\|\nabla e^{it\Delta}u_2\|_{L^{2n/(n-2)-}(\mathbf{R}^n)} \leq C(1+t)^{-1+\epsilon}, \tag{4.9.32}$$

$$E(w(t)) \leq C\int(|\nabla v| + |\nabla w|)|\nabla v||v+w|^{\alpha}dx, \tag{4.9.33}$$

$$\begin{aligned} &\int |\nabla v|^2|v+w|^{\alpha}dx \leq \|\nabla v\|_p^2\|v+w\|_2^{\alpha} \leq \\ &C(\|u_0\|_{H^{1/2+,1}(\mathbf{R}^n)} + \|u_0\|_{H^{\sigma+,1}(\mathbf{R}^n)})(1+t)^{-\beta}, \end{aligned} \tag{4.9.34}$$

where $\frac{1}{p} = \frac{1}{2} - \frac{\alpha}{4} < \frac{1}{2} - \frac{1}{n}$, and $\beta > 1$.

$$\begin{aligned} &\int |\nabla v||\nabla w||v+w|^{\alpha}dx \leq \|\nabla v\|_q\|\nabla w\|_2\|v+w\|_2^{\alpha} \\ &\leq C(\|u_0\|_{H^{1/2+,1}(\mathbf{R}^n)} + \|u_0\|_{H^{\sigma+,1}(\mathbf{R}^n)})(1+t)^{-\gamma}. \end{aligned} \tag{4.9.35}$$

In this case $\frac{1}{q} = \frac{1}{2} - \frac{\alpha}{2} < \frac{1}{2} - \frac{2}{n}$. In this case $\gamma > 1$ also. Thus by Gronwall's inequality $E(w(t))$ is finite. This completes the proof of Theorem [4.9.3] □.

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