AN INVESTIGATION OF NON-TRAPPING, ASYMPTOTICALLY EUCLIDEAN WAVE EQUATIONS

Robert Booth

A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics.

Chapel Hill
2018

Approved by:

Jason Metcalfe

Yaiza Canzani

Hans Christianson

Jeremy Marzuola

Michael Taylor

ABSTRACT

Robert Booth: An Investigation of Non-Trapping, Asymptotically
Euclidean Wave Equations
(Under the direction of Jason Metcalfe)

In this dissertation, we demonstrate almost global existence for a class of variable coefficient, non-trapping, asymptotically Euclidean, quasilinear wave equations with small initial data. A novel feature is that the wave operator may be a large perturbation of the usual D'Alembertian operator. The key step is developing a local energy estimate for an appropriately linearized version of our wave equation. The linearized wave operator is a combination of a stationary, non-trapping, asymptotically Euclidean wave operator and a small time-dependent perturbation. The time-dependent perturbation need not be asymptotically Euclidean.

To my wife, Kim.

ACKNOWLEDGEMENTS

I cannot express enough gratitude towards my advisor, Jason Metcalfe, whose training has made this endeavor possible. Jason's instruction and guidance has been integral in further developing my mathematical intuition and maturity. His infinite patience and steadfast support has been greatly appreciated during my graduate school experience. Jason's contributions to my professional growth cannot be overstated.

I would like to thank my committee members: Professors Yaiza Canzani, Hans Christianson, Jeremy Marzuola, Jason Metcalfe, and Michael Taylor for their careful reading of this document. I am very grateful for the research support that I have received from Jeremy Marzuola, Jason Metcalfe, and Michael Taylor during my doctoral studies. I am appreciative of the many fruitful conversations that I have had with Jeremy and Michael. I would especially like to thank Yaiza and Hans, whose guidance, training, and encouragement has been a crucial component in my studies.

I am grateful for the conversations and guidance that I have received from Professors Peter Hislop, Hans Lindblad, Chris Sogge, and Mihai Tohaneanu. I am very appreciative towards the faculty of Rowan University and in particular Professors Hong Ling, Sam Lofland, and Hieu Nguyen for their encouragement and friendship.

Last, but certainly not least, I would like to thank my family and friends for all of their love and support. I am particularly grateful for the friendship of Travis Flemming and Andrew Fabian. The encouragement of my father, grandparents, Aunt Donna, and sisters has been very important to me. I must especially thank my first teacher, my mother, whose support has been critical in any success I have enjoyed. My wife, Kim, deserves an acknowledgments page of her own. I must thank her for her patience, understanding, and unconditional support throughout all of my pursuits.

TABLE OF CONTENTS

Chapter 1: Introduction

Our first goal is to obtain a local energy estimate for a large class of non-trapping variable coefficient wave operators. The wave operators can be thought of as being a combination of a stationary, non-trapping, asymptotically Euclidean wave operator and a small time-dependent perturbation. The time-dependent perturbation need not be asymptotically Euclidean. Even though the stationary component is asymptotically Euclidean, it can still be a large perturbation of the typical D'Alembertian operator $\Box := \partial_t^2 - \Delta$. Upon obtaining the said local energy estimate, we will apply it to prove a long-time existence theorem for solutions to quasilinear variable coefficient wave equations.

1.1 Statement of Problem and Setup

We are interested in the following initial value problem:

$$\begin{cases} P_h u(t,x) = F(t,x) \quad (t,x) \in (\mathbb{R}_+, \mathbb{R}^3) \\ u(0,\cdot) = f_1 \in C^2(\mathbb{R}^3), \quad \partial_t u(0,\cdot) = f_2 \in C^1(\mathbb{R}^3). \end{cases} \tag{1.1}$$

Here, P_h is a time-dependent variable coefficient wave operator:

$$P_h \equiv P_g + h^{\alpha\beta}(t,x) D_\alpha D_\beta, \tag{1.2}$$

where $h = (h^{\alpha\beta})$ is a smooth, symmetric, matrix-valued function and P_g is the stationary wave operator:

$$P_g = -D_t^2 + D_i g^{ij}(x) D_j. \tag{1.3}$$

Note the use of Einstein summation convention. Here $D_\alpha = \frac{1}{i}\partial_\alpha$ and is interpreted as an operator. Greek indices range from 0 to 3, with 0 denoting time and $1, 2, 3$ denoting spatial dimensions. Latin indices range from 1 to 3. Our background geometry is $\mathbb{R}_+ \times \mathbb{R}^3$ equipped with the usual Minkowski metric $\mathrm{diag}(-1, 1, 1, 1)$.

Note that $g = (g^{ij})$ is assumed to be a stationary, smooth, symmetric, matrix-valued function. Further, $D_i g^{ij} D_j$ is strictly elliptic in the sense that

$$0 < \lambda_0 |\xi|^2 \leq \lambda(x)|\xi|^2 \leq g^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^3 - \{0\}. \tag{1.4}$$

For more on elliptic operators, see [11]. Note that (1.4) implies the following useful lower bound:

$$\langle D_i g^{ij} D_j u, u\rangle_{L^2} \gtrsim \|\nabla_x u\|_{L^2}^2. \tag{1.5}$$

In addition, we assume that $D_i g^{ij} D_j$ is non-trapping and asymptotically Euclidean. By asymptotically Euclidean, we assume that $g^{ij}(x)$ has the following spatial decay:

$$\sum_{|\mu|\le 2}\sum_{i,j=1}^{3} \|\langle x\rangle^{|\mu|}\partial_x^\mu (g^{ij}-\delta^{ij})\|_{\ell_l^1 L^\infty(A_l)} = O(1), \tag{1.6}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $\delta = \delta_{ij}$ is the Kronecker delta function, and A_l denotes dyadic regions. Specifically, A_l denotes regions where $\langle x\rangle \approx 2^l$ and l runs over nonnegative integers. So, the L^∞ norms are taken over each dyadic region and then we sum over all such norms.

Our asymptotically Euclidean assumption is similar to that of [31], [29], among others. Observe the use of multi-index notation. Here $\partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

For the remainder of this paper, we fix the constant M_0 so that

$$|||g||| \equiv \sum_{|\mu|\le 2}\sum_{i,j=1}^{3} \|\langle x\rangle^{|\mu|}\partial_x^\mu (g^{ij}-\delta^{ij})\|_{\ell_l^1 L^\infty(A_l)} \le M_0. \tag{1.7}$$

Further, for any $0 < c \ll 1$, we can find R_{AF} such that

$$|||g|||_{>R_{AF}} \equiv \sum_{|\mu|\le 2}\sum_{i,j=1}^{3} \|\langle x\rangle^{|\mu|}\partial_x^\mu (g^{ij}-\delta^{ij})\|_{\ell_l^1 L^\infty(A_l\cap\{|x|>R_{AF}\})} \le c. \tag{1.8}$$

The idea is that while $D_i g^{ij} D_j$ can be a large perturbation of $-\Delta$, for $|x| > R_{AF}$, it is a small perturbation of $-\Delta$, due to our asymptotic Euclidean condition.

We assume that P_g is non-trapping in that upon setting up a Hamiltonian flow with respect to the principal symbol of the elliptic portion of our operator $g^{ij}\xi_i\xi_j$:

$$\begin{aligned}\dot{x}_k^s &= p_{\xi_k} = 2g^{ik}(x^s)\xi_i^s\\ \dot{\xi}_k^s &= -p_{x_k} = -(\partial_k g^{ij})(x^s)\xi_i^s\xi_j^s,\end{aligned}$$

with $(x^0, \xi^0) = (x, \xi)$, all geodesics escape to infinity. Hence, no geodesic stays in a compact set for all time.

We use the following multi-index notation to describe derivatives of a function:

$$\partial^{\leq N} f = \sum_{|\mu| \leq N} \partial^\mu f. \tag{1.9}$$

Here ∂ denotes the full space-time gradient (∂_t, ∇_x).

For future reference, it is useful to introduce a smooth cut-off function $\chi(|x|)$ that is monotonically decreasing as a function of $|x|$ such that $\chi \equiv 1$ for $|x| \leq 1$ and $\chi \equiv 0$ for $|x| > 2$ and to define

$$\chi_R(|x|) = \chi(|x|/R). \tag{1.10}$$

The notation $A \lesssim B$ means that $A \leq CB$ for some positive constant, C, that is independent of all important parameters. Further $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Lastly, $h^{\alpha\beta}$ and its derivatives will usually be small in the sense

$$\| \langle x \rangle \partial^{\leq 2} h \|_{\ell_l^\infty L^\infty L^\infty([0,T] \times A_l)} < \delta, \tag{1.11}$$

for some $\delta > 0$ to be chosen later. Here we are using the mixed norm notation $L^p L^q \equiv L^p_t L^q_x$. That is the L^q norm is taken with respect to spatial variables while the L^p norm is taken with respect to time. Further, we make use of the following notation:

$$|h| = \sum_{\alpha,\beta=0}^{3} |h^{\alpha\beta}| \qquad \text{and} \qquad |\partial^{\leq N} h| = \sum_{|\mu| \leq N} \sum_{\alpha,\beta=0}^{3} |\partial^\mu h^{\alpha\beta}|.$$

Observe that $h^{\alpha\beta}(t,x)$ is time dependent and not necessarily asymptotically flat. Hence we can think of P_h as a small Lipschitz perturbation of P_g.

Obtaining a local energy estimate for P_h is the key innovation in this work. In other words, our primary objective is to find a local energy estimate for the stationary wave operator P_g with such a (non asymptotically flat) perturbation. We will then apply this estimate to obtain a long-time existence theorem for solutions to quasilinear wave equations of the form $P_g u = Q(\partial u, \partial^2 u)$, where Q is quadratic in its arguments and linear in $\partial^2 u$.

1.2 Local Energy Norms

In order to state the main local energy estimate for the operator P_h, we will need to define a few local energy norms.

Definition 1.1. Local Energy Norms

We define the following local energy norms for functions $u(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$, as used in [26] and [31]:

$$\|u\|_{LE} = \| \langle x \rangle^{-1/2} u \|_{\ell_l^\infty L^2 L^2([0,T] \times A_l)}, \quad \|u\|_{LE^1} = \| (\partial u, \langle x \rangle^{-1} u) \|_{LE}.$$

For forcing terms, F, we use the following dual norm:

$$\|F\|_{LE^*} = \|\langle x\rangle^{1/2} F\|_{\ell_l^1 L^2 L^2([0,T]\times A_l)}.$$

All of the above L^2 norms are taken over both time and space. Further, the spatial L^2 norms are localized to a single dyadic region. We then take the ℓ^∞ or ℓ^1 norm with respect to all such dyadic regions, A_l. Note that all time integrations are over the interval $[0, T]$.

As in (1.8), we use the notation $\|u\|_{LE_{>R}}$ ($\|u\|_{LE^1_{>R}}$, $\|u\|_{L^2_{>R}}$) to denote the restriction of the LE (LE^1, L^2) norm to regions such that $|x| > R$. Similarly, we use the notation $\|u\|_{LE_R}$ ($\|u\|_{LE^1_R}$, $\|u\|_{L^2_R}$) to denote the restriction of the LE (LE^1, L^2) norm to the region such that $|x| \approx R$.

On occasion it will be useful to examine similar local energy norms, but at a fixed time. We use the notation $\mathcal{LE}$ and $\mathcal{LE}^*$ to denote the fixed time version of the LE and LE^* norms, respectively. More specifically,

$$\|u\|_{\mathcal{LE}} = \|\langle x\rangle^{-1/2} u\|_{\ell_l^\infty L^2(A_l)}, \qquad \text{and} \qquad \|F\|_{\mathcal{LE}^*} = \|\langle x\rangle^{1/2} F\|_{\ell_l^1 L^2(A_l)}.$$

We use the notation $\|u\|_{\mathcal{LE}^1}$ to mean something similar, except we will not measure time derivatives. Specifically,

$$\|u\|_{\mathcal{LE}^1} = \|(\nabla_x u, \langle x\rangle^{-1} u)\|_{\mathcal{LE}}.$$

1.3 The Main Theorems

We are now in a position to state the local energy decay theorem for our wave operator P_h. This is a major result of this dissertation.

Theorem 1.1. *Let P_h be as in (1.2), where $h^{\alpha\beta}$ is smooth, symmetric, and satisfies (1.11) for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is smooth, symmetric, non-trapping, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). Fix $T > 0$. Suppose $u(t, x) \in C^2([0, T] \times \mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\le 2} u(t, x)| \to 0$ as $|x| \to \infty$ for all $t \in [0, T]$. Then*

$$\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \lesssim \|\partial u(0)\|_{L^2} + \|P_h u\|_{LE^*} + \delta \log(2 + T)\|u\|_{LE^1}. \tag{1.12}$$

Here the implicit constant is independent of T and δ.

Theorem 1.1 is the principal result in this work. When $h = 0$, Theorem 1.1 has recently been proven by Metcalfe, Sterbenz, and Tataru in [29] for a larger class of stationary, asymptotically flat, non-trapping wave operators. Further, their result is stable with respect to small, asymptotically flat perturbations. Theorem 1.1 is not merely a corollary of the work of [29], however. This is because the perturbation term $h^{\alpha\beta}D_\alpha D_\beta$ is not asymptotically flat. The perturbation h is small but does not gain additional spatial decay when it is differentiated. Getting good bounds on such non-asymptotically flat time-dependent perturbations is a major difficulty and important for many applications, as we will discuss. Still, we are influenced by the methods in [29].

One important application of local energy estimates is to prove long-time existence results for solutions to nonlinear wave equations. Indeed, the primary motivation for developing Theorem 1.1 was such a problem. We consider the following quasilinear, variable coefficient wave equation:

$$\begin{cases} P_g u(t,x) = Q(\partial u, \partial^2 u) & (t,x) \in (\mathbb{R}_+, \mathbb{R}^3) \\ u(0,\cdot) = f_1 \in C^\infty(\mathbb{R}^3), & \partial_t u(0,\cdot) = f_2 \in C^\infty(\mathbb{R}^3), \end{cases} \tag{1.13}$$

where the nonlinearity $Q(\partial u, \partial^2 u)$ in (1.13) is quadratic in its arguments and linear in $\partial^2 u$:

$$Q(\partial u, \partial^2 u) = B(\partial u) + B_\gamma^{\alpha\beta}\partial_\gamma u \partial_\alpha \partial_\beta u, \tag{1.14}$$

where $B(\partial u)$ is a constant coefficient quadratic form and $B_\gamma^{\alpha\beta}$ are real constants. We also require that $B_\gamma^{\alpha\beta} = B_\gamma^{\beta\alpha}$. Here P_g is as in (1.3). Further, $D_i g^{ij} D_j$ is smooth, non-trapping, symmetric, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). As the proof of the long-time existence theorem will require commuting with special classes of vector fields, we have the following addition assumptions on $D_i g^{ij} D_j$. Specifically, we assume that we can write $D_i g^{ij} D_j$ as

$$D_i g^{ij}(x) D_j = -\Delta + D_r g_r(r) D_r - g_\omega(r)\frac{1}{r^2}\Delta_{\mathbb{S}^2} + D_i g_{sr}^{ij}(x) D_j. \tag{1.15}$$

Here $\Delta_{\mathbb{S}^2}$ is the Laplacian on the unit sphere and $D_r = \frac{1}{i}\partial_r = \frac{1}{i}\frac{x_j}{r}\partial_j$ and is interpreted as an operator. Further $g_r(r)$ and $g_\omega(r)$ are smooth, radial functions, while $g_{sr}^{ij}(x)$ is a smooth, symmetric, matrix-valued function such that

$$\begin{aligned} \sum_{|\mu|\le 2} \|\langle x\rangle^{|\mu|}\partial_x^\mu g_r\|_{\ell_l^1 L^\infty(A_l)} + \sum_{|\mu|\le 2} \|\langle x\rangle^{|\mu|}\partial_x^\mu g_\omega\|_{\ell_l^1 L^\infty(A_l)} + \sum_{3\le|\mu|\le 15} \|\langle x\rangle^{2}\partial_x^\mu g_r\|_{\ell_l^1 L^\infty(A_l)} \\ + \sum_{3\le|\mu|\le 15} \|\langle x\rangle^{2}\partial_x^\mu g_\omega\|_{\ell_l^1 L^\infty(A_l)} + \sum_{|\mu|\le 15} \|\langle x\rangle^{1+|\mu|}\partial_x^\mu g_{sr}\|_{\ell_l^1 L^\infty(A_l)} = O(1). \end{aligned} \tag{1.16}$$

So, we can think of $D_r g_r(r) D_r - g_\omega(r)\frac{1}{r^2}\Delta_{\mathbb{S}^2}$ as a long-range radial perturbation of the Laplacian and $D_i g_{sr}^{ij}(x) D_j$ as a short-range perturbation of the Laplacian. Observe $D_i g_{sr}^{ij}(x) D_j$ need not be radial.

We use the notation

$$\Omega_{ij} = x_i \partial_j - x_j \partial_i \tag{1.17}$$

to denote the generators of rotations and

$$\{Z\} = \{\partial_\alpha, \Omega\} \tag{1.18}$$

to denote the generators of translations and rotations, where $\Omega \in \{\Omega_{ij}\}$.

We can now state our small data, long-time existence theorem to solutions of (1.13):

Theorem 1.2. *Consider* (1.13)*, where P_g is as in* (1.3)*. Further, $D_i g^{ij} D_j$ is smooth, non-trapping, symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and can be written as* (1.15) *where g_r, g_ω, and g_{sr}^{ij} satisfy* (1.16)*. The nonlinearity $Q(\partial u, \partial^2 u)$ is as in* (1.14)*, where where $B(\partial u)$ is a constant coefficient quadratic form and $B_\gamma^{\alpha\beta}$ are real constants. We also require that $B_\gamma^{\alpha\beta} = B_\gamma^{\beta\alpha}$. Then, there exists real positive constants κ and ε_0 such that for all $\varepsilon < \varepsilon_0$ and initial data $f_1, f_2 \in C^\infty(\mathbb{R}^3)$ satisfying*

$$\sum_{|k|+|\gamma|\le 15} \|\partial_x^k \Omega^\gamma \nabla_x f_1\|_{L^2} + \sum_{|k|+|\gamma|\le 15} \|\partial_x^k \Omega^\gamma f_2\|_{L^2} \le \varepsilon, \tag{1.19}$$

(1.13) *has a unique solution $u \in C^\infty([0, T_\varepsilon) \times \mathbb{R}^3)$ with*

$$T_\varepsilon = \exp(\kappa/\varepsilon). \tag{1.20}$$

Remark. *Prior work of [14] and [41] shows that the lifespan result of Theorem 1.2 is sharp.*

Remark. *Lifespans such as* (1.20) *are often referred to as almost global existence theorems since, while we cannot expect a global solution to* (1.13)*, the solution does have a lifespan that grows at an exponential rate as the size of the initial data approaches zero.*

1.4 A Brief Review of Prior Results

We now report prior key work, beginning by highlighting local energy estimates and then moving on to nonlinear theory.

1.4.1 History of Local Energy Estimates

Local energy decay theorems such as Theorem 1.1 have a long and rich history, with initial introduction by Morawetz for the Klein-Gordon equation and then later the wave equation, all on Minkowski space-time [35], [36], [37]. Key advances were made by Keel, Smith, and Sogge in [15] and by Metcalfe and Tataru in [31]. Other contributions have been made in [1], [2], [3], [4], [6], [7], [13], [18], [27], [28], [29], [34], [38], [45], [49], [50].

Local energy estimates are a fundamental object of study for dispersive partial differential equations in the asymptotically flat regime. They are known to be somewhat stable with respect to small time-dependent, asymptotically flat, geometric perturbations [1], [27], [28], [29], [30], [31], as well as stationary, non-trapping perturbations [2], [6], [47].

Further, other measures of dispersion can be derived from local energy estimates. In particular local energy estimates imply global-in-time Strichartz estimates for wave equations in the asymptotically flat regime [26], [30], [31], [51], [54]. Additionally, weak local energy estimates (in the asymptotically flat regime) imply pointwise decay estimates [32], [33], [52]. In particular, they have been used to prove Price's law ([39], [40]), which was a conjecture that solutions to the wave equation on a Schwarzschild space-time should decay at a rate of t^{-3} for fixed x. For local energy estimates in the context of exterior domain problems, see [5], [12], [27], [34], [45].

1.4.2 Some Quasilinear Theory

Many nonlinear results are known to follow provided one is able to obtain a good local energy estimate for an appropriately linearized version of the problem [2], [15], [16], [27], [28], [47], [56], [57], [58], [59]. Indeed, this is a large motivation for developing the estimate in Theorem 1.1.

Quasilinear wave equations can be thought of as arising from nontrivial background geometry where the metric of the manifold depends on the solution of the equation. Lifespan estimates for quasilinear wave equations with small initial data on Minkowski space-time are now well understood [46]. Here the solution to the linear problem decays at a rate of $t^{-(n-1)/2}$, where n denotes spatial dimension. This is integrable for $n \geq 4$, allowing an iteration argument to prove a global existence result. For $n = 3$, this misses being integrable by a logarithm, implying almost global existence, instead of global existence.

The breakthrough work of [19] introduced the now standard invariant vector field method to obtain sufficient decay to close an iteration argument to prove long time existence results for quasilinear wave equations on Minkowski space-time. Here the generators of the full Poincaré group plus the scaling vector field are used. These are the vector fields that preserve the homogeneous linear wave equation. The use of Lorentz boosts (and to a lesser extent, the scaling vector field) are problematic in many important settings such as multiple speed systems of equations, exterior domain problems, and equations on background metrics where commuting the Lorentz boosts (and scaling vector field) with the linear operator (needed for the iteration argument) may yield terms that grow large for long times.

For quasilinear wave equations on exterior domains, the authors of [16] developed methods that avoid the use of Lorentz boosts. See also the related work of [20], [42], [43], and [44], who first developed similar methods in different contexts. Later, the authors of [27], based on the work of [15], simplified the argument of [16]. The methodology of [27] uses the generators of translations and rotations to obtain decay in $|x|$ over the more standard decay in t. Pairing this decay with local energy estimates for solutions to the linear problem with time-dependent perturbations allowed them to close an iteration argument. This is an improvement over prior work in that fewer symmetries on the background geometry are needed. The methodology of [27] provides a method for proving almost global existence for quasilinear wave equations with small initial data in fairly generic scenarios, provided that an appropriate local energy decay theorem for a "background" operator can be obtained. A key facet to mention is that the background operator must incorporate small time-dependent perturbations as done in Theorem 1.1. This perturbation essentially incorporates the quasilinear nature of our equation. Hence, upon attaining Theorem 1.1, it is natural to expect a result such as Theorem 1.2 to hold.

There are several other related works to Theorem 1.2. The authors of [2] and [47] investigate related problems for semilinear equations with a product manifold structure and short range perturbations. The works of [23], [56], [57], and [58] consider semilinear wave equations satisfying the null condition. The work of [22] focuses on quasilinear wave equations close to Schwarzschild that satisfy the weak null condition (it should be noted that here there are complications involving trapping at the photon sphere that we need not consider). The works of [55] and [59] require the wave operator to be a small perturbation of the d'Alembertian operator on Minkowski space-time, although the framework of [59] holds in more general settings, provided one is able to prove an appropriate local energy estimate for the linear problem. Further, [59] again assumes the null condition. We shall make no such assumptions in this work.

1.5 Outline of the Proof of Theorem 1.1

The bulk of this document is devoted to proving Theorem 1.1. This is done by observing that for sufficiently large $|x|$, our wave operator is a small perturbation of $\Box$ and good local energy decay should hold with errors localized to a dyadic region as in [24], [29]. From here, it will suffice to work on a compact region and find estimates that have errors that can be absorbed for high, medium, and low time frequencies, respectively. The high frequency analysis is a positive commutator argument, using appropriately constructed pseudo-differential operators and the Gårding inequality. The medium frequency analysis uses Carleman estimates, which are weighted L^2L^2 estimates where, morally, the weights are convex. The low frequency analysis is essentially an argument in elliptic regularity. Upon attaining the high, medium, and low frequency local energy estimates, we piece them together using time frequency cut-offs. After proving Theorem 1.1, we obtain additional energy estimates by applying vector fields to the solution of our wave equation. We then follow the work of [27] to use an iteration argument to prove Theorem 1.2.

Chapter 2: Uniform Energy Estimates

Observe that the standard uniform energy estimate for stationary wave operators is built into Theorem 1.1. We now state this uniform energy estimate as a proposition. This is similar to estimates found in [46].

Proposition 2.1. *Let P_g be as in* (1.3)*, where $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_g u = F \in L^1 L^2$, with initial data $(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then, the following uniform energy estimate holds:*

$$\|\partial u\|^2_{L^\infty L^2} \lesssim \|\partial u(0)\|^2_{L^2} + \int_0^T \int_{\mathbb{R}^3} |P_g u||\partial_t u| \, dxdt. \tag{2.1}$$

Proof. We begin by defining an energy functional

$$E_g[u](t) = \int_{\mathbb{R}^3} (D_i g^{ij} D_j u)\overline{u} + |D_t u|^2 \, dx, \tag{2.2}$$

and observe

$$\frac{d}{dt} E_g[u](t) = 2\mathrm{Re} \int_{\mathbb{R}^3} \partial_t u \overline{F} \, dx.$$

Integrating the above expression shows that for any $t \in [0,T]$

$$E_g[u](t) \lesssim E_g[u](0) + \int_0^T \int_{\mathbb{R}^3} |F||\partial_t u| \, dxdt.$$

We note in the absence of forcing, the above equation becomes a statement of energy conservation. Proposition 2.1 will follow if we can show $\|\partial u\|^2_{L^2} \approx E_g[u](t)$. But $D_i g^{ij} D_j$ is strictly elliptic and asymptotically Euclidean, and so this is immediate, completing the proof. □

Since h will be small in an appropriate sense, we have an "almost" uniform energy bound for solutions to $P_h u = F$. Before stating this as a proposition, we note the following lemma which will be repeatedly used throughout this work.

Lemma 2.1. *Let $h^{\alpha\beta}$ be a smooth, symmetric, matrix-valued function that satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Suppose $u(t,x) \in LE^1 \cap L^\infty L^2$. Then*

$$\int_0^T \int_{\mathbb{R}^3} \big(|\partial^2 h| + |\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle x\rangle}\big)^2 \,dxdt \lesssim \delta \log(2+T)\|u\|_{LE^1}^2 + \delta \|\partial u\|_{L^\infty L^2}^2. \tag{2.3}$$

Here the implicit constant is independent of T and δ.

Proof. We first observe

$$\int_0^T \int_{\mathbb{R}^3} \big(|\partial^2 h| + |\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle x\rangle}\big)^2 \,dxdt \lesssim \sum_{l=0}^{\log(2+T)} \|\langle x\rangle \partial^{\le 2} h\|_{L^\infty L^\infty([0,T]\times A_l)} \|u\|_{LE^1}^2$$
$$+ T\|\partial^{\le 2} h\|_{L^\infty L^\infty([0,T]\times\{|x|>T\})} \|\partial u\|_{L^\infty L^2}^2.$$

Note that we have applied a Hardy inequality on the lower order terms. Specifically, we have used

$$\||x|^{-1} f(x)\|_{L^2} \lesssim \|\nabla_x f(x)\|_{L^2}, \tag{2.4}$$

where $f(x) \in H^1$. Observe that (1.11) implies

$$T\|\partial^{\le 2} h\|_{L^\infty L^\infty([0,T]\times\{|x|>T\})} < \delta.$$

Hence, we have shown

$$\int_0^T \int_{\mathbb{R}^3} \big(|\partial^2 h| + |\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle x\rangle}\big)^2 \,dxdt$$
$$\lesssim \delta \|\partial u\|_{L^\infty L^2}^2 + \sum_{j=0}^{\log(2+T)} \|\langle x\rangle \partial^{\le 2} h\|_{L^\infty L^\infty([0,T]\times A_l)} \|u\|_{LE^1}^2$$
$$\lesssim \delta \log(2+T)\ \|u\|_{LE^1}^2 + \delta \|\partial u\|_{L^\infty L^2}^2,$$

as desired. This completes the proof of Lemma 2.1. □

Armed with Lemma 2.1, we now state and prove the "almost" uniform energy bound for solutions to $P_h u = F$.

Proposition 2.2. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T]\times\mathbb{R}^3)$ solves $P_h u = F \in L^1 L^2$, with initial data*

$(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ *such that* $|\partial^{\leq 2} u(t,x)| \to 0$ *as* $|x| \to \infty$ *for all* $t \in [0,T]$. *Then*

$$\|\partial u\|_{L^\infty L^2}^2 \lesssim \|\partial u(0)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |P_h u||\partial_t u| \, dxdt + \delta \log(2+T) \|u\|_{LE^1}^2. \tag{2.5}$$

Here the implicit constant is independent of T *and* δ.

Remark. *Proposition 2.2 essentially holds for weaker assumptions on* h. *Indeed, if we assume only that* $|\partial^{\leq 1} h| < \varepsilon$, *for some* $\varepsilon > 0$ *chosen sufficiently small, then the same proof demonstrates*

$$\|\partial u\|_{L^\infty L^2}^2 \lesssim \|\partial u(0)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |P_h u||\partial_t u| \, dxdt + \int_0^T \int_{\mathbb{R}^3} |\partial h||\partial u|^2 \, dxdt.$$

Proof. We modify the energy functional $E_g[u](t)$ defined in (2.2) to

$$E[u](t) = \int_{\mathbb{R}^3} (D_i g^{ij} D_j u)\overline{u} + (D_i h^{ij} D_j u)\overline{u} + (1 + h^{00})|\partial_t u|^2 \, dx$$

and compute

$$\begin{aligned}\frac{d}{dt} E[u](t) =& 2\mathrm{Re} \int_{\mathbb{R}^3} \partial_t u \overline{P_h u} \, dx + \int_{\mathbb{R}^3} \partial_t h^{ij} \partial_j u \overline{\partial_i u} \, dx + \int_{\mathbb{R}^3} (h^{0j} \partial_j \partial_t u) \overline{\partial_t u} \, dx \\ &+ \int_{\mathbb{R}^3} \partial_t u \overline{h^{0j} \partial_j \partial_t u} \, dx + \int_{\mathbb{R}^3} \partial_t u \overline{\partial_i h^{ij} \partial_j u} \, dx + \int_{\mathbb{R}^3} (\partial_i h^{ij} \partial_j u) \overline{\partial_t u} \, dx \\ &+ \int_{\mathbb{R}^3} \partial_t h^{00} |\partial_t u|^2 \, dx.\end{aligned}$$

We integrate the above expression in time to find for any $t \in [0,T]$:

$$\begin{aligned}E[u](t) \lesssim E[u](0) + \int_0^T \int_{\mathbb{R}^3} |P_h u||\partial_t u| + \int_0^T \int_{\mathbb{R}^3} |\partial h||\partial u|^2 \, dxdt \\ + \int_0^T \int_{\mathbb{R}^3} (h^{0j} \partial_j \partial_t u) \overline{\partial_t u} \, dx + \int_0^T \int_{\mathbb{R}^3} \partial_t u \overline{h^{0j} \partial_j \partial_t u} \, dx.\end{aligned}$$

Integrating the last term on the right-hand side by parts, we find

$$\int_0^T \int_{\mathbb{R}^3} \partial_t u \overline{h^{0j} \partial_j \partial_t u} \, dx = -\int_0^T \int_{\mathbb{R}^3} (h^{0j} \partial_j \partial_t u) \overline{\partial_t u} \, dx - \int_0^T \int_{\mathbb{R}^3} \partial_t u \overline{\partial_j h^{0j} \partial_t u}.$$

So, we have shown

$$E[u](t) \lesssim E[u](0) + \int_0^T \int_{\mathbb{R}^3} |P_h u||\partial_t u| + \int_0^T \int_{\mathbb{R}^3} |\partial h||\partial u|^2 \, dxdt.$$

Using that $D_i g^{ij} D_j$ is strictly elliptic, we have

$$\int_{\mathbb{R}^3} \Big((D_i g^{ij} D_j u)\overline{u} + |\partial_t u|^2 \Big)\, dx \approx \|\partial u\|_{L^2}^2.$$

Since

$$\int_{\mathbb{R}^3} h^{00} |\partial_t u|^2 + h^{ij} \partial_j u \overline{\partial_i u}\, dx \lesssim \|h\|_{L^\infty L^\infty} \|\partial u\|_{L^2}^2,$$

by choosing δ sufficiently small, we find $E[u](t) \approx \|\partial u\|_{L^2}^2$. Therefore we have demonstrated

$$\|\partial u\|_{L^\infty L^2}^2 \lesssim \|\partial u(0)\|_{L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |P_h u||\partial_t u|\, dxdt + \int_0^T \int_{\mathbb{R}^3} |\partial h||\partial u|^2\, dxdt.$$

An application of Lemma 2.1 completes the proof. □

Chapter 3: Exterior Estimates

We begin with a variation of a theorem found in [27], [28], and [31] which essentially states that good local energy decay holds for small, asymptotically flat perturbations of the D'Alembertian.

Theorem 3.1. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further assume $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and a small, asymptotically Euclidean perturbation of $-\Delta$ in the sense that $|||g||| \leq c \ll 1$ for some $c > 0$, sufficiently small. Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in L^1 L^2$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then*

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big)\, dx dt + \delta \log(2+T) \|u\|_{LE^1}^2. \tag{3.1}$$

Here the implicit constant is independent of T and δ.

Remark. *Theorem 3.1 essentially holds for weaker assumptions on h. Indeed, if we assume only that $|\partial^{\leq 1} h| < \varepsilon$, for some $\varepsilon > 0$ chosen sufficiently small, then the same proof demonstrates*

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big)\, dx dt + \int_0^T \int_{\mathbb{R}^3} \big(|\partial h| + \frac{|h|}{\langle x \rangle} \big) |\partial u| \big(|\partial u| + \frac{|u|}{\langle x \rangle} \big)\, dx dt.$$

Proof. The proof is a positive commutator argument. A direct computation shows

$$\begin{aligned}
2\mathrm{Im}\, \langle P_h u, Qu \rangle_{L^2L^2} =& \langle i[D_j g^{jk} D_k, Q]u, u \rangle_{L^2L^2} - 2\mathrm{Re}\, \langle D_t u, Qu \rangle \big|_0^T - 2\langle h^{0j} D_j u, Qu \rangle_{L^2} \big|_0^T \\
&+ 2\mathrm{Re}\, \langle \partial_i h^{ij} D_j u, Qu \rangle_{L^2L^2} - 2\langle \partial_t h^{0j} D_j u, Qu \rangle_{L^2L^2} + 2\langle \partial_j h^{0j} Qu, D_t u \rangle_{L^2L^2} \\
&+ \frac{1}{i} \langle h^{ij} D_j u, [D_i, Q]u \rangle_{L^2L^2} + \frac{1}{i} \langle [Q, h^{ij} D_i]u, D_j u \rangle_{L^2L^2} \\
&+ \frac{2}{i} \langle [Q, h^{0j} D_j]u, D_t u \rangle_{L^2L^2} + \frac{1}{i} \langle [Q, h^{00}] D_t u, D_t u \rangle_{L^2L^2} \\
&+ 2\mathrm{Im}\, \langle \partial_t h^{00} \partial_t u, Qu \rangle_{L^2L^2} - 2\mathrm{Im}\, \langle h^{00} \partial_t u, Qu \rangle_{L^2} \big|_0^T,
\end{aligned} \tag{3.2}$$

if $Q(x, D_x)$ is self-adjoint. We choose

$$Q = f(r) \frac{x_l}{r} g^{lm} D_m + D_m f(r) \frac{x_l}{r} g^{lm}$$

and

$$f(r) = \frac{r}{r+\rho}.$$

For now we leave $\rho \geq 1$ as a parameter to be chosen later. We record some useful computations for later:

$$f'(r) = \frac{\rho}{(r+\rho)^2}, \qquad f''(r) = \frac{-2\rho}{(r+\rho)^3}, \qquad f'''(r) = \frac{6\rho}{(r+\rho)^4}. \tag{3.3}$$

We begin bounding the terms in (3.2). For the left-hand side, we note

$$2\text{Im}\,\langle P_h u, Qu\rangle_{L^2L^2} \lesssim \int_0^T \int_{\mathbb{R}^3} |P_h u|\big(|\partial u| + \frac{|u|}{\langle x\rangle}\big)\, dxdt.$$

The time boundary terms are controlled by $\|\partial u\|^2_{L^\infty L^2}$. Indeed, applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
-2\text{Re}\,\langle D_t u, Qu\rangle\big|_0^T + \langle h^{0j} D_j u, Qu\rangle_{L^2}\big|_0^T - 2\text{Im}\,\langle h^{00}\partial_t u, Qu\rangle_{L^2}\big|_0^T \\
\lesssim \|\partial u\|^2_{L^2}\big|_{t=0} + \|\partial u\|^2_{L^2}\big|_{t=T} + \|Qu\|^2_{L^2}\big|_{t=0} + \|Qu\|^2_{L^2}\big|_{t=T} \\
\lesssim \|\partial u\|^2_{L^\infty L^2} + \|\langle x\rangle^{-1} u\|^2_{L^\infty L^2} \lesssim \|\partial u\|^2_{L^\infty L^2}.
\end{aligned}$$

Note that we have applied a Hardy inequality (2.4) on the lower order terms.

Appealing to (2.3), we can bound the remaining h dependent terms directly from above:

$$\begin{aligned}
\text{Re}\,\langle \partial_i h^{ij} D_j u, Qu\rangle_{L^2L^2} - 2\langle \partial_t h^{0j} D_j u, Qu\rangle_{L^2L^2} + 2\langle \partial_j h^{0j} Qu, D_t u\rangle_{L^2L^2} \\
+ \frac{1}{i}\langle h^{ij} D_j u, [D_i, Q]u\rangle_{L^2L^2} + \frac{1}{i}\langle [Q, h^{ij} D_i]u, D_j u\rangle_{L^2L^2} + \frac{2}{i}\langle [Q, h^{0j} D_j]u, D_t u\rangle_{L^2L^2} \\
+ \frac{1}{i}\langle [Q, h^{00}] D_t u, D_t u\rangle_{L^2L^2} + 2\text{Im}\,\langle \partial_t h^{00}\partial_t u, Qu\rangle_{L^2L^2}
\end{aligned}$$

by

$$\int_0^T \int_{\mathbb{R}^3} \Big(|\partial h| + \frac{|h|}{\langle x\rangle}\Big)|\partial u|\Big(|\partial u| + \frac{|u|}{\langle x\rangle}\Big)\, dxdt \lesssim \delta \log(2+T)\|u\|^2_{LE^1} + \delta\|u\|^2_{L^\infty L^2},$$

as desired. This is clear since

$$[D_i, Q] = 2\frac{1}{i}\partial_i\big(f(r)\frac{x_l}{r} g^{lm}\big) D_m - \partial_i\partial_m\big(f(r)\frac{x_l}{r} g^{lm}\big) = O(\langle x\rangle^{-1}) D_x + O(\langle x\rangle^{-2})$$

and

$$[Q, h^{ij} D_i] = 2\frac{1}{i} f(r)\frac{x_l}{r} g^{lm}\partial_m h^{ij} D_i - 2\frac{1}{i} h^{ij}\partial_i\big(f(r)\frac{x_l}{r} g^{lm}\big) D_m + h^{ij}\partial_i\partial_m\big(f(r)\frac{x_l}{r} g^{lm}\big)$$

$$= O(|\partial h|)D_x + O(|h|\langle x\rangle^{-1})D_x + O(|h|\langle x\rangle^{-2}).$$

Observe that we have used that g is asymptotically Euclidean so that $|\partial g| \lesssim \langle x\rangle^{-1}$ and $|\partial g| \lesssim \langle x\rangle^{-2}$. Similarly

$$[Q, h^{0j}D_j] = O(|\partial h|)D_x + O(|h|\langle x\rangle^{-1})D_x + O(|h|\langle x\rangle^{-2}) \quad \text{and} \quad [Q, h^{00}] = O(|\partial h|),$$

where we have again made use of our asymptotically Euclidean assumption.

All that remains is to bound $\langle i[D_j g^{jk} D_k, Q]u, u\rangle_{L^2L^2}$ from below. Writing $g^{jk} = m^{jk} + \tilde{g}^{jk}$, we find

$$\langle i[D_j g^{jk} D_k, Q]u, u\rangle_{L^2L^2} \gtrsim \langle i[-\Delta, f(r)\frac{x_l}{r}D_l + D_l\frac{x_l}{r}f(r)]u, u\rangle_{L^2L^2} - |||g|||(\|\partial_x u\|_{LE}^2 + \|\langle x\rangle^{-1}u\|_{LE}^2). \tag{3.4}$$

Perhaps the $|||g|||(\|\partial_x u\|_{LE}^2 + \|\langle x\rangle^{-1}u\|_{LE}^2)$ term deserves additional comment. This is controlling the following exact terms:

$$\begin{aligned}\langle i[-\Delta, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m + D_m f(r)\frac{x_l}{r}\tilde{g}^{lm}]u, u\rangle_{L^2L^2} \\ + \langle iD_j\tilde{g}^{jk}D_k, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m + D_m f(r)\frac{x_l}{r}\tilde{g}^{lm}]u, u\rangle_{L^2L^2} \\ + \langle i[D_j\tilde{g}^{jk}D_k, f(r)\frac{x_l}{r}D_l + D_l\frac{x_l}{r}f(r)]u, u\rangle_{L^2L^2}.\end{aligned} \tag{3.5}$$

Now

$$\begin{aligned}\langle i[-\Delta, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m]u, u\rangle_{L^2L^2} = &- \langle i\partial_j\partial_j(f(r)\frac{x_l}{r}\tilde{g}^{lm})D_m u, u\rangle_{L^2L^2} \\ &+ 2\langle\partial_j(f(r)\frac{x_l}{r}\tilde{g}^{lm})D_jD_m u, u\rangle_{L^2L^2}.\end{aligned}$$

Integrating the last term in the above line by parts, we obtain

$$\begin{aligned}\langle i[-\Delta, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m]u, u\rangle_{L^2L^2} &= 2\langle\partial_j(f(r)\frac{x_l}{r}\tilde{g}^{lm})D_m u, D_j u\rangle_{L^2L^2} + \langle i\partial_j\partial_j(f(r)\frac{x_l}{r}\tilde{g}^{lm})D_m u, u\rangle_{L^2L^2} \\ &\lesssim |||g|||(\|\partial_x u\|_{LE}^2 + \|\langle x\rangle^{-1}u\|_{LE}^2),\end{aligned}$$

where we have applied Cauchy's inequality and made use of our asymptotically Euclidean assumption and the fact that $\partial_x^k(f(r)\frac{x_l}{r}) = O(\langle x\rangle^{-|k|})$ for $0 \le |k| \le 2$.

We now bound $\langle i[-\Delta, D_m f(r)\frac{x_l}{r}\tilde{g}^{lm}]u, u\rangle_{L^2L^2}$ by noting

$$\langle i[-\Delta, D_m f(r)\frac{x_l}{r}\tilde{g}^{lm}]u, u\rangle_{L^2L^2} = \langle i[-\Delta, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m]u, u\rangle_{L^2L^2} + \langle[-\Delta, \partial_m(f(r)\frac{x_l}{r}\tilde{g}^{lm})]u, u\rangle_{L^2L^2}.$$

We have already bound $\langle i[-\Delta, f(r)\frac{x_l}{r}\tilde{g}^{lm}D_m]u, u\rangle_{L^2L^2}$ and so it suffices to bound $\langle [-\Delta, \partial_m\big(f(r)\frac{x_l}{r}\tilde{g}^{lm}\big)]u, u\rangle_{L^2L^2}$. Integrating by parts, we find

$$\langle [-\Delta, \partial_m\big(f(r)\frac{x_l}{r}\tilde{g}^{lm}\big)]u, u\rangle_{L^2L^2} = \langle i\partial^2_{jm}\big(f(r)\frac{x_l}{r}\tilde{g}^{lm}\big)D_j u, u\rangle_{L^2L^2} - \langle i\partial^2_{jm}\big(f(r)\frac{x_l}{r}\tilde{g}^{lm}\big)u, D_j u\rangle_{L^2L^2}.$$

Again this is controlled by $|\!|\!|g|\!|\!|\|u\|^2_{LE^1}$. The remaining terms in (3.5) are bounded very similarly.

We return to $[-\Delta, f(r)\frac{x_l}{r}D_l + D_l\frac{x_l}{r}f(r)]$ coming from the right-hand side of (3.4). A direct computation shows

$$\begin{aligned} i[-\Delta, f(r)\frac{x_l}{r}D_l + D_l\frac{x_l}{r}f(r)] =& 4D_k\frac{x_k}{r}f'(r)\frac{x_j}{r}D_j \\ &+ 4D_l\big(\delta_{lk} - \frac{x_kx_l}{r^2}\big)\frac{f(r)}{r}\big(\delta_{jk} - \frac{x_kx_j}{r^2}\big)D_j - \Delta\partial_k(\frac{x_k}{r}f(r)). \end{aligned} \tag{3.6}$$

Continuing, we compute

$$-\Delta\partial_k(\frac{x_k}{r}f(r)) = \frac{2\rho r + 8\rho^2}{r(r+\rho)^4}.$$

Combining this and our explicit formulas for $f(r)$ and $f'(r)$ and integrating by parts, we see

$$\begin{aligned} &\langle i\big(\frac{4}{i}D_k\frac{x_k}{r}f'(r)\frac{x_j}{r}D_j + \frac{4}{i}D_l\big(\delta_{lk} - \frac{x_kx_l}{r^2}\big)\frac{f(r)}{r}\big(\delta_{jk} - \frac{x_kx_j}{r^2}\big)D_j - \frac{1}{i}\Delta\partial_k(\frac{x_k}{r}f(r))\big)u, u\rangle_{L^2L^2} \\ &\quad\gtrsim \int_0^T\int_{\mathbb{R}^3}\Big(\frac{\rho}{(r+\rho)^2}|\partial_r u|^2 + \frac{1}{r+\rho}|\not\nabla u|^2 + \frac{\rho}{(\rho+r)^4}u^2\Big)\,dxdt, \end{aligned} \tag{3.7}$$

where $\not\nabla = \nabla - \frac{x}{r}\partial_r$ denotes angular derivatives. So, we have shown

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^3}\Big(\frac{\rho}{(r+\rho)^2}|\partial_r u|^2 + \frac{1}{r+\rho}|\not\nabla u|^2 + \frac{\rho}{(\rho+r)^4}u^2\Big)\,dxdt& \\ \lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T\int_{\mathbb{R}^3}|P_h u|\Big(|\partial u| + \frac{|u|}{\langle x\rangle}\Big)\,dxdt& \\ + \delta\log(2+T)\|u\|^2_{LE^1} + |\!|\!|g|\!|\!|\|u\|^2_{LE^1}&. \end{aligned} \tag{3.8}$$

We recover time derivatives via a Lagrangian correction. To this end, we compute:

$$(3.9)\qquad \begin{aligned}\langle P_h u, -f'(r)u\rangle_{L^2L^2} = &- \langle \partial_t u, f'(r)u\rangle_{L^2}\big|_0^T + \langle \partial_t u, f'(r)\partial_t u\rangle_{L^2L^2}\\ &- \langle g^{ij} D_j u, D_i f'(r)u\rangle_{L^2L^2} - \langle \partial_i h^{ij}\partial_j u, f'(r)u\rangle_{L^2L^2}\\ &- \langle h^{ij} D_j u, D_i f'(r)u\rangle_{L^2L^2} - 2\langle \partial_j h^{0j}\partial_t u, f'(r)u\rangle_{L^2L^2}\\ &- 2\langle h^{0j} D_t u, D_j f'(r)u\rangle_{L^2L^2} - \langle h^{00}\partial_t u, f'(r)u\rangle_{L^2}\big|_0^T\\ &+ \langle h^{00}\partial_t u, f'(r)\partial_t u\rangle_{L^2L^2} + \langle \partial_t h^{00}\partial_t u, f'(r)u\rangle_{L^2L^2}.\end{aligned}$$

We note

$$\langle P_h u, f'(r)u\rangle_{L^2L^2} \lesssim \int_0^T \int_{\mathbb{R}^3} |P_h u| \frac{|u|}{\langle x\rangle}\, dxdt.$$

The time boundary terms are controlled by $\|\partial u\|^2_{L^\infty L^2}$, where we have employed (2.4) on lower order terms. We immediately observe

$$\begin{aligned}&\langle \partial_i h^{ij}\partial_j u, f'(r)u\rangle_{L^2L^2} + 2\langle h^{0j} D_t u, D_j f'(r)u\rangle_{L^2L^2}\\ &\qquad + \langle h^{ij} D_j u, D_i f'(r)u\rangle_{L^2L^2} + 2\langle \partial_j h^{0j}\partial_t u, f'(r)u\rangle_{L^2L^2}\\ &\qquad\qquad + \langle h^{00}\partial_t u, f'(r)\partial_t u\rangle_{L^2L^2} + \langle \partial_t h^{00}\partial_t u, f'(r)u\rangle_{L^2L^2}\end{aligned}$$

is bounded from above by

$$\int_0^T \int_{\mathbb{R}^3} \Big(|\partial h| + \frac{|h|}{\langle x\rangle}\Big)|\partial u|\Big(|\partial u| + \frac{|u|}{\langle x\rangle}\Big)\, dxdt \lesssim \delta \log(2+T)\|u\|^2_{LE^1} + \delta\|u\|^2_{L^\infty L^2},$$

where we have utilized (2.3). Now

$$\langle \partial_t u, f'(r)\partial_t u\rangle_{L^2L^2} \approx \int_0^T \int_{\mathbb{R}^3} \frac{\rho}{(r+\rho)^2}|\partial_t u|^2\, dxdt,$$

and so our analysis shows

$$\int_0^T \int_{\mathbb{R}^3} \frac{\rho}{(r+\rho)^2} |\partial_t u|^2 \, dxdt \lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \frac{|u|}{\langle x \rangle} \, dxdt \tag{3.10}$$

$$+ \delta \log(2+T) \|u\|^2_{LE^1} + \big| \langle g^{ij} D_j u, D_i f'(r) u \rangle_{L^2 L^2} \big|$$

$$\lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \frac{|u|}{\langle x \rangle} \, dxdt$$

$$+ \int_0^T \int_{\mathbb{R}^3} \Big(\frac{\rho}{(r+\rho)^2} |\partial_r u|^2 + \frac{1}{r+\rho} |\not\nabla u|^2 + \frac{\rho}{(\rho+r)^4} u^2 \Big) \, dxdt$$

$$+ |||g||| \|u\|^2_{LE^1} + \delta \log(2+T) \|u\|^2_{LE^1}.$$

Multiplying (3.10) by a small constant and adding it to (3.8), we obtain the bound

$$\int_0^T \int_{\mathbb{R}^3} \Big(\frac{\rho}{(r+\rho)^2} |\partial_r u|^2 + \frac{\rho}{(r+\rho)^2} |\partial_t u|^2 + \frac{1}{r+\rho} |\not\nabla u|^2 + \frac{\rho}{(\rho+r)^4} u^2 \Big) \, dxdt$$

$$\lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big) \, dxdt + |||g||| \|u\|^2_{LE^1}$$

$$+ \delta \log(2+T) \|u\|^2_{LE^1}.$$

By choosing $\rho = 1$ and restricting the range of integration on the left-hand side of the equation, we obtain the bound

$$\int_0^T \int_{|x|\le 1} \Big(|\partial u|^2 + u^2 \Big) \, dxdt \lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big) \, dxdt + |||g||| \|u\|^2_{LE^1} \tag{3.11}$$

$$+ \delta \log(2+T) \|u\|^2_{LE^1},$$

while choosing $\rho = 2^k$ for an integer $k \ge 1$, we find

$$\int_0^T \int_{2^{k-1} \le |x| \le 2^k} \Big(\frac{|\partial u|^2}{\langle x \rangle} + \frac{u^2}{\langle x \rangle^3} \Big) \, dxdt \lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big) \, dxdt + |||g||| \|u\|^2_{LE^1} \tag{3.12}$$

$$+ \delta \log(2+T) \|u\|^2_{LE^1}.$$

Taking the supremum over all such dyadic regions and absorbing the error term $|||g||| \|u\|^2_{LE^1}$ using the smallness of $|||g|||$, we have shown

$$\|u\|^2_{LE^1} \lesssim \|\partial u\|^2_{L^\infty L^2} + \int_0^T \int_{\mathbb{R}^3} |P_h u| \Big(|\partial u| + \frac{|u|}{\langle x \rangle} \Big) \, dxdt + \delta \log(2+T) \|u\|^2_{LE^1},$$

completing the proof of Theorem 3.1. □

We now return to the operator P_h as described in the introduction, where g^{ij} can be a large perturbation of the identity matrix. Since g is still asymptotically Euclidean, we expect a statement similar to Theorem 3.1 to hold for sufficiently large $|x|$. Our next theorem develops such an exterior estimate. This theorem will be useful for our medium and high frequency analysis. A similar estimate can be found in [29]. See also the earlier related work in [24]. We now state the theorem:

Theorem 3.2. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix $T > 0$ and further fix R_{AF} sufficiently large. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in L^1L^2$, with initial data $(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then for any $R > 2R_{AF}$*

$$\|u\|^2_{LE^1_{>R}} \lesssim \|\partial u\|^2_{L^\infty L^2} + R^{-2}\|u\|^2_{LE_R} + \int_0^T \int_{\mathbb{R}^3} |P_h u|\Big(|\partial u| + \frac{|u|}{\langle x \rangle}\Big)\, dxdt + \delta \log(2+T)\|u\|^2_{LE^1}. \tag{3.13}$$

Here the implicit constant is independent of T and δ.

Remark. *Theorem 3.1 essentially holds for weaker assumptions on h. Indeed, if we assume only that $|\partial^{\leq 1} h| < \varepsilon$, for some $\varepsilon > 0$ chosen sufficiently small, then the same proof demonstrates*

$$\begin{aligned}\|u\|^2_{LE^1_{>R}} \lesssim \|\partial u\|^2_{L^\infty L^2} + R^{-2}\|u\|^2_{LE_R} + \int_0^T \int_{\mathbb{R}^3} |P_h u|\Big(|\partial u| + \frac{|u|}{\langle x \rangle}\Big)\, dxdt \\ + \int_0^T \int_{\mathbb{R}^3} \big(|\partial h| + \frac{|h|}{\langle x \rangle}\big)|\partial u|\big(|\partial u| + \frac{|u|}{\langle x \rangle}\big)\, dxdt.\end{aligned}$$

Proof. The proof follows the proof of Theorem 3.1 and the work of [29]. It suffices to replace the self-adjoint operator Q with the following operator

$$Q = \big(1 - \chi_{R/2}(|x|)\big) f(r) \frac{x_l}{r} g^{lm} D_m + D_m \big(1 - \chi_{R/2}(|x|)\big) f(r) \frac{x_l}{r} g^{lm},$$

where $\chi_{R/2}(|x|)$ is as in (1.10). For our Lagrangian correction, we modify the multiplier used in (3.9) from $-f'(r)$ to $-\big(1 - \chi_{R/2}(|x|)\big) f'(r)$. We have already proved a large portion of this result. Indeed, (3.2) still holds and $2\mathrm{Im}\, \langle P_h u, Qu \rangle_{L^2L^2} \lesssim \int_0^T \int_{\mathbb{R}^3} |P_h u|\big(|\partial u| + \frac{|u|}{\langle x \rangle}\big)\, dxdt$. Again, the time boundary terms arising on the right-hand side of (3.2) are controlled by $\|\partial u\|^2_{L^\infty L^2}$. The remaining h-dependent terms on the right-hand side of (3.2):

$$\begin{aligned}
\mathrm{Re}\,&\langle \partial_i h^{ij} D_j u, Qu\rangle_{L^2L^2} - 2\langle \partial_t h^{0j} D_j u, Qu\rangle_{L^2L^2} + 2\langle \partial_j h^{0j} Qu, D_t u\rangle_{L^2L^2} \\
&+ \frac{1}{i}\langle h^{ij} D_j u, [D_i, Q]u\rangle_{L^2L^2} + \frac{1}{i}\langle [Q, h^{ij} D_i]u, D_j u\rangle_{L^2L^2} + \frac{2}{i}\langle [Q, h^{0j} D_j]u, D_t u\rangle_{L^2L^2} \\
&+ \frac{1}{i}\langle [Q, h^{00}] D_t u, D_t u\rangle_{L^2L^2} + 2\mathrm{Im}\,\langle \partial_t h^{00} \partial_t u, Qu\rangle_{L^2L^2}
\end{aligned}$$

are controlled by

$$\int_0^T \int_{\mathbb{R}^3} \Big(|\partial h| + \frac{|h|}{\langle x\rangle}\Big)|\partial u|\Big(|\partial u| + \frac{|u|}{\langle x\rangle}\Big)\,dxdt \lesssim \delta \log(2+T)\|u\|_{LE^1}^2 + \delta\|\partial u\|_{L^\infty L^2}^2, \tag{3.14}$$

as before. Further, investigating terms with h dependence in our Lagrangian correction in (3.9):

$$\begin{aligned}
&- 2\mathrm{Re}\,\langle \partial_i h^{ij} \partial_j u, \big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2} - 2\mathrm{Re}\,\langle h^{ij} D_j u, D_i\big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2} \\
&- 4\mathrm{Re}\,\langle \partial_j h^{0j} \partial_t u, \big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2} - 4\mathrm{Re}\,\langle h^{0j} D_t u, D_j\big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2} \\
&+ 2\mathrm{Re}\,\langle h^{00} \partial_t u, \big(1-\chi_{R/2}(|x|)\big) f'(r) \partial_t u\rangle_{L^2L^2} + 2\mathrm{Re}\,\langle \partial_t h^{00} \partial_t u, \big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2} \\
&- 2\mathrm{Re}\,\langle h^{00} \partial_t u, \big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2}\Big|_0^T,
\end{aligned}$$

we see that these are again controlled by $\|\partial u\|_{L^\infty L^2}^2 + \delta\log(2+T)\|u\|_{LE^1}^2$, as desired.

So to finish the proof, it suffices to bound the portions of the following terms that contain derivatives of $\chi_{R/2}(|x|)$:

$$\langle i[D_j g^{jk} D_k, Q]u, u\rangle_{L^2L^2} - \tilde{c}\langle g^{ij} D_j u, D_i\big(1-\chi_{R/2}(|x|)\big) f'(r) u\rangle_{L^2L^2}. \tag{3.15}$$

Here $\tilde{c}$ is the small constant that (3.10) was multiplied by in the proof of Theorem 3.1. We investigate $[D_j g^{jk} D_k, Q]$ and observe

(3.16)

$$\begin{aligned}
[D_j g^{jk} D_k, Q] = {} & [D_j g^{jk} D_k, \big(1-\chi_{R/2}(|x|)\big)] f(r)\frac{x_l}{r} g^{lm} D_m + D_m f(r)\frac{x_l}{r} g^{lm} [D_j g^{jk} D_k, \big(1-\chi_{R/2}(|x|)\big)] \\
&+ [[D_j g^{jk} D_k, D_m f(r)\frac{x_l}{r} g^{lm}], \big(1-\chi_{R/2}(|x|)\big)] \\
&+ \big(1-\chi_{R/2}(|x|)\big)[D_j g^{jk} D_k, f(r)\frac{x_l}{r} g^{lm} D_m + D_m f(r)\frac{x_l}{r} g^{lm}].
\end{aligned}$$

Derivatives can land on $\chi_{R/2}(|x|)$ directly from the terms:

(3.17) $$[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)]f(r)\frac{x_l}{r}g^{lm}D_m+D_mf(r)\frac{x_l}{r}g^{lm}[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)] + [[D_jg^{jk}D_k,D_mf(r)\frac{x_l}{r}g^{lm}],\big(1-\chi_{R/2}(|x|)\big)]$$

or from the second order terms in

(3.18) $$\big(1-\chi_{R/2}(|x|)\big)[D_jg^{jk}D_k,f(r)\frac{x_l}{r}g^{lm}D_m+D_mf(r)\frac{x_l}{r}g^{lm}]$$

when we consider

(3.19) $$\int_0^T\int_{\mathbb{R}^3} i\Big(\big(1-\chi_{R/2}(|x|)\big)[D_jg^{jk}D_k,f(r)\frac{x_l}{r}g^{lm}D_m+D_mf(r)\frac{x_l}{r}g^{lm}]u\Big)\overline{u}\,dxdt,$$

and integrate by parts, as in the proof of Theorem 3.1.

We first bound the terms in (3.17):

$$\begin{aligned}[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)] &= -\frac{1}{i}\frac{4}{R}D_jg^{jk}\frac{x_k}{r}\chi'(2|x|/R)-\frac{2}{R}\partial_j\big(g^{jk}\chi'(2|x|/R)\frac{x_k}{r}\big)\\ &= -\frac{1}{i}\frac{4}{R}g^{jk}\chi'(2|x|/R)\frac{x_j}{r}D_k+\frac{2}{R}\partial_j\big(g^{jk}\chi'(2|x|/R)\frac{x_k}{r}\big),\end{aligned}$$

which shows

$$\begin{aligned}&[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)]f(r)\frac{x_l}{r}g^{lm}D_m+D_mf(r)\frac{x_l}{r}g^{lm}[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)]\\ &\quad= -\frac{1}{i}\frac{8}{R}D_jg^{jk}\frac{x_k}{r}\chi'(2|x|/R)f(r)\frac{x_l}{r}g^{lm}D_m+\frac{1}{i}\frac{2}{R}\partial_m\big(f(r)\frac{x_l}{r}g^{lm}\partial_j\big(g^{jk}\chi'(2|x|/R)\frac{x_k}{r}\big)\big).\end{aligned}$$

The above line yields the following bound from below:

(3.20) $$\begin{aligned}&\langle i[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)]f(r)\frac{x_l}{r}g^{lm}D_m+D_mf(r)\frac{x_l}{r}g^{lm}[D_jg^{jk}D_k,\big(1-\chi_{R/2}(|x|)\big)]u,u\rangle_{L^2L^2}\\ &\quad\gtrsim\int_0^T\int_{\mathbb{R}^3}-R^{-1}f(r)\chi'(2|x|/R)(g^{jk}\frac{x_k}{r}\partial_ju)^2\,dxdt-R^{-2}\|u\|_{LE_R}.\end{aligned}$$

Note that the first term on the right-hand side of (3.20) is nonnegative since $\chi_{R/2}(|x|)$ is a monotonically decreasing function of $|x|$.

We claim

$$\langle i[[D_jg^{jk}D_k,D_mf(r)\frac{x_l}{r}g^{lm}],\big(1-\chi_{R/2}(|x|)\big)]u,u\rangle_{L^2L^2}$$

is also bounded by $R^{-2}\|u\|_{LE_R}$. Indeed, the commutator contains first order and zeroth order terms. The zeroth order terms are controlled by $R^{-2}\|u\|_{LE_R}$. The first order terms are controlled by this quantity as well, which we see by writing $\partial_x u u = \frac{1}{2}\partial_x(u^2)$ and integrating by parts. Indeed, $[D_j g^{jk} D_k, D_m f(r)\frac{x_l}{r} g^{lm}]$ is an operator of the following form $i\omega_{-1}(x)D_x^2 + \omega_{-2}(x)D_x + i\alpha(x)$, where $\omega_k(x)$ are real-valued functions of order $O(\langle x\rangle^{-k})$ and $\alpha(x)$ is a bounded real-valued function. Hence

$$[[D_j g^{jk} D_k, D_m f(r)\frac{x_l}{r} g^{lm}], (1-\chi_{R/2}(|x|))] = \tilde{\omega}_{-2}(x)\chi'(2|x|/R)D_x + i\tilde{\omega}_{-3}(x)\chi''(2|x|/R),$$

where $\tilde{\omega_k}$ are real-valued functions of order $O(\langle x\rangle^{-k})$. Therefore

$$\begin{aligned}\langle i[[D_j g^{jk} D_k, D_m f(r)\frac{x_l}{r} g^{lm}], \big(1-\chi_{R/2}(|x|)\big)]u, u\rangle_{L^2L^2} &= \langle -\tilde{\omega}_{-3}(x)\chi''(2|x|/R)u, u\rangle_{L^2L^2} \\ &\quad + \langle \tilde{\omega}_{-2}(x)\chi'(2|x|/R)\partial_x u, u\rangle_{L^2L^2}.\end{aligned}$$

We immediately see $\langle -\tilde{\omega}_{-3}(x)\chi''(2|x|/R)u, u\rangle_{L^2L^2}$ is controlled by $R^{-2}\|u\|_{LE_R}$ as claimed. For the remaining part, utilizing the chain rule and integrating by parts, we find

$$\begin{aligned}\langle \tilde{\omega}_{-2}(x)\chi'(2|x|/R)\partial_x u, u\rangle_{L^2L^2} &= \int_0^T \int_{\mathbb{R}^3} \frac{1}{2}\tilde{\omega}_{-2}(x)\chi'(2|x|/R)\partial_x(u^2)\, dxdt \\ &= -\int_0^T \int_{\mathbb{R}^3} \frac{1}{2}\partial_x(\tilde{\omega}_{-2}(x)\chi'(2|x|/R))u^2\, dxdt \\ &\lesssim R^{-2}\|u\|_{LE_R}^2,\end{aligned} \tag{3.21}$$

as desired.

We must still bound the terms with $\chi'(2|x|/R)$ arising from integrating the second order terms by parts in (3.19). We begin by observing that (3.19) can be written as

$$\begin{aligned}&2\int_0^T \int_{\mathbb{R}^3} i\Big(\big(1-\chi_{R/2}(|x|)\big)D_j[g^{jk}D_k, f(r)\frac{x_l}{r}g^{lm}D_m]u\Big)\overline{u}\, dxdt \\ &\qquad + 2\int_0^T \int_{\mathbb{R}^3} i\Big(\big(1-\chi_{R/2}(|x|)\big)D_k g^{jk}[D_j, f(r)\frac{x_l}{r}g^{lm}D_m]u\Big)\overline{u}\, dxdt \\ &\qquad - 2\int_0^T \int_{\mathbb{R}^3} \Big(\big(1-\chi_{R/2}(|x|)\big)\partial_k\big(g^{jk}[D_j, f(r)\frac{x_l}{r}g^{lm}D_m]\big)u\Big)\overline{u}\, dxdt \\ &\qquad\qquad + \int_0^T \int_{\mathbb{R}^3} \Big(\big(1-\chi_{R/2}(|x|)\big)[D_j g^{jk} D_k, \partial_m(f(r)\frac{x_l}{r}g^{lm})]u\Big)\overline{u}\, dxdt.\end{aligned}$$

Note that second order terms are only present in the first two integrals and can be expressed as

$$2\int_0^T\int_{\mathbb{R}^3}\Big(\big(1-\chi_{R/2}(|x|)\big)D_jg^{jk}\partial_k\big(f(r)\frac{x_l}{r}g^{lm}\big)D_mu\Big)\overline{u}\,dxdt$$
$$-2\int_0^T\int_{\mathbb{R}^3}\Big(\big(1-\chi_{R/2}(|x|)\big)D_jg^{lm}\partial_mg^{jk}f(r)\frac{x_l}{r}D_ku\Big)\overline{u}\,dxdt$$
$$+2\int_0^T\int_{\mathbb{R}^3}\Big(\big(1-\chi_{R/2}(|x|)\big)D_kg^{jk}\partial_j\big(f(r)\frac{x_l}{r}g^{lm}\big)D_mu\Big)\overline{u}\,dxdt.$$

Integrating by parts, we obtain the following error terms involving derivatives of our spatial cut-off function:

$$-4R^{-1}\int_0^T\int_{\mathbb{R}^3}\chi'(2|x|/R)\frac{x_j}{r}g^{jk}\partial_k\big(f(r)\frac{x_l}{r}g^{lm}\big)\partial_muu\,dxdt$$
$$+4R^{-1}\int_0^T\int_{\mathbb{R}^3}\chi'(2|x|/R)\frac{x_j}{r}g^{lm}\partial_mg^{jk}f(r)\frac{x_l}{r}\partial_kuu\,dxdt$$
$$-4R^{-1}\int_0^T\int_{\mathbb{R}^3}\chi'(2|x|/R)\frac{x_k}{r}g^{jk}\partial_j\big(f(r)\frac{x_l}{r}g^{lm}\big)\partial_muu\,dxdt. \tag{3.22}$$

These can now be controlled by using the method from (3.21). Indeed, utilizing the chain rule and integrating by parts, (3.22) becomes:

$$2R^{-1}\int_0^T\int_{\mathbb{R}^3}\partial_m\Big(\chi'(2|x|/R)\frac{x_j}{r}g^{jk}\partial_k\big(f(r)\frac{x_l}{r}g^{lm}\big)\Big)u^2\,dxdt$$
$$-2R^{-1}\int_0^T\int_{\mathbb{R}^3}\partial_k\Big(\chi'(2|x|/R)\frac{x_j}{r}g^{lm}\partial_mg^{jk}f(r)\frac{x_l}{r}\Big)u^2\,dxdt$$
$$+2R^{-1}\int_0^T\int_{\mathbb{R}^3}\partial_m\Big(\chi'(2|x|/R)\frac{x_k}{r}g^{jk}\partial_j\big(f(r)\frac{x_l}{r}g^{lm}\big)\Big)u^2\,dxdt,$$

which is bounded by $R^{-2}\|u\|_{LE_R}$, as desired.

Finally, we investigate the remaining term in (3.15). The term with a $\chi'_{R/2}$ from $\langle g^{ij}D_ju, D_i\big(1-\chi_{R/2}(|x|)\big)f'(r)u\rangle_{L^2L^2}$ is bounded via the method from (3.21). This completes the proof. ☐

We now develop a second exterior estimate which will be useful for $|x|$ sufficiently large, where P_g is a small, asymptotically Euclidean perturbation of $\Box$. This estimate will be applied in our low frequency analysis section. Again, we follow the prior work of [29].

Theorem 3.3. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta>0$ sufficiently small. Further $D_ig^{ij}D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix $T>0$ and further fix R_{AF} sufficiently large. Suppose $u(t,x)\in C^2([0,T]\times\mathbb{R}^3)$ solves $P_hu=F\in LE^*$, with initial data $(u,\partial_tu)(0,\cdot)=(f_1,f_2)\in C^2(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$ such that $|\partial^{\le 2}u(t,x)|\to 0$ as $|x|\to\infty$ for all $t\in[0,T]$. Then for any $R>2R_{AF}$*

$$\|u\|^2_{LE^1_{>R}}\lesssim\|\partial u\|^2_{L^\infty L^2}+\|P_hu\|^2_{LE^*_{>R}}+\|\partial u\|^2_{LE_R}+\delta\log(2+T)\|u\|^2_{LE^1}. \tag{3.23}$$

Here the implicit constant is independent of T and δ.

Proof. Instead of working with the function u, we will consider

$$u_{ext}(t,r,\omega) = (1-\chi_R)(|x|)u(t,r,\omega) + \chi_R(|x|)\bar{u}_R(t), \tag{3.24}$$

where $\chi_R(|x|)$ is as in (1.10) and $\bar{u}_R(t)$ is a local space-time average of u adapted to the annulus $R \le |x| \le 2R$. Restricting our analysis to $R > 2R_{AF}$ allows us to assume that P_g is a small asymptotically Euclidean perturbation of $\Box$ globally. We define $\bar{u}_R(t)$:

$$\bar{u}_R(t) = R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty u(s,r,\omega)\gamma\left(\frac{t-s}{R},\frac{r}{R}\right)r^2drd\sigma(\omega)ds, \tag{3.25}$$

where $\gamma(t,x)$ is a smooth, nonnegative bump function, with supp$\gamma \subset [-1,1]\times[1,2]$ such that

$$\int_{\mathbb{R}}\int_0^\infty \gamma(t,r)\, r^2drdt = 1.$$

It will be necessary to bound time derivatives of $\bar{u}_R$ in L^2. Taking $j \ge 1$ time derivatives of (3.25), integrating by parts, and applying the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
\partial_t^j\bar{u}_R(t) =& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty u(s,r,\omega)\partial_t^j\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right)r^2drd\sigma(\omega)dt\\
=& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty u(s,r,\omega)\partial_t^{j-1}(-\partial_s)\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right)r^2drd\sigma(\omega)dt\\
=& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty \partial_s u(s,r,\omega)\partial_t^{j-1}\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right)r^2drd\sigma(\omega)dt\\
\le& R^{-4}\int \|\partial u\|_{L^2(R\le r\le 2R)}\|\partial_t^{j-1}\gamma(\frac{t-s}{R},\frac{\cdot}{R})\|_{L^2(R\le r\le 2R)}ds\\
\lesssim& R^{-4}\int R^{1/2}\|\langle x\rangle^{-1/2}\partial u\|_{L^2(R\le r\le 2R)}R^{3/2}\|\partial_t^{j-1}\gamma(\frac{t-s}{R},\cdot)\|_{L^2(1\le r\le 2)}ds.
\end{aligned} \tag{3.26}$$

An application of Young's convolution inequality yields the desired bound:

$$\begin{aligned}
\|\partial_t^j\bar{u}_R\|_{L^2} \lesssim& R^{-2}\|\langle x\rangle^{-1/2}\partial u\|_{L^2L^2(R\le r\le 2R)}\|\partial_t^{j-1}\gamma(\frac{\cdot}{R},\cdot)\|_{L^1L^2(1\le r\le 2)}\\
=& R^{-2}\|\langle x\rangle^{-1/2}\partial u\|_{L^2L^2(R\le r\le 2R)}R^{2-j}\|\partial_t^{j-1}\gamma(\cdot,\cdot)\|_{L^1L^2(1\le r\le 2)}\\
\le& R^{-j}\|\partial u\|_{LE_R}.
\end{aligned} \tag{3.27}$$

A similar argument shows

$$\|\bar{u}_R\|_{L^2} \lesssim R^{-1}\|u\|_{LE_R}. \tag{3.28}$$

Indeed,

$$\begin{aligned}\bar{u}_R(t) =& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty u(s,r,\omega)\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right) r^2 dr d\sigma(\omega) dt\\ \leq& R^{-4}\int \|u\|_{L^2(R\leq r\leq 2R)}\|\gamma(\frac{t-s}{R},\frac{\cdot}{R})\|_{L^2(R\leq r\leq 2R)}ds\\ \lesssim& R^{-4}\int R^{1/2}\|\langle x\rangle^{-1/2}u\|_{L^2(R\leq r\leq 2R)}R^{3/2}\|\gamma(\frac{t-s}{R},\cdot)\|_{L^2(1\leq r\leq 2)}ds.\end{aligned}$$

Again, an application of Young's convolution inequality yields the desired bound:

$$\begin{aligned}\|\bar{u}_R\|_{L^2} \lesssim& R^{-2}\|\langle x\rangle^{-1/2}u\|_{L^2L^2(R\leq r\leq 2R)}\|\gamma(\frac{\cdot}{R},\cdot)\|_{L^1L^2(1\leq r\leq 2)}\\ =& R^{-2}\|\langle x\rangle^{-1/2}u\|_{L^2L^2(R\leq r\leq 2R)}R\|\gamma(\cdot,\cdot)\|_{L^1L^2(1\leq r\leq 2)}\\ \leq& R^{-1}\|u\|_{LE_R}.\end{aligned}$$

We also note a Poincaré-type inequality (see, for example [29], [9]):

$$\|\langle x\rangle^{-1}(\bar{u}_R - u)\|_{LE_R} \lesssim \|\partial u\|_{LE_R}. \tag{3.29}$$

Applying Theorem 3.1 to u_{ext}, we see

$$\|u_{ext}\|^2_{LE^1} \lesssim \|\partial u_{ext}\|^2_{L^\infty L^2} + \|P_h u_{ext}\|^2_{LE^*} + \delta\log(2+T)\|u_{ext}\|^2_{LE^1}. \tag{3.30}$$

Using the support of our cut-off functions, we find

$$\|u_{ext}\|_{LE^1} \approx \|(1-\chi_R)(|x|)u\|_{LE^1} + \|\bar{u}_R\|_{L^2} + R\|\partial_t\bar{u}_R\|_{L^2}. \tag{3.31}$$

Perhaps this line deserves some explanation. The triangle inequality yields the upper bound.

To obtain the lower bound, we integrate over a dyadic radius of size approximately R that is outside of the support of $(1-\chi_R)(|x|)$ to obtain

$$\|\bar{u}_R\|_{L^2} + R\|\partial_t\bar{u}_R\|_{L^2} \approx R^{-3/2}\|\bar{u}_R\|_{L^2L^2_{R/2}} + R^{-1/2}\|\partial_t\bar{u}_R\|_{L^2L^2_{R/2}} \lesssim \|u_{ext}\|_{LE^1}.$$

The above line combined with the triangle inequality yields the desired lower bound, proving (3.31).

We claim

$$\|\partial u_{ext}\|_{L^\infty L^2} \lesssim \|\partial u\|_{L^\infty L^2}. \tag{3.32}$$

To begin, observe:

$$|\partial u_{ext}| \leq |(1-\chi_R)(|x|)\partial u| + |\chi'(|x|/R)R^{-1}u| + |\chi_R(|x|)\partial_t \bar{u}_R| + |\chi'(|x|/R)R^{-1}\bar{u}_R|.$$

We bound each term separately. The bound on the first term is immediate. The bound for the second term follows after the use of a Hardy inequality (2.4). For the third term, we use the work in (3.26), to observe

$$\begin{aligned}\partial_t \bar{u}_R(t) =& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty \partial_s u(s,r,\omega)\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right) r^2 dr d\sigma(\omega) dt \\ \lesssim& R^{-5/2}\int \|\partial u\|_{L^2}\|\gamma(\frac{t-s}{R},\cdot)\|_{L^2(1\leq r\leq 2)} ds.\end{aligned}$$

Applying Young's Convolution inequality, we find

$$\|\partial_t \bar{u}_R(t)\|_{L^\infty} \lesssim R^{-3/2}\|\partial u\|_{L^\infty L^2}\|\gamma(\cdot,\cdot)\|_{L^1 L^2(1\leq r\leq 2)} \lesssim R^{-3/2}\|\partial u\|_{L^\infty L^2}.$$

Hence,

$$\|\chi_R(|x|)\partial_t \bar{u}_R\|_{L^\infty L^2} \lesssim R^{-3/2}\|\partial u\|_{L^\infty L^2}\|\chi_R(|x|)\|_{L^2} \lesssim \|\partial u\|_{L^\infty L^2},$$

as desired.

The bound for the remaining term $\chi'(|x|/R)R^{-1}\bar{u}$ follows from a similar argument. Indeed, a variant of the work in (3.26) and a Hardy inequality (2.4) show

$$\begin{aligned}\bar{u}_R(t) =& R^{-4}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\int_0^\infty u(s,r,\omega)\gamma\left(\frac{t-s}{R},\frac{|x|}{R}\right) r^2 dr d\sigma(\omega) dt \\ \lesssim& R^{-3/2}\int \|\langle x\rangle^{-1}u\|_{L^2}\|\gamma(\frac{t-s}{R},\cdot)\|_{L^2(1\leq r\leq 2)} ds \\ \lesssim& R^{-3/2}\int \|\partial u\|_{L^2}\|\gamma(\frac{t-s}{R},\cdot)\|_{L^2(1\leq r\leq 2)} ds.\end{aligned}$$

As in the prior case, an application of Young's convolution inequality gives the bound, proving (3.32).

We now calculate $P_h u_{ext}$:

$$P_h u_{ext} = (1-\chi_R)(|x|)P_h u + \chi_R(|x|)P_h \bar{u}_R - [P_h, \chi_R(|x|)](u-\bar{u}_R). \tag{3.33}$$

The commutator term is first order:

$$\begin{aligned}[P_h, \chi_R(|x|)](u-\bar{u}_R) \lesssim R^{-1}|\chi'(|x|/R)||\partial u| + R^{-1}|\chi'(|x|/R)||\partial_t \bar{u}_R| \\ + R^{-2}\big(|\chi''(|x|/R)| + |\chi'(|x|/R)|\big)|u-\bar{u}_R|.\end{aligned}$$

This is supported where $r \approx R$ and so we can bound it in the LE^* norm via:

$$\begin{aligned}\|[P_h, \chi_R(|x|)](u-\bar{u}_R)\|_{LE^*} \lesssim & \|R^{-1/2}\partial u\|_{L^2L^2_R} + \|R^{-3/2}(u-\bar{u}_R)\|_{L^2L^2_R} \\ & + \|R^{-1/2}\partial_t \bar{u}_R\|_{L^2L^2_R} \\ \lesssim & \|\partial u\|_{LE_R} + \|\langle x\rangle^{-1}(u-\bar{u}_R)\|_{LE_R} \lesssim \|\partial u\|_{LE_R},\end{aligned} \tag{3.34}$$

where we have applied (3.27) and (3.29) in the last line. Applying (3.27) yields the desired bound for $\|\chi_R(|x|)P_h\bar{u}_R\|_{LE^*}$:

$$\|\chi_R(|x|)P_h\bar{u}_R\|_{LE^*} \lesssim R^2\|\partial_t^2\bar{u}_R\|_{L^2} \lesssim \|\partial u\|_{LE_R}. \tag{3.35}$$

Therefore, we have shown

$$\|P_h u_{ext}\|_{LE^*} \lesssim \|P_h u\|_{LE^*_{>R}} + \|\partial u\|_{LE_R}. \tag{3.36}$$

Lastly, we observe

$$\|u_{ext}\|_{LE^1} \leq \|u\|_{LE^1} + \|\chi_R(|x|)\bar{u}_R\|_{LE^1} \lesssim \|u\|_{LE^1}, \tag{3.37}$$

where we have applied (3.27) and (3.28) in the above line. Theorem 3.3 now follows from combining (3.30), (3.31), (3.32), (3.36), and (3.37). □

Chapter 4: High Frequency Analysis

4.1 High Frequency Background Estimate

We begin by stating the main theorem of this section, which is a local energy estimate for our background operator P_g with an error term that can be absorbed for high time frequencies.

Theorem 4.1. *Let P_g be as in* (1.3)*, where $D_i g^{ij} D_j$ is smooth, symmetric, non-trapping, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix R_{AF} sufficiently large. Further, fix $R_1 > 2R_{AF}$ and $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_g u = F \in LE^* + L^1L^2$, with initial data $(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then*

$$\|u\|_{LE^1} \lesssim \|\partial u\|_{L^\infty L^2} + \|u\|_{L^2 L^2_{<2R_1}} + \|P_g u\|_{LE^* + L^1 L^2}. \tag{4.1}$$

Remark. *The implicit constant in* (4.1) *can depend on R_1.*

We devote the remainder of this section to proving Theorem 4.1 and will then consider the perturbation separately in the next section.

Now, via the triangle inequality

$$\|u\|_{LE^1} \leq \|\chi_{R_1}(|x|)u\|_{LE^1} + \|(1 - \chi_{R_1})(|x|)u\|_{LE^1}.$$

Here $\chi_{R_1}(|x|)$ is as in (1.10). We choose $R_1 > 2R_{AF}$ so that we can bound $\|(1 - \chi_{R_1})(|x|)u\|_{LE^1}$ via our exterior estimate Theorem 3.2. Hence, for the remainder of this section, we will focus on developing a good bound for $\|\chi_{R_1}(|x|)u\|_{LE^1}$. We will prove the following:

$$\begin{aligned}\|\chi_{R_1}(|x|)u\|^2_{LE^1} \lesssim \|u\|^2_{L^2L^2_{<2R_1}} + \|\partial u\|^2_{L^\infty L^2} + \|[P_g, \chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1} \\ + \|P_g u\|_{LE^*+L^1L^2}(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}).\end{aligned} \tag{4.2}$$

Combining (4.2) with Theorem 3.2 proves Theorem 4.1. Observe that we are applying Cauchy's inequality and Theorem 3.2 on the error term $\|[P_g, \chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1}$. We now focus on proving (4.2), which essentially follows from the following two lemmas.

Lemma 4.1. *Under the same assumptions of Theorem 4.1, there exists a smooth, real valued symbol* $q \in S^1$ *such that*

$$\{g^{ij}\xi_i\xi_j, q\} \gtrsim \beta_{\geq\lambda}(|\xi|)\chi_{R_2}(|x|)|\xi|^2 + r(x,\xi), \tag{4.3}$$

where $R_2 \gg R_1$ *and* $\beta_{\geq\lambda}(|\xi|) = 1 - \chi_{\lambda/2}(|\xi|)$ *. Here* $\lambda \gg 1$ *to be chosen later and* $r(x,\xi) \in S^0$*.*

Lemma 4.2. *Let* q *be as in Lemma 4.1 and* P_g *as in Theorem 4.1. Then*

$$\begin{aligned} 2\mathrm{Im}\int_0^T \int_{\mathbb{R}^3} & P_g\chi_{R_1}(|x|)u\overline{q^w\chi_{R_1}(|x|)u}\,dxdt \\ & \lesssim \|P_g u\|_{LE^*+L^1L^2}(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}) + \|[P_g, \chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1}. \end{aligned}$$

We will hold off on the proofs of Lemmas 4.1 and 4.2 momentarily and instead show how they imply (4.2).

Proof of Theorem 4.1. To simplify notation, set $v = \chi_{R_1}(|x|)u$. Observe

$$2\mathrm{Im}\,\langle P_g v, q^w v\rangle_{L^2L^2} = i\langle \partial_t v, q^w v\rangle_{L^2}\Big|_0^T - i\langle q^w v, \partial_t v\rangle_{L^2}\Big|_0^T + i\langle [D_i g^{ij} D_j, q^w]v, v\rangle_{L^2L^2}. \tag{4.4}$$

The time boundary terms are controlled by $\|\partial u\|^2_{L^\infty L^2}$. Indeed by applying the Cauchy-Schwarz inequality and noting $q^w \in OPS^1$ so that it is a bounded linear operator from $H^1 \to L^2$ (see Appendix A, Theorem A.6), we find

$$\begin{aligned} \left| i\langle \partial_t v, q^w v\rangle_{L^2}\Big|_0^T - i\langle q^w v, \partial_t v\rangle_{L^2}\Big|_0^T \right| &\lesssim \|\partial_t v\|^2_{L^2}\Big|_{t=0} + \|\partial_t v\|^2_{L^2}\Big|_{t=T} + \|q^w v\|^2_{L^2}\Big|_{t=0} + \|q^w v\|^2_{L^2}\Big|_{t=T} \\ &\lesssim \|\partial_t v\|^2_{L^2}\Big|_{t=0} + \|\partial_t v\|^2_{L^2}\Big|_{t=T} + \|v\|^2_{H^1}\Big|_{t=0} + \|v\|^2_{H^1}\Big|_{t=T} \\ &\lesssim \|\partial v\|^2_{L^2}\Big|_{t=0} + \|\partial v\|^2_{L^2}\Big|_{t=T} \lesssim \|\partial u\|^2_{L^2}\Big|_{t=0} + \|\partial u\|^2_{L^2}\Big|_{t=T} \\ &\lesssim \|\partial u\|^2_{L^\infty L^2}, \end{aligned}$$

where we used the compact support of v to bound all lower order terms via the Hardy inequality (2.4).

Now from Theorem A.7 in Appendix A, we have

$$i\langle [D_i g^{ij} D_j, q^w]v, v\rangle_{L^2L^2} = \langle \{g^{ij}\xi_i\xi_j, q\}^w v, v\rangle_{L^2L^2} + \langle A^0 v, v\rangle_{L^2L^2},$$

where $A^0 \in OPS^0$. Applying the Cauchy-Schwarz inequality, utilizing that A^0 is a bounded linear operator from $L^2 \to L^2$ (see Appendix A, Theorem A.6), and using the compact support of v, we can bound the error term

$\langle A^0 v, v\rangle_{L^2L^2}$ appropriately:

$$\langle A^0 v, v\rangle_{L^2L^2} \lesssim \|A^0 v\|_{L^2L^2}\|v\|_{L^2L^2} \lesssim \|v\|^2_{L^2L^2} \lesssim \|u\|^2_{L^2L^2_{<2R_1}}.$$

We now investigate $\langle \{g^{ij}\xi_i\xi_j, q\}^w v, v\rangle_{L^2L^2}$ with the goal of applying Lemma 4.1 to bound this term from below. Consider the new symbol $\{g^{ij}\xi_i\xi_j, q\} + (1-\chi_{R_2}(|x|))|\xi|^2$ and observe

$$\{g^{ij}\xi_i\xi_j, q\} + (1-\chi_{R_2}(|x|))|\xi|^2 \gtrsim \langle \xi\rangle^2, \qquad \text{for } |\xi| \gg 1.$$

Therefore, an application of the Gårding inequality (Theorem A.8) shows

$$\langle \{g^{ij}\xi_i\xi_j, q\}^w v, v\rangle_{L^2L^2} + \langle \big((1-\chi_{R_2}(|x|))|\xi|^2\big)^w v, v\rangle_{L^2L^2} \gtrsim \|v\|^2_{L^2H^1} - \|v\|^2_{L^2L^2}.$$

As $R_2 \gg R_1$, $\langle \big((1-\chi_{R_2}(|x|))|\xi|^2\big)^w v, v\rangle_{L^2L^2} = 0$. Therefore, we have shown the desirable bound:

$$\langle \{g^{ij}\xi_i\xi_j, q\}^w v, v\rangle_{L^2L^2} \gtrsim \|v\|^2_{L^2H^1} - \|v\|^2_{L^2L^2}.$$

Making use of the compact support of v, we have shown

$$\begin{aligned}(4.5)\quad \|\partial_x(\chi_{R_1}(|x|)u)\|^2_{LE} + \|\langle x\rangle^{-1}\chi_{R_1}(|x|)u\|^2_{LE} \lesssim \|u\|^2_{L^2L^2_{<2R_1}} + \|\partial u\|^2_{L^\infty L^2} + \|[P_g, \chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1} \\ + \|P_g u\|_{LE^*+L^1L^2}(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}).\end{aligned}$$

To obtain (4.2), we need to add in time derivative terms. We accomplish this by considering the following integral and integrating by parts:

$$(4.6)\qquad -\varepsilon\int_0^T\int_{\mathbb{R}^3}(P_g v)\overline{v}\,dxdt = \varepsilon\int_0^T\int_{\mathbb{R}^3}|\partial_t v|^2\,dxdt - \varepsilon\int_0^T\int_{\mathbb{R}^3}(g^{ij}D_j v)\overline{D_i v}\,dxdt - \varepsilon\int_{\mathbb{R}^3}\partial_t v v\,dx\Big|_0^T.$$

The above time boundary terms are controlled by $\|\partial u\|^2_{L^\infty L^2}$, where we have used the compact support of v to apply (2.4) on lower order terms. We immediately see

$$\int_0^T\int_{\mathbb{R}^3}(P_g v)\overline{v}\,dxdt \lesssim \|[P_g, \chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1} + \|P_g u\|_{LE^*+L^1L^2}(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}).$$

Hence choosing $\varepsilon > 0$ small enough so that we can absorb the $-\varepsilon \int_0^T \int_{\mathbb{R}^3} g^{ij} D_j v \overline{D_i v}\, dxdt$ term into (4.5), and utilizing the compact support of v to convert L^2L^2 norms to LE norms as needed, we find combining (4.5) and (4.6) implies (4.2), completing the proof of Theorem 4.1. $\square$

We now prove Lemma 4.1.

Proof of Lemma 4.1. Let $p(x,\xi)$ denote the principal symbol for the spatial components of our background operator. That is

$$p(x,\xi) = g^{ij}\xi_i\xi_j. \tag{4.7}$$

To construct our multiplier, we utilize a construction related to prior work in [8], [17], [24], [29], and [48]. We set up the following Hamiltonian flow with respect to $p(x,\xi)$. Let (x^s, ξ^s) solve

$$\begin{aligned} \dot{x}_k^s &= p_{\xi_k} = 2g^{ik}(x^s)\xi_i^s \\ \dot{\xi}_k^s &= -p_{x_k} = -(\partial_k g^{ij})(x^s)\xi_i^s\xi_j^s, \end{aligned} \tag{4.8}$$

with $(x^0, \xi^0) = (x, \xi)$. We note that due to the homogeneity of p in ξ, (4.8) enjoys the following scaling relation, as discussed in [17]:

$$\begin{aligned} x^s(x, t\xi) &= x^{ts}(x,\xi) \\ \xi^s(x, t\xi) &= t\xi^{ts}(x,\xi). \end{aligned} \tag{4.9}$$

We define a multiplier useful for an interior region:

$$q_{in} = -\beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\, ds. \tag{4.10}$$

Here $R_2 \gg R_1$ is chosen sufficiently large and $\beta_{\geq\lambda}(|\xi|)$ is included because of the homogeneity of the symbol in $|\xi|$. The argument of [25] shows that q_{in} is symbolic while an argument rooted in utilizing the scaling in (4.9) ensures that $q_{in} \in S^1$. Indeed, by choosing $t = |\xi|^{-1}$, (4.9) and a u-substitution shows q_{in} is equivalent to

$$q_{in} = -\beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)|\xi|\int_0^\infty |\xi^s(x, \frac{\xi}{|\xi|})|^2\chi_{R_2}(|x^s(x, \frac{\xi}{|\xi|})|)\, ds.$$

Using the non-trapping and asymptotically Euclidean assumptions, we see the above integral is bounded, as in [29]. The methods of [25] now carry through.

We now compute $H_p q_{in}$:

$$\begin{aligned} H_p q_{in} &= -H_p\Big(\beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\,ds\Big) \\ &= -H_p\Big(\beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)\Big)\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\,ds \\ &\qquad\qquad - \beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)\int_0^\infty \frac{d}{ds}\big(|\xi^s|^2\chi_{R_2}(|x^s|)\big)\,ds \\ &= -\{p, \beta_{\geq\lambda}(|\xi|)\chi_{4R_2}(|x|)\}\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\,ds + \beta_{\geq\lambda}(|\xi|)\chi_{R_2}(|x|)|\xi|^2, \end{aligned}$$

where we have used the Fundamental Theorem of Calculus. Computing the above Poisson bracket, we find

$$\begin{aligned} H_p q_{in} = \beta_{\geq\lambda}(|\xi|)\chi_{R_2}(|x|)|\xi|^2 - 2g^{ik}\xi_i\beta_{\geq\lambda}(|\xi|)\frac{1}{4R_2}\frac{x_k}{|x|}\chi'(|x|/4R_2)\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\,ds \\ + \partial_k g^{ij}\xi_i\xi_j\beta'(|\xi|/\lambda)\frac{\xi_k}{\lambda|\xi|}\chi_{4R_2}(|x|)\int_0^\infty |\xi^s|^2\chi_{R_2}(|x^s|)\,ds. \end{aligned} \tag{4.11}$$

We note that due to the support of $\beta'(|\xi|/\lambda)$, the third term on the right-hand side of the above equation is in the symbol class S^0. That is

$$r(x,\xi) \equiv \partial_k g^{ij}\xi_i\xi_j\beta'(|\xi|/\lambda)\frac{\xi_k}{\lambda|\xi|}\chi_{4R_2}(|x|)\int_0^\infty |\xi^s|^2\chi_{2R_2}(|x^s|)\,ds \in S^0. \tag{4.12}$$

Motivated by the multiplier used in [24] and [29], we also consider an exterior multiplier:

$$q_{out} = (1-\chi_{R_2/2})(|x|)f(|x|)\frac{x_i}{|x|}g^{ij}\xi_j,$$

where

$$f(|x|) = \frac{|x|}{|x|+R_2}. \tag{4.13}$$

We need to compute $H_p q_{out}$. Taking the Poisson bracket $\{p, q_{out}\}$, one finds:

$$H_pq_{out} = -4\frac{x_m}{|x|}g^{mj}\xi_j\frac{f(|x|)}{R_2}\chi'(2|x|/R_2)\frac{x_k}{|x|}g^{kl}\xi_l + 2\frac{x_m}{|x|}g^{mj}\xi_j(1-\chi_{R_2/2})(|x|)f'(|x|)\frac{x_k}{|x|}g^{kl}\xi_l$$
$$+ 2g^{mj}\xi_j(\delta_{im} - \frac{x_i}{|x|}\frac{x_m}{|x|})(1-\chi_{R_2/2})(|x|)\frac{f(|x|)}{|x|}(\delta_{ik} - \frac{x_i}{|x|}\frac{x_k}{|x|})g^{kl}\xi_l$$
$$+ 2g^{mj}\partial_m(g^{kl})\frac{x_k}{|x|}\xi_j\xi_l(1-\chi_{R_2/2})(|x|)f(|x|) - \partial_m(g^{ij})g^{km}\frac{x_k}{|x|}\xi_i\xi_j(1-\chi_{R_2/2})(|x|)f(|x|). \tag{4.14}$$

We note that H_pq_{out} is nonnegative everywhere and strictly positive for $|x| > R_2$, as we are in the regime where $|||g|||$ is small and the first term on the right-hand side is beneficially signed.

We now combine symbols to define

$$q(x,\xi) = \tilde{\delta}q_{in}(x,\xi) + q_{out}(x,\xi), \tag{4.15}$$

where $\tilde{\delta}$ is a small positive constant such that

$$H_pq \gtrsim \beta_{\geq\lambda}(|\xi|)\chi_{R_2}(|x|)|\xi|^2 + r(x,\xi). \tag{4.16}$$

Indeed, $\tilde{\delta}$ is chosen small enough so that the error term

$$-2g^{ik}\xi_i\beta_{\geq\lambda}(|\xi|)\frac{1}{4R_2}\frac{x_k}{|x|}\chi'(|x|/4R_2)\int_0^\infty |\xi^s|^2\chi_{2R_2}(|x^s|)\,ds$$

in (4.11) can be absorbed into (4.14). This completes the proof of the lemma. □

All that remains to show is Lemma 4.2. We do so by first proving a boundedness lemma for q^w.

Lemma 4.3. *Let q be as in Lemma 4.1. Then,*

$$\|q^w\chi_{R_1}(|x|)u\|^2_{L^2L^2} \lesssim \|u\|^2_{LE^1} \tag{4.17}$$

and

$$\|q^w\chi_{R_1}(|x|)u\|^2_{L^\infty L^2} \lesssim \|\partial u\|^2_{L^\infty L^2}. \tag{4.18}$$

Proof. Again, let $v = \chi_{R_1}(|x|)u$. We first prove (4.17). Now $q^w \in OPS^1$ and hence it is a bounded linear operator from $H^1 \to L^2$ (see Appendix A, Theorem A.6). Therefore,

$$\|q^w v\|^2_{L^2L^2} \lesssim \|v\|^2_{L^2H^1} \lesssim \|u\|^2_{LE^1},$$

where we used the compact support of v to convert to a LE^1 norm. The proof of (4.18) is essentially the same as the proof of (4.17), except that we bound all lower order terms by a derivative via a Hardy inequality (2.4). □

Armed with Lemma 4.3, we are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. We begin by integrating $P_g\chi_{R_1}(|x|)u$ against $q^w\chi_{R_1}(|x|)u$:

$$
\begin{aligned}
&2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}(P_g\chi_{R_1}(|x|)u)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt\\
&=2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}\chi_{R_1}(|x|)(P_gu)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt+2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}([P_g,\chi_{R_1}(|x|)]u)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt.
\end{aligned}
$$

Now

$$
\begin{aligned}
2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}([P_g,\chi_{R_1}(|x|)]u)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt &\lesssim \int_0^T\int_{\mathbb{R}^3}\Big|[P_g,\chi_{R_1}(|x|)]u\Big|\Big|q^w\chi_{R_1}(|x|)u\Big|\,dxdt\\
&\lesssim \|[P_g,\chi_{R_1}(|x|)]u\|_{L^2L^2}\|q^w\chi_{R_1}(|x|)u\|_{L^2L^2}\\
&\lesssim \|[P_g,\chi_{R_1}(|x|)]u\|_{LE^*}\|u\|_{LE^1},
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality and applied Lemma 4.3. Hence, it suffices to establish

$$
2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}\chi_{R_1}(|x|)(P_gu)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt\lesssim\|P_gu\|_{LE^*+L^1L^2}(\|u\|_{LE^1}+\|\partial u\|_{L^\infty L^2}).
$$

We write $P_gu = F_1 + F_2$ where $F_1 \in L^1L^2$ and $F_2 \in LE^*$ and proceed as in the bound for $2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}[P_g,\chi_{R_1}(|x|)]u\overline{q^w\chi_{R_1}(|x|)u}\,dxdt$. Again, applying the Cauchy-Schwarz inequality and Lemma 4.3, proves the result. Indeed,

$$
\begin{aligned}
&2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3}\chi_{R_1}(|x|)(P_gu)\overline{q^w\chi_{R_1}(|x|)u}\,dxdt\\
&\qquad\lesssim\int_0^T\int_{\mathbb{R}^3}|F_1||q^w\chi_{R_1}(|x|)u|\,dxdt+\int_{\mathbb{R}^3}|F_2||q^w\chi_{R_1}(|x|)u|\,dxdt\\
&\qquad\lesssim\|q^w\chi_{R_1}(|x|)u\|_{L^\infty L^2}\|F_1\|_{L^1L^2}+\|q^w\chi_{R_1}(|x|)u\|_{L^2L^2}\|F_2\|_{L^2L^2}.
\end{aligned}
$$

An application of Lemma 4.3 completes the proof. □

4.2 High Frequency Estimate with Perturbation

The goal of this section is to add the perturbation $h^{\alpha\beta}D_\alpha D_\beta$ to our high frequency estimate. We now state the main theorem of this section.

Theorem 4.2. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is smooth, symmetric, and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is smooth, symmetric, non-trapping, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix R_{AF} sufficiently large. Further, fix $R_1 > 2R_{AF}$ and $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in LE^* + L^1L^2$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then*

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + \|u\|_{L^2 L^2_{<2R_1}}^2 + \|P_h u\|_{LE^* + L^1L^2}^2 + \delta \log(2+T) \|u\|_{LE^1}^2. \tag{4.19}$$

Here the implicit constant is independent of T and δ.

Remark. *The implicit constant in* (4.19) *can depend on R_1.*

Proof. This theorem follows by combining the exterior estimate Theorem 3.2 with the proof of Theorem 4.1, where we track the h-dependent terms that we are now including. Indeed,

$$2\text{Im} \int_0^T \int_{\mathbb{R}^3} (P_h \chi_{R_1}(|x|)u) \overline{q^w \chi_{R_1}(|x|)u} \, dxdt - \varepsilon \int_0^T \int_{\mathbb{R}^3} (P_h \chi_{R_1}(|x|)u) \chi_{R_1}(|x|)u \, dxdt$$
$$\lesssim \|P_h u\|_{LE^* + L^1L^2} \big(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2} \big) + \|[P_h, \chi_{R_1}(|x|)]u\|_{LE^*} \|u\|_{LE^1},$$

just as before. So it suffices to bound

$$2\text{Im} \int_0^T \int_{\mathbb{R}^3} (h^{\alpha\beta} D_\alpha D_\beta \chi_{R_1}(|x|)u) \overline{q^w \chi_{R_1}(|x|)u} \, dxdt - \varepsilon \int_0^T \int_{\mathbb{R}^3} (h^{\alpha\beta} D_\alpha D_\beta \chi_{R_1}(|x|)u) \overline{\chi_{R_1}(|x|)u} \, dxdt, \tag{4.20}$$

as all other terms are controlled via the proof of Theorem 4.1. We handle each term separately.

For the first term, it suffices to bound

$$2\Big| \text{Im} \int_0^T \int_{\mathbb{R}^3} (h^{i\beta} D_i D_\beta \chi_{R_1}(|x|)u) \overline{q^w \chi_{R_1}(|x|)u} \, dxdt \Big| \tag{4.21}$$

and

$$2\Big| \text{Im} \int_0^T \int_{\mathbb{R}^3} (h^{00} D_t D_t \chi_{R_1}(|x|)u) \overline{q^w \chi_{R_1}(|x|)u} \, dxdt \Big|. \tag{4.22}$$

Again, we set $v = \chi_{R_1}(|x|)u$ to simplify notation. Expanding the first integral above and integrating by parts, we see that it is equal to

$$(4.23)\quad \langle \partial_i h^{i\beta} D_\beta v, q^w v\rangle_{L^2L^2} + \langle q^w v, \partial_\beta h^{i\beta} D_i v\rangle_{L^2L^2} + \langle i[D_\beta, q^w]v, h^{i\beta} D_i v\rangle_{L^2L^2}$$
$$+ \langle h^{i\beta} D_\beta v, i[D_i, q^w]v\rangle_{L^2L^2} - \langle D_\beta v, i[q^w, h^{i\beta}]D_i v\rangle_{L^2L^2} + \langle q^w v, h^{i0} D_i v\rangle_{L^2}\Big|_0^T.$$

The above time boundary term is controlled by $\|\partial u\|^2_{L^\infty L^2}$. We control $\langle \partial_i h^{i\beta} D_\beta v, q^w v\rangle_{L^2L^2}$ by applying Cauchy's inequality, utilizing the compact support of v, and using the fact that $q^w \in OPS^1$. This ensures that q^w a bounded linear operator from $H^1 \to L^2$ (see Appendix A, Theorem A.6). Therefore:

$$\langle \partial_i h^{i\beta} D_\beta v, q^w v\rangle_{L^2L^2} \lesssim \delta\Big(\|\partial v\|^2_{L^2L^2} + \|q^w v\|^2_{L^2L^2}\Big)$$
$$\lesssim \delta\Big(\|\partial v\|^2_{L^2L^2} + \|v\|^2_{L^2H^1}\Big) \lesssim \delta\|u\|^2_{LE^1},$$

where we have used the compact support of v in the last line. The bound for $\langle q^w v, \partial_\beta h^{i\beta} D_i v\rangle_{L^2L^2}$ is similar.

We control $\langle i[D_\beta, q^w]v, h^{i\beta} D_i v\rangle_{L^2L^2}$ by noting $[D_t, q^w] = 0$ and $[D_j, q^w] \in OPS^1$ (see Appendix A, Theorem A.5). Applying similar steps to our prior bound, we immediately see

$$\langle i[D_\beta, q^w]v, h^{i\beta} D_i v\rangle_{L^2L^2} \lesssim \delta\Big(\|\partial v\|^2_{L^2L^2} + \|v\|^2_{L^2H^1}\Big) \lesssim \delta\|u\|^2_{LE^1}.$$

The term $\langle h^{i\beta} D_\beta v, i[D_i, q^w]v\rangle_{L^2L^2}$ is controlled similarly.

We now examine $\langle D_\beta v, i[q^w, h^{i\beta}]D_i v\rangle_{L^2L^2}$. We use Theorem A.7 to write this as

$$\langle D_\beta v, i[q^w, h^{i\beta}]D_i v\rangle_{L^2L^2} = \langle D_\beta v, \{q, h^{i\beta}\}^w D_i v\rangle_{L^2L^2} + \langle D_\beta v, Z_{-2} D_i v\rangle_{L^2L^2},$$

where $Z_{-2} \in OPS^{-2}$. Continuing, we use Theorem A.4 to switch quantizations to the Kohn-Nirenberg quantization to write this as

$$(4.24)\quad \langle D_\beta v, i[q^w, h^{i\beta}]D_i v\rangle_{L^2L^2} = \langle D_\beta v, \{q, h^{i\beta}\}_{KN} D_i v\rangle_{L^2L^2} + \langle D_\beta v, Z_{-1} D_i v\rangle_{L^2L^2} + \langle D_\beta v, Z_{-2} D_i v\rangle_{L^2L^2},$$

where $Z_{-1} \in OPS^{-1}$. Now

$$\langle D_\beta v, \{q, h^{i\beta}\}_{KN} D_i v\rangle_{L^2L^2} = \int_0^T \int_{\mathbb{R}^3} D_\beta v \overline{\partial_x h^{i\beta}(x) \mathcal{F}^{-1} \partial_\xi q \mathcal{F} D_i v}\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^3} \partial_x h^{i\beta} D_\beta v \overline{(\partial_\xi q)_{KN} D_i v}\, dxdt.$$

Now $\partial_\xi q \in S^0$ and so $(\partial_\xi q)_{KN}$ is a bounded linear operator from $L^2 \to L^2$. Utilizing this fact, and applying Cauchy's inequality, we are able to bound the above by $\delta\|\partial v\|^2_{L^2L^2} \lesssim \delta\|u\|^2_{LE^1}$. It remains to bound the error terms

$\langle D_\beta v, Z_{-1}D_i v\rangle_{L^2L^2} + \langle D_\beta v, Z_{-2}D_i v\rangle_{L^2L^2}$. We note it suffices to bound $\langle D_\beta v, Z_{-1}D_i v\rangle_{L^2L^2}$. Now $Z_{-1}D_i \in OPS^0$, and so we utilize this, Cauchy's inequality, and the compact support of v to see

$$\langle D_\beta v, Z_{-1}D_i v\rangle_{L^2L^2} \lesssim \varepsilon\|\partial v\|^2_{L^2L^2} + \frac{1}{\varepsilon}\|Z_{-1}D_i v\|_{L^2L^2} \lesssim \varepsilon\|\partial v\|^2_{L^2L^2} + \frac{1}{\varepsilon}\|v\|_{L^2L^2}.$$

We choose ε small enough to absorb $\varepsilon\|\partial v\|^2_{L^2L^2}$ into the left-hand side of (4.1). So we have controlled the integral in (4.21) by

$$\varepsilon\|\partial(\chi_{R_1}(|x|)u)\|^2_{L^2L^2} + \varepsilon^{-1}\|u\|^2_{L^2L^2_{<2R_1}} + \|\partial u\|^2_{L^\infty L^2} + \delta\|u\|^2_{LE^1}, \tag{4.25}$$

which suffices.

We need to obtain a similar bound for (4.22). To this end, we investigate

$$2\mathrm{Im}\int_0^T\int_{\mathbb{R}^3} h^{00}D_tD_tv\overline{q^wv}\,dxdt$$

and note that this is equal to

$$-\langle iq^wv, h^{00}\partial_tv\rangle_{L^2}\big|_0^T + \langle i\partial_tv, q^wh^{00}v\rangle_{L^2}\big|_0^T + \langle iq^wv, \partial_th^{00}\partial_tv\rangle_{L^2L^2} - \langle i\partial_tv, q^w\partial_th^{00}v\rangle_{L^2L^2} + \langle\partial_t^2v, i[q^w, h^{00}]v\rangle_{L^2L^2}.$$

The time boundary terms are controlled via $\|\partial u\|^2_{L^\infty L^2}$, as usual. We have already controlled terms very similar to $\langle iq^wv, \partial_th^{00}\partial_tv\rangle_{L^2L^2}$. We write

$$\langle i\partial_tv, q^w\partial_th^{00}v\rangle_{L^2L^2} = \langle i\partial_th^{00}\partial_tv, q^wv\rangle_{L^2L^2} + \langle i\partial_tv, [q^w, \partial_th^{00}]v\rangle_{L^2L^2}.$$

We have already bound terms in the form $\langle i\partial_th^{00}\partial_tv, q^wv\rangle_{L^2L^2}$. For $\langle i\partial_tv, [q^w, \partial_th^{00}]v\rangle_{L^2L^2}$, we note $[q^w, \partial_th^{00}] \in OPS^0$ and so

$$\begin{aligned}\langle i\partial_tv, [q^w, \partial_th^{00}]v\rangle_{L^2L^2} &\lesssim \varepsilon\|\partial_tv\|^2_{L^2L^2} + \frac{1}{\varepsilon}\|[q^w, \partial_th^{00}]v\|^2_{L^2L^2}\\ &\lesssim \varepsilon\|\partial_tv\|^2_{L^2L^2} + \frac{1}{\varepsilon}\|v\|^2_{L^2L^2}.\end{aligned}$$

We choose ε small enough to absorb the $\varepsilon\|\partial_tv\|^2_{L^2L^2}$ term as needed.

We now turn to $\langle\partial_t^2v, i[q^w, h^{00}]v\rangle_{L^2L^2}$ and write this term as

$$\langle \partial_t^2 v, i[q^w, h^{00}]v\rangle_{L^2L^2} = \langle P_h v, i[q^w, h^{00}]v\rangle_{L^2L^2} - \langle D_i g^{ij} D_j v, i[q^w, h^{00}]v\rangle_{L^2L^2} - \langle h^{\alpha\beta} D_\alpha D_\beta v, i[q^w, h^{00}]v\rangle_{L^2L^2}. \tag{4.26}$$

Using that $[q^w, h^{00}] \in OPS^0$, the compact support of v, and the Hardy inequality used in (2.4), we see

$$\langle P_h v, i[q^w, h^{00}]v\rangle_{L^2L^2} \lesssim \|P_h \chi_{R_1}(|x|)u\|_{LE^*+L^1L^2}\|(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}).$$

Now

$$- \langle h^{\alpha\beta} D_\alpha D_\beta v, i[q^w, h^{00}]v\rangle_{L^2L^2} = \langle h^{0\beta}\partial_\beta v, i[q^w, h^{00}]v\rangle_{L^2L^2}\big|_0^T - \langle \partial_\alpha h^{\alpha\beta}\partial_\beta v, i[q^w, h^{00}]v\rangle_{L^2L^2} + \langle h^{\alpha\beta}\partial_\beta v, D_\alpha[q^w, h^{00}]v\rangle_{L^2L^2}. \tag{4.27}$$

The time boundary term is controlled by $\|\partial u\|_{L^\infty L^2}^2$, as usual. Recognizing $[q^w, h^{00}] \in OPS^0$, we see the last two terms in the above equation are similar to terms already bounded. Therefore (4.27) is controlled by $\|\partial u\|^2_{L^\infty L^2} + \delta\|u\|^2_{LE^1}$.

We now turn to the remaining term in (4.26) and find:

$$\begin{aligned} -\langle D_i g^{ij} D_j v, i[q^w, h^{00}]v\rangle_{L^2L^2} &= -\langle D_i g^{ij} D_j v, \{q, h^{00}\}^w v\rangle_{L^2L^2} + \langle D_i g^{ij} D_j v, \Omega_{-2} v\rangle_{L^2L^2} \\ &= -\langle D_i g^{ij} D_j v, \{q, h^{00}\}_{KN} v\rangle_{L^2L^2} + \langle D_i g^{ij} D_j v, \Omega_{-1} v\rangle_{L^2L^2}, \end{aligned}$$

where we have applied Theorems A.8 and Theorems A.4. Here $\Omega_{-2} \in OPS^{-2}$ and $\Omega_{-1} \in OPS^{-1}$. Now $\langle D_i g^{ij} D_j v, \Omega_{-1} v\rangle_{L^2L^2} = \langle g^{ij} D_j v, D_i \Omega_{-1} v\rangle_{L^2L^2}$ which is in the form of terms we have already bounded. Continuing, we find

$$\begin{aligned} -\langle D_i g^{ij} D_j v, \{q, h^{00}\}_{KN} v\rangle_{L^2L^2} &= \langle D_i g^{ij} D_j v, \partial_x h^{00} (\partial_\xi q)_{KN} v\rangle_{L^2L^2} \\ &= \langle \partial_x h^{00} g^{ij} D_j v, D_i (\partial_\xi q)_{KN} v\rangle_{L^2L^2} \\ &\quad - \langle \partial_i \partial_x h^{00} g^{ij} \partial_j v, (\partial_\xi q)_{KN} v\rangle_{L^2L^2}. \end{aligned}$$

Note that $(\partial_\xi q)_{KN}$ is a bounded linear operator from $L^2 \to L^2$ and $D_i(\partial_\xi q)_{KN}$ is a bounded linear operator from $H^1 \to L^2$ (see Appendix A, Theorem A.6), and so we have already controlled terms of this form.

So we have bound the first term in (4.20) by

$$\varepsilon\|\partial(\chi_{R_1}(|x|)u)\|^2_{L^2L^2} + \varepsilon^{-1}\|u\|^2_{L^2L^2_{<2R_1}} + \|\partial u\|^2_{L^\infty L^2} + \delta\|u\|^2_{LE^1} + \|P_h\chi_{R_1}(|x|)u\|_{LE^*+L^1L^2}\big(\|u\|_{LE^1} + \|\partial u\|_{L^\infty L^2}\big), \tag{4.28}$$

which suffices.

For the second term in (4.20), we must bound

$$\int_0^T \int_{\mathbb{R}^3} (h^{\alpha\beta} D_\alpha D_\beta) v\overline{v}\, dxdt.$$

We integrate by parts to obtain

$$\int_0^T \int_{\mathbb{R}^3} \partial_\alpha h^{\alpha\beta} \partial_\beta v \overline{v}\, dxdt + \int_0^T \int_{\mathbb{R}^3} h^{\alpha\beta} \partial_\beta v \overline{\partial_\alpha v}\, dxdt - \int_{\mathbb{R}^3} h^{0\beta} \partial_\beta v \overline{v}\, dx\Big|_0^T.$$

The time boundary term is controlled by $\|\partial u\|^2_{L^\infty L^2}$, while the remaining terms are bounded by

$$\int_0^T \int_{\mathbb{R}^3} \big(|\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle x\rangle}\big)^2\, dxdt,$$

where we have utilized the compact support of v in the last line. An application of (2.3) completes the proof of Theorem 4.2. □

Chapter 5: Medium Frequency Analysis

We begin by stating the main theorem of this section, which is a weighted local energy estimate with an error term that can be absorbed for any bounded range of time frequencies supported away from zero. This theorem is related to a similar theorem in [29]. The key difference is obtaining good bounds for our time-dependent perturbation $h^{\alpha\beta}D_\alpha D_\beta$, as it is not asymptotically flat.

Theorem 5.1. *Let P_h be as in (1.2), where $h^{\alpha\beta}$ is symmetric and satisfies (1.11) for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Fix R_{AF} sufficiently large and fix $R_{ext} > R_{out} > R_{in} > R_{AF}$. Then, for any radial weight $\phi(r)$ that is constant for $r \geq 2R_{ext}$ and compatible with the constructions in Appendix B in that (B.1) holds for $\{R_{in}/2 \leq r \leq 2R_{out}\}$ and (B.4) holds for $\{r \leq R_{in}/2\}$, the following bound is true:*

$$
\begin{aligned}
(5.1)\quad & \|\langle r\rangle^{-1}(1+\phi''_+)^{1/2} e^{\phi}(\langle r\rangle^{-1}(1+\phi')u, \nabla_x u)\|^2_{L^2L^2_{<2R_{ext}}} + \|\langle r\rangle^{-1}(1+\phi')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2_{<2R_{ext}}} \\
&+ R_{ext}^{-1}\|e^{\phi}u\|^2_{LE^1_{>R_{ext}}} \lesssim \|e^{\phi}P_h u\|^2_{L^2L^2_{<2R_{ext}}} + R_{ext}^{-1}\|e^{\phi}P_h u\|^2_{LE^*} + \|(\phi'')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2(1\lesssim r<2R_{in})} \\
&\qquad + C_{R_{in},R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big) \\
&\qquad\qquad + \|\langle r\rangle^{-2}(1+\phi')^{3/2}e^{\phi}u\|^2_{L^2L^2(R_{out}<r<2R_{ext})}.
\end{aligned}
$$

Here the implicit constant is independent of T and δ.

Remark. *The implicit constant in (5.1) is independent of R_{in}, R_{out}, R_{ext}, ϕ, and its derivatives unless explicitly stated in the form $C_{R_{in},R_{out},R_{ext},\phi}$.*

This is a medium frequency estimate since it implies the following local energy estimate for u with appropriate time frequency support:

Corollary 5.1. *Let P_h be as in (1.2), where $h^{\alpha\beta}$ is symmetric and satisfies (1.11) for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Further* *assume $u(t,x)$ has time frequency support in some bounded interval away from zero: $0 < \tau_0 \le |\tau| \le \tau_1$, where τ is the Fourier dual of t. Then, the following bound holds:*

$$\|u\|^2_{LE^1} \lesssim \|P_h u\|^2_{LE^*} + \|\partial u\|^2_{L^\infty L^2} + \delta \log(2+T)\|u\|^2_{LE^1}.$$

Here the implicit constant is independent of T and δ.

We first show Theorem 5.1 implies Corollary 5.1, before proving the theorem.

Proof of Corollary 5.1. The corollary follows by applying Theorem 5.1 to u. We need to absorb the following two error terms on the right-hand side of (5.1): $\|(\phi'')^{1/2}e^\phi \partial_t u\|^2_{L^2L^2(1\lesssim r<2R_{in})}$ and $\|\langle r\rangle^{-2}(1+\phi')^{3/2}e^\phi u\|^2_{L^2L^2(R_{out}<r<2R_{ext})}$. The term $\|(\phi'')^{1/2}e^\phi \partial_t u\|^2_{L^2L^2(1\lesssim r<2R_{in})}$ is absorbed into the first term on the left-hand side of (5.1) by observing $\phi' \gtrsim \lambda \gg 1$ in this regime and choosing $\lambda \gg R^2_{in}\tau_1$. We are now free to choose R_{out} sufficiently large so that $\tau_0 \gg \sup_{R_{out}<r<2R_{ext}} \langle r\rangle^{-1}(1+\phi)'$. Indeed this is possible provided $\phi'(\log(r))/r \to 0$ as $r \to \infty$, which is certainly true for the weight function constructed in Appendix B. This ensures that the error term $\|\langle r\rangle^{-2}(1+\phi')^{3/2}e^\phi u\|^2_{L^2L^2(R_{out}<r<2R_{ext})}$ can be absorbed into the second term on the left-hand side of (5.1). This completes the proof of the corollary. □

We now turn to Theorem 5.1 which is proved via two separate Carleman estimates. The first estimate is useful for when our "background" operator $-D_t^2 + D_i g^{ij} D_j$ is a small, asymptotically Euclidean perturbation of $\Box$, i.e. for $r > R_{AF}$. The second estimate is useful on a compact set around the origin. Our Carleman estimates are weighted L^2L^2 estimates obtained by conjugating the wave operator with e^ϕ, where $\phi = \phi(r)$ is a convex weight function. For more on this, we refer the reader to [21], [29], and the references therein.

We begin with a preliminary estimate that is applied for r sufficiently large. The authors of [29] prove a similar estimate by working on the symbol side, quantitizing, and absorbing errors. We work directly on the differential operator side but are motivated by the methods found in [29].

Proposition 5.1. *Let P_h be as in (1.2), where $h^{\alpha\beta}$ is symmetric and satisfies (1.11) for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). Fix R_{AF} sufficiently large and fix $R_{m2} > R_{m1} > R_{AF}$. Further, fix $T > 0$. Suppose $u(t,x) \in C^2([0,T]\times\mathbb{R}^3)$ solves $P_h u = F \in LE^*$ with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$, such that u is supported in $R_{m_1} < r < R_{m_2}$. Then, for any weight $\phi(s)$, where $s = \log(r)$, that satisfies*

$$\lambda \lesssim \phi''(s) \le \phi'(s)/2 \lesssim \phi''(s), \qquad |\phi'''(s)| \lesssim \phi'(s), \qquad \lambda \gg 1, \tag{5.2}$$

the following bound holds:

$$\|r^{-1}(\phi'')^{1/2}e^{\phi}(r^{-1}\phi' u, \nabla_x u)\|^2_{L^2L^2} + \|r^{-1}(\phi')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2} \lesssim \|e^{\phi}P_h u\|^2_{L^2L^2} + C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big). \tag{5.3}$$

Here the implicit constant is independent of T and δ.

Remark. *The only reason that we require u to be compactly supported is to ensure that all constants $C_{R_{m1},R_{m2},\phi}$ are finite since $\phi \to \infty$ as $r \to \infty$. We will actually bend this weight in a later estimate at the expense of an error term, which we later absorb for bounded time frequencies.*

Remark. *To see the existence of such a weight function ϕ, we refer the reader to [29] and especially [21], where such a function is explicitly constructed. A similar construction can be found in Appendix B, where a weight function is constructed for the next proposition. By choosing $R_{m2} \leq 2R_{out}$, this function satisfies all the desired properties in Proposition 5.1 on the support of u.*

Proof. To prove Proposition 5.1, we will conjugate P_h by e^{ϕ} and break the resulting operator into its self-adjoint and skew-adjoint components. We then form the commutator of the resulting self-adjoint and skew-adjoint operators and obtain the proposition via a positive commutator argument. In the sequel, it is useful to consider the new differential operator:

$$\tilde{P}_h = -D_t^2 + D_i g^{ij} D_j + D_\alpha h^{\alpha\beta} D_\beta \equiv D_\alpha \tilde{g}^{\alpha\beta} D_\beta. \tag{5.4}$$

The operator $\tilde{P}_h$ has the benefit that it is symmetric (up to time boundary terms). We now conjugate this operator, calculating

$$P_\phi \equiv e^{\phi}\tilde{P}_h e^{-\phi} = P^{self} + P^{skew}. \tag{5.5}$$

Breaking the result into self-adjoint and skew-adjoint components, we find

$$P^{self} = D_\alpha \tilde{g}^{\alpha\beta} D_\beta - \tilde{g}^{ij}\frac{(\phi')^2}{r^2}\frac{x_i}{r}\frac{x_j}{r} \tag{5.6}$$

and

$$P^{skew} = -\frac{1}{i}\frac{\phi'}{r}\frac{x_i}{r}\tilde{g}^{i\beta}D_\beta - \frac{1}{i}D_\beta \tilde{g}^{i\beta}\frac{\phi'}{r}\frac{x_i}{r}. \tag{5.7}$$

Setting $v = e^{\phi}u$, we see it suffices to show

$$\|r^{-1}(\phi'')^{1/2}(r^{-1}\phi' v, \nabla_x v)\|^2_{L^2L^2} + \|r^{-1}(\phi')^{1/2}\partial_t v\|^2_{L^2L^2} \lesssim \|P_h v\|^2_{L^2L^2} + \tilde{E}_1[v]\Big|_0^T + \tilde{H}_1[v], \tag{5.8}$$

where $\tilde{E}_1[v]\Big|_0^T$ and $\tilde{H}_1[v]$ have good bounds in terms of u:

$$\tilde{E}_1[v]\Big|_0^T \lesssim C_{R_{m1},R_{m1},\phi}\|\partial u\|^2_{L^\infty L^2}$$

and

$$\tilde{H}_1[v] \lesssim C_{R_{m1},R_{m1},\phi}\delta\Big(\log(2+T)\|u\|^2_{LE^1} + \|\partial u\|^2_{L^\infty L^2}\Big).$$

Morally, we want to reduce the proof to a positive commutator argument by observing

$$\|P_\phi v\|^2_{L^2L^2} = \|P^{self}v\|^2_{L^2L^2} + \|P^{skew}v\|^2_{L^2L^2} + \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + \text{time boundary terms.}$$

We can control the time boundary terms via $C_{R_{m1},R_{m2},\phi}\|\partial u\|^2_{L^\infty L^2}$.

It will not actually suffice to work with $\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2}$ alone. Rather, we will show that the left-hand side of (5.8) is controlled by

$$\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + 2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2} + \langle (\frac{1}{2}\frac{\phi''}{r^2} - 4\frac{\phi'}{r^2})v, P^{self}v\rangle_{L^2L^2} + \tilde{E}_1[v]\Big|_0^T + \tilde{H}_1[v],$$

and then obtain (5.8) via Cauchy's inequality.

We now compute the aforementioned time boundary terms via integration by parts:

$$\begin{aligned}\|P_\phi v\|^2_{L^2L^2} = \|P^{self}v\|^2_{L^2L^2} &+ \|P^{skew}v\|^2_{L^2L^2} + \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} \\ &- 2\int_{\mathbb{R}^3}\frac{\phi'^3}{r^3}\tilde{g}^{l0}\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}v\overline{v}\,dx\Big|_0^T + 2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_\alpha\tilde{g}^{\alpha\beta}D_\beta v)\overline{v}\,dx\Big|_0^T \\ &- \int_{\mathbb{R}^3}(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v + D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v)\overline{\tilde{g}^{0\beta}D_\beta v}\,dx\Big|_0^T \\ &- \int_{\mathbb{R}^3}\tilde{g}^{\alpha 0}D_\alpha\Big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v + D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\Big)\overline{v}\,dx\Big|_0^T.\end{aligned}$$

Observe that the above time boundary terms are equal to:

$$
\begin{aligned}
&-2\int_{\mathbb{R}^3}\frac{\phi'^3}{r^3}\tilde{g}^{l0}\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}v\overline{v}\,dx\Big|_0^T+2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_i\tilde{g}^{i\beta}D_\beta v)\overline{v}\,dx\Big|_0^T\\
&\qquad+2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_t\tilde{g}^{0j}D_jv)\overline{v}\,dx\Big|_0^T+2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_t\tilde{g}^{00}D_tv)\overline{v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v+D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\big)\overline{\tilde{g}^{0\beta}D_\beta v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\tilde{g}^{i0}\Big(D_i\big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v+D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\big)\Big)\overline{v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\tilde{g}^{00}\Big(D_t\big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v+D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\big)\Big)\overline{v}\,dx\Big|_0^T\\
&=-2\int_{\mathbb{R}^3}\frac{\phi'^3}{r^3}\tilde{g}^{l0}\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}v\overline{v}\,dx\Big|_0^T+2\int_{\mathbb{R}^3}(\tilde{g}^{i\beta}D_\beta v)\overline{D_i\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}v}\,dx\Big|_0^T\\
&\qquad-2\int_{\mathbb{R}^3}\frac{1}{i}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_t\partial_j\tilde{g}^{0j}v)\overline{v}\,dx\Big|_0^T-2\int_{\mathbb{R}^3}(D_t\tilde{g}^{0j}v)\overline{D_j\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}v}\,dx\Big|_0^T\\
&\qquad+2\int_{\mathbb{R}^3}\frac{1}{i}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(\partial_t\tilde{g}^{00}D_tv)\overline{v}\,dx\Big|_0^T+2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(\tilde{g}^{00}D_t^2v)\overline{v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v+D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\big)\overline{\tilde{g}^{0\beta}D_\beta v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\Big(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma v+D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}v\Big)\overline{D_i\tilde{g}^{i0}v}\,dx\Big|_0^T\\
&\qquad-\int_{\mathbb{R}^3}\tilde{g}^{00}\frac{1}{i}\big(D_t\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})v\big)\overline{v}\,dx\Big|_0^T-2\int_{\mathbb{R}^3}\tilde{g}^{00}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0}(D_t^2v)\overline{v}\,dx\Big|_0^T\\
&\qquad-2\int_{\mathbb{R}^3}\tilde{g}^{00}\frac{1}{i}\partial_t(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})(D_\gamma v)\overline{v}\,dx\Big|_0^T-2\int_{\mathbb{R}^3}(D_tv)\overline{D_j\tilde{g}^{00}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{lj}v}\,dx\Big|_0^T.
\end{aligned}
$$

By taking the derivative of v, the supremum of ϕ-dependent terms, applying Cauchy's inequality, and a Hardy inequality (2.4) on lower order terms, we can bound this from above by $C_{R_{m1},R_{m2},\phi}\|\partial u\|^2_{L^\infty L^2}$, as desired.

We now consider the commutator $[P^{self},P^{skew}]$ and observe

$$
\begin{aligned}
&[-\tilde{g}^{ij}\frac{(\phi')^2}{r^2}\frac{x_i}{r}\frac{x_j}{r},-\frac{1}{i}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\beta}D_\beta-\frac{1}{i}D_\beta\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\beta}]\\
&\quad=4\frac{(\phi')^2\phi''}{r^4}(\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+4\frac{(\phi')^3}{r^4}\tilde{g}^{lk}\tilde{g}^{kj}\frac{x_j}{r}\frac{x_l}{r}-8\frac{(\phi')^3}{r^4}(\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+2\frac{(\phi')^3}{r^4}\partial_\beta\tilde{g}^{ij}\tilde{g}^{l\beta}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}.
\end{aligned}
\tag{5.9}
$$

Calculating the other terms present in $[P^{self},P^{skew}]$, we find

$$
\begin{aligned}
&[D_\alpha\tilde{g}^{\alpha\beta}D_\beta,-\frac{1}{i}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}D_\gamma-\frac{1}{i}D_\gamma\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}]=2D_\alpha\tilde{g}^{\alpha\beta}\partial_\beta(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})D_\gamma+2D_\gamma\tilde{g}^{\alpha\beta}\partial_\alpha(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})D_\beta\\
&\qquad-2D_\alpha\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}\partial_\gamma\tilde{g}^{\alpha\beta}D_\beta-\frac{1}{i}\tilde{g}^{\alpha\beta}\partial_\alpha\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})D_\beta+\frac{1}{i}D_\alpha\tilde{g}^{\alpha\beta}\partial_\beta\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}).
\end{aligned}
\tag{5.10}
$$

Remark. *We note that a more standard form of the above commutator would have no first order terms. It is indeed possible to put* (5.10) *in this form, as it is equal to*

$$2D_\alpha\tilde{g}^{\alpha\beta}\partial_\beta(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})D_\gamma + 2D_\gamma\tilde{g}^{\alpha\beta}\partial_\alpha(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})D_\beta - 2D_\alpha\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}\partial_\gamma\tilde{g}^{\alpha\beta}D_\beta - \partial_\alpha\big(\tilde{g}^{\alpha\beta}\partial_\beta\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})\big).$$

Note that the last term in the above expression allows for three derivatives to land on g and h (contained in $\tilde{g}$). It is preferable for this not to occur. Therefore, we work with (5.10) *instead of the above expression.*

Utilizing (5.9) and (5.10), we see

$$\begin{aligned}(5.11)\quad \int_0^T\int_{\mathbb{R}^3}([P^{self},P^{skew}]v)\overline{v}\,dxdt &= \int_0^T\int_{\mathbb{R}^3}\Big(4\frac{(\phi')^2\phi''}{r^4}(\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r})^2 + 4\frac{(\phi')^3}{r^4}\tilde{g}^{lk}\tilde{g}^{kj}\frac{x_j}{r}\frac{x_l}{r}\Big)v^2\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}\Big(-8\frac{(\phi')^3}{r^4}(\tilde{g}^{ij}\frac{x_i}{r}\frac{x_j}{r})^2 + 2\frac{(\phi')^3}{r^4}\partial_\beta\tilde{g}^{ij}\tilde{g}^{l\beta}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}\Big)v^2\,dxdt\\
&+4\int_0^T\int_{\mathbb{R}^3}\tilde{g}^{\alpha\beta}\partial_\beta(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})\partial_\gamma v\partial_\alpha v\,dxdt - 2\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}\partial_\gamma\tilde{g}^{\alpha\beta}\partial_\beta v\partial_\alpha v\,dxdt\\
&+2\int_0^T\int_{\mathbb{R}^3}\tilde{g}^{\alpha\beta}\partial_\alpha\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})\partial_\beta vv\,dxdt - \int_{\mathbb{R}^3}\tilde{g}^{\alpha0}\partial_\alpha\partial_\gamma(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})v^2\,dx\Big|_0^T\\
-2\int_{\mathbb{R}^3}\tilde{g}^{0\beta}\partial_\beta(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma})\partial_\gamma vv\,dx\Big|_0^T &- 2\int_{\mathbb{R}^3}\tilde{g}^{\alpha\beta}\partial_\alpha(\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l0})\partial_\beta vv\,dx\Big|_0^T + 2\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}\tilde{g}^{l\gamma}\partial_\gamma\tilde{g}^{0\beta}\partial_\beta vv\,dx\Big|_0^T.\end{aligned}$$

We can bound the above time boundary terms via $C_{R_{m1},R_{m2},\phi}\|\partial u\|^2_{L^\infty L^2}$, using very similar steps as before.

We must control the remaining terms in (5.11) and observe the immediate bound for all h-dependent terms:

$$\begin{aligned}&\langle[D_\alpha h^{\alpha\beta}D_\beta - h^{ij}\frac{(\phi')^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}, P^{skew}]v,v\rangle_{L^2L^2}\\
&\quad+\langle[-D_t^2 + D_i g^{ij}D_j - g^{ij}\frac{(\phi')^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}, -\frac{1}{i}\frac{\phi'}{r}\frac{x_i}{r}h^{i\beta}D_\beta - \frac{1}{i}D_\beta\frac{\phi'}{r}\frac{x_i}{r}h^{i\beta}]v,v\rangle_{L^2L^2}\\
&\quad\lesssim C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \int_0^T\int_{\mathbb{R}^3}\big(|\partial^2 h| + |\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle r\rangle}\big)^2\,dxdt\Big)\\
&\quad\lesssim C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big),\end{aligned}$$

where we have applied (2.3) in the last line by choosing δ sufficiently small.

Therefore we need consider only the following terms from $\langle[P^{self},P^{skew}]v,v\rangle_{L^2L^2}$:

$$\begin{aligned}(5.12)\quad &\int_0^T\int_{\mathbb{R}^3}\big(4\frac{(\phi')^2\phi''}{r^4}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+4\frac{(\phi')^3}{r^4}g^{lk}g^{kj}\frac{x_j}{r}\frac{x_l}{r}\big)v^2\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}\big(-8\frac{(\phi')^3}{r^4}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+2\frac{(\phi')^3}{r^4}\partial_k g^{ij}g^{lk}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}\big)v^2\,dxdt\\
&+4\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j(\frac{\phi'}{r}\frac{x_l}{r}g^{lk})\partial_k v\partial_i v\,dxdt-2\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}g^{lk}\partial_k g^{ij}\partial_i v\partial_j v\,dxdt\\
&+2\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j\partial_k(\frac{\phi'}{r}\frac{x_l}{r}g^{lk})\partial_i vv\,dxdt.\end{aligned}$$

We now calculate $2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}$:

$$\begin{aligned}2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}&=8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\tilde g^{i\beta}\frac{x_i}{r}\partial_\beta v)^2\,dxdt+8\int_0^T\int_{\mathbb{R}^3}(\frac{1}{r}\frac{x_l}{r}\tilde g^{l\alpha}\partial_\alpha v)\partial_\beta(\frac{\phi'}{r}\frac{x_i}{r}\tilde g^{i\beta})v\,dxdt\\
&+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{\phi'}\big(\partial_\beta(\frac{\phi'}{r}\frac{x_i}{r}\tilde g^{i\beta})v\big)^2\,dxdt.\end{aligned}$$

We now investigate the h-dependent terms in $2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}$ and find the immediate bound:

$$C_{R_{m1},R_{m2},\phi}\int_0^T\int_{\mathbb{R}^3}\big(|\partial h|+|h|\big)\big(|\partial u|+\frac{|u|}{\langle r\rangle}\big)^2\,dxdt\lesssim C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2}+\delta\log(2+T)\|u\|^2_{LE^1}\Big),$$

where we have applied (2.3) in the last line.

So it suffices to consider the following terms from $2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}$:

$$\begin{aligned}(5.13)\quad &8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(g^{ij}\frac{x_i}{r}\partial_j v)^2\,dxdt+8\int_0^T\int_{\mathbb{R}^3}\frac{1}{r}\frac{x_l}{r}g^{lk}\partial_j(\frac{\phi'}{r}\frac{x_i}{r}g^{ij})\partial_k vv\,dxdt\\
&+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{\phi'}\big(\partial_j(\frac{\phi'}{r}\frac{x_i}{r}g^{ij})v\big)^2\,dxdt.\end{aligned}$$

We now investigate $\langle(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})v,P^{self}v\rangle_{L^2L^2}$ and integrate by parts to obtain:

$$\begin{aligned}(5.14)\quad &\langle(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})v,P^{self}v\rangle_{L^2L^2}=\int_0^T\int_{\mathbb{R}^3}(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})\tilde g^{\alpha\beta}\partial_\beta v\partial_\alpha v\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}\partial_i\Big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\Big)\tilde g^{i\beta}\partial_\beta vv\,dxdt+\int_{\mathbb{R}^3}\Big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\Big)\tilde g^{0\beta}\partial_\beta vv\,dx\Big|_0^T\\
&+\int_0^T\int_{\mathbb{R}^3}\Big(4\tilde g^{ij}\frac{\phi'^3}{r^4}\frac{x_i}{r}\frac{x_j}{r}-\frac{1}{2}\tilde g^{ij}\frac{\phi''}{r^2}\frac{\phi'^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}\Big)v^2\,dxdt.\end{aligned}$$

Applying our typical analysis on the time boundary terms, utilizing Cauchy's inequality, and a Hardy inequality (2.4), we control them by $C_{R_{m1},R_{m2},\phi}\|\partial u\|^2_{L^\infty L^2}$.

Further, we bound h-dependent terms in (5.14) (i.e., when $\tilde{g}^{\alpha\beta} = h^{\alpha\beta}$) by

$$C_{R_{m1},R_{m2},\phi}\int_0^T\int_{\mathbb{R}^3}|h|\big(|\partial u|+\frac{|u|}{\langle r\rangle}\big)^2\,dxdt \lesssim C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2}+\delta\log(2+T)\|u\|^2_{LE^1}\Big).$$

Hence, we need only consider the following terms from (5.14):

$$\begin{aligned}(5.15)\quad &\int_0^T\int_{\mathbb{R}^3}\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)g^{ij}\partial_i v\partial_j v\,dxdt+\int_0^T\int_{\mathbb{R}^3}\big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\big)(\partial_t v)^2\,dxdt\\ &+\int_0^T\int_{\mathbb{R}^3}\partial_i\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)g^{ij}\partial_j v v\,dxdt+\int_0^T\int_{\mathbb{R}^3}\Big(4g^{ij}\frac{\phi'^3}{r^4}\frac{x_i}{r}\frac{x_j}{r}-\frac{1}{2}g^{ij}\frac{\phi''}{r^2}\frac{\phi'^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}\Big)v^2\,dxdt.\end{aligned}$$

Combining (5.12), (5.13), and (5.15) leads to the following bound:

$$\begin{aligned}(5.16)\quad &\langle[P^{self},P^{skew}]v+\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)v,P^{self}v\rangle_{L^2L^2}+2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}\\ &\geq\int_0^T\int_{\mathbb{R}^3}\big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\big)(\partial_t v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)g^{ij}\partial_i v\partial_j v\,dxdt\\ &\quad+8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}\big(g^{ij}\frac{x_i}{r}\partial_j v\big)^2\,dxdt+4\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j\big(\frac{\phi'}{r}\frac{x_l}{r}g^{lk}\big)\partial_k v\partial_i v\,dxdt\\ &\quad-2\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}g^{lk}\partial_k g^{ij}\partial_i v\partial_j v\,dxdt+2\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j\partial_k\big(\frac{\phi'}{r}\frac{x_l}{r}g^{lk}\big)\partial_i v v\,dxdt\\ &\quad+8\int_0^T\int_{\mathbb{R}^3}\frac{1}{r}\frac{x_l}{r}g^{lk}\partial_j\big(\frac{\phi'}{r}\frac{x_i}{r}g^{ij}\big)\partial_k v v\,dxdt+\int_0^T\int_{\mathbb{R}^3}\partial_i\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)g^{ij}\partial_j v v\,dxdt\\ &\quad+\int_0^T\int_{\mathbb{R}^3}\big(4\frac{(\phi')^2\phi''}{r^4}\big(g^{ij}\frac{x_i}{r}\frac{x_j}{r}\big)^2+4\frac{(\phi')^3}{r^4}g^{lk}g^{kj}\frac{x_j}{r}\frac{x_l}{r}\big)v^2\,dxdt\\ &\quad+\int_0^T\int_{\mathbb{R}^3}\Big(-8\frac{(\phi')^3}{r^4}\big(g^{ij}\frac{x_i}{r}\frac{x_j}{r}\big)^2+2\frac{(\phi')^3}{r^4}\partial_k g^{ij}g^{lk}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}\Big)v^2\,dxdt\\ &\quad+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{\phi'}\big(\partial_j\big(\frac{\phi'}{r}\frac{x_i}{r}g^{ij}\big)v\big)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\Big(4g^{ij}\frac{\phi'^3}{r^4}\frac{x_i}{r}\frac{x_j}{r}-\frac{1}{2}g^{ij}\frac{\phi''}{r^2}\frac{\phi'^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}\Big)v^2\,dxdt\\ &\quad+\tilde{E}_1[v]\Big|_0^T+\tilde{H}_1[v].\end{aligned}$$

Applying Cauchy's inequality on first order terms, we can bound this from below by

$$
\begin{aligned}
(5.17)\quad &\int_0^T\int_{\mathbb{R}^3}\big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\big)(\partial_t v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}\big)g^{ij}\partial_i v\partial_j v\,dxdt\\
&+8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(g^{ij}\frac{x_i}{r}\partial_j v)^2\,dxdt+4\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j(\frac{\phi'}{r}\frac{x_l}{r}g^{lk})\partial_k v\partial_i v\,dxdt\\
&-2\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}\frac{x_l}{r}g^{lk}\partial_k g^{ij}\partial_i v\partial_j v\,dxdt-\frac{11}{2}\int_0^T\int_{\mathbb{R}^3}\frac{1}{r^2}|\nabla_x v|^2\,dxdt\\
&-\sum_i\int_0^T\int_{\mathbb{R}^3}\Big(rg^{ij}\partial_j\partial_k(\frac{\phi'}{r}\frac{x_l}{r}g^{lk})\Big)^2v^2\,dxdt-4\sum_k\int_0^T\int_{\mathbb{R}^3}\Big(\frac{x_l}{r}g^{lk}\partial_j(\frac{\phi'}{r}\frac{x_i}{r}g^{ij})\Big)^2v^2\,dxdt\\
&-\frac{1}{2}\sum_j\int_0^T\int_{\mathbb{R}^3}\Big(r\partial_i(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})g^{ij}\Big)^2v^2\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}\big(4\frac{(\phi')^2\phi''}{r^4}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+4\frac{(\phi')^3}{r^4}g^{lk}g^{kj}\frac{x_j}{r}\frac{x_l}{r}\big)v^2\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}\Big(-8\frac{(\phi')^3}{r^4}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2+2\frac{(\phi')^3}{r^4}\partial_k g^{ij}g^{lk}\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}\Big)v^2\,dxdt\\
&+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{\phi'}(\partial_j(\frac{\phi'}{r}\frac{x_i}{r}g^{ij})v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\Big(4g^{ij}\frac{\phi'^3}{r^4}\frac{x_i}{r}\frac{x_j}{r}-\frac{1}{2}g^{ij}\frac{\phi''}{r^2}\frac{\phi'^2}{r^2}\frac{x_i}{r}\frac{x_j}{r}\Big)v^2\,dxdt\\
&+\tilde{E}_1[v]\Big|_0^T+\tilde{H}_1[v].
\end{aligned}
$$

By choosing $R_{m1} > R_{AF}$ sufficiently large, we may freely replace g^{ij} with m^{ij} at the expense of additional error terms involving $|||g|||_{>R_{m1}}$. So, we must consider

$$
\begin{aligned}
(5.18)\quad &\int_0^T\int_{\mathbb{R}^3}\big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\big)(\partial_t v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}-\frac{11}{2}\frac{1}{r^2}\big)|\nabla_x v|^2\,dxdt\\
&+8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_r v)^2\,dxdt+4\int_0^T\int_{\mathbb{R}^3}\partial_i(\frac{\phi'}{r}\frac{x_k}{r})\partial_k v\partial_i v\,dxdt\\
&-\sum_i\int_0^T\int_{\mathbb{R}^3}\Big(r\partial_i\partial_l(\frac{\phi'}{r}\frac{x_l}{r})\Big)^2v^2\,dxdt-4\sum_k\int_0^T\int_{\mathbb{R}^3}\Big(\frac{x_k}{r}\partial_i(\frac{\phi'}{r}\frac{x_i}{r})\Big)^2v^2\,dxdt\\
&-\frac{1}{2}\sum_j\int_0^T\int_{\mathbb{R}^3}\Big(r\partial_j(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})\Big)^2v^2\,dxdt\\
&+\frac{7}{2}\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^2\phi''}{r^4}v^2\,dxdt+2\int_0^T\int_{\mathbb{R}^3}\frac{1}{\phi'}(\partial_i(\frac{\phi'}{r}\frac{x_i}{r})v)^2\,dxdt\\
&-|||g|||_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\,dxdt-|||g|||_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^3}{r^4}v^2\,dxdt.
\end{aligned}
$$

Utilizing the largeness assumptions on the first two derivatives of ϕ in (5.2) and the smallness of $|||g|||_{>R_{m1}}$, we are able to bound the above equation from below by

$$
\begin{aligned}
(5.19)\quad & \int_0^T\int_{\mathbb{R}^3}\big(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2}\big)(\partial_t v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\big(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}-\frac{11}{2}\frac{1}{r^2}\big)|\nabla_x v|^2\,dxdt\\
&+8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_r v)^2\,dxdt+4\int_0^T\int_{\mathbb{R}^3}\partial_i\big(\frac{\phi'}{r}\frac{x_k}{r}\big)\partial_k v\partial_i v\,dxdt+\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^2\phi''}{r^4}v^2\,dxdt\\
&-|||g|||_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\,dxdt.
\end{aligned}
$$

Observe, thus far, we have only simplified our bound for the lower order terms.

Making use of (5.2), we see

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}(4\frac{\phi'}{r^2}-\frac{1}{2}\frac{\phi''}{r^2})(\partial_t v)^2\,dxdt+\int_0^T\int_{\mathbb{R}^3}\frac{\phi''(\phi')^2}{r^4}v^2\,dxdt\\
&\qquad\gtrsim\|r^{-2}(\phi'')^{1/2}\phi' v\|^2_{L^2L^2}+\|r^{-1}(\phi')^{1/2}\partial_t u\|^2_{L^2L^2},
\end{aligned}
$$

as desired. Now the remaining terms in (5.19) can be expressed as

$$
\begin{aligned}
(5.20)\quad & 4\int_0^T\int_{\mathbb{R}^3}\partial_i\big(\frac{\phi'}{r}\frac{x_k}{r}\big)\partial_k v\partial_i v\,dxdt+8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_r v)^2\,dxdt\\
&+\int_0^T\int_{\mathbb{R}^3}(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2}-\frac{11}{2}\frac{1}{r^2})|\nabla_x v|^2\,dxdt-|||g|||_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\,dxdt\\
&=4\int_0^T\int_{\mathbb{R}^3}\frac{\phi''}{r^2}(\partial_r v)^2\,dxdt+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}(\frac{\phi''}{r^2}-\frac{11}{r^2})|\nabla_x v|^2\,dxdt\\
&\quad-|||g|||_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\,dxdt\\
&\gtrsim\|r^{-1}(\phi'')^{1/2}\nabla_x u\|^2_{L^2L^2},
\end{aligned}
$$

provided λ and R_{m_1} are sufficiently large.

So, we have shown

$$
\begin{aligned}
(5.21)\quad & \|r^{-1}(\phi'')^{1/2}(r^{-1}\phi' v,\nabla_x v)\|^2_{L^2L^2}+\|r^{-1}(\phi')^{1/2}\partial_t v\|^2_{L^2L^2}\\
&\lesssim\langle[P^{self},P^{skew}]v,v\rangle_{L^2L^2}+\langle(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})v,P^{self}v\rangle_{L^2L^2}+2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}+\tilde{E}_1[v]\Big|_0^T+\tilde{H}_1[v].
\end{aligned}
$$

Applying Cauchy's inequality on $\langle(\frac{1}{2}\frac{\phi''}{r^2}-4\frac{\phi'}{r^2})v,P^{self}v\rangle_{L^2L^2}$ and making use of the properties of ϕ in (5.2), we can bootstrap error terms into the left-hand side of (5.21) to obtain

$$
(5.22)\qquad \|r^{-1}(\phi'')^{1/2}(r^{-1}\phi' v,\nabla_x v)\|^2_{L^2L^2}+\|r^{-1}(\phi')^{1/2}\partial_t v\|^2_{L^2L^2}\lesssim\|P_\phi v\|^2_{L^2L^2}+\tilde{E}_1[v]\Big|_0^T+\tilde{H}_1[v].
$$

We can switch back to our original coordinates to find

$$\begin{aligned}(5.23)\quad &\|r^{-1}(\phi'')^{1/2}e^{\phi}(r^{-1}\phi' u,\nabla_x u)\|^2_{L^2L^2}+\|r^{-1}(\phi')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2}\\ &\qquad\lesssim \|e^{\phi}\tilde{P}u\|^2_{L^2L^2}+C_{R_{m1},R_{m2},\phi}\Big(\|\partial u\|^2_{L^\infty L^2}+\delta\log(2+T)\|u\|^2_{LE^1}\Big).\end{aligned}$$

Finally we estimate the error associated with working with $\tilde{P}$ as opposed to P_h:

$$\begin{aligned}(5.24)\qquad \|e^{\phi}(P_h-\tilde{P})u\|^2_{L^2L^2}=\|e^{\phi}\partial_\alpha h^{\alpha\beta}\partial_\beta u\|^2_{L^2L^2}&\lesssim C_{R_{m1},R_{m2},\phi}\int_0^T\int_{\mathbb{R}^3}|\partial h|^2|\partial u|^2\,dxdt\\ &\lesssim C_{R_{m1},R_{m2}}\delta\Big(\|\partial u\|^2_{L^\infty L^2}+\log(2+T)\|u\|^2_{LE^1}\Big),\end{aligned}$$

where we have applied (2.3) in the last line. This completes the proof of Proposition 5.1. □

We would like to pair Proposition 5.1 with the exterior estimate, Theorem 3.2, but this requires that our weight ϕ be constant at infinity, which breaks the convexity assumption. To overcome this, we permit a lower order error term localized to a region $R_{out}<r<2R_{ext}$ and modify ϕ such that ϕ is as in Proposition 5.1 for $r\le 2R_{out}$ and ϕ is constant for $r\ge 2R_{ext}$. Please see Appendix B for an explicit construction of such a function, which is based on the prior work of [21].

With this set up, we have the following proposition:

Proposition 5.2. *Let P_h be as in* (1.2), *where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta>0$ sufficiently small. Further $D_i g^{ij}D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4), *and asymptotically Euclidean in the sense of* (1.6). *Fix R_{AF} sufficiently large and fix $R_{ext}>R_{out}>R_{m1}>R_{AF}$, where R_{ext} and R_{out} are defined precisely in Appendix B and R_{m1} is as in Proposition 5.1. Let $\phi=\phi(s)$, where $s=\log(r)$, be the weight constructed in Appendix B. Observe ϕ satisfies*

$$\lambda\lesssim\phi''(s)\le\phi'(s)/2\lesssim\phi''(s),\qquad |\phi'''(s)|\lesssim\phi'(s),\qquad \lambda\gg 1$$

for $r\le 2R_{out}$ and is constant for $r\ge 2R_{ext}$.

Fix $T>0$. Suppose $u(t,x)\in C^2([0,T]\times\mathbb{R}^3)$ solves $P_h u=F\in LE^*$, with initial data $(u,\partial_t u)(0,\cdot)=(f_1,f_2)\in C^2(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$ such that $|\partial^{\le 2}u(t,x)|\to 0$ as $|x|\to\infty$ for all $t\in[0,T]$ and further assume $u(t,x)$ is supported in $\{r>R_{m1}\}$. Then the following bound holds:

$$\begin{aligned}(5.25)\quad \|r^{-1}(1+\phi_+'')^{1/2}e^{\phi}(r^{-1}(1+\phi')u,\nabla_x u)\|^2_{L^2L^2_{<2R_{ext}}} &+ \|r^{-1}(1+\phi')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2_{<2R_{ext}}} \\ &+ R_{ext}^{-1}\|e^{\phi}u\|^2_{LE^1_{>R_{ext}}} \\ \lesssim \|e^{\phi}P_h u\|^2_{L^2L^2_{<2R_{ext}}} + R_{ext}^{-1}\|e^{\phi}P_h u\|^2_{LE^*} &+ \|\langle r\rangle^{-2}(1+\phi')^{3/2}e^{\phi}u\|^2_{L^2L^2(R_{out}<r<2R_{ext})} \\ &+ C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).\end{aligned}$$

Here the implicit constant is independent of T and δ.

Remark. *Observe R_{out} corresponds to $R_{m2}/2$ in Proposition 5.1.*

Remark. *In the transition region between R_{out} and $2R_{ext}$, we may break the conditions on ϕ' and ϕ'', but will still have good bounds for ϕ' and ϕ''. In particular for $R_{out} \le r \le 3R_{ext}/2$, $\phi' \gtrsim 1$, but ϕ'' will be in part negative, with a bound from below $|\phi''| \le \phi'/2$. For $r \ge R_{ext}$, $|\phi'| + |\phi''| + |\phi'''| \ll 1$.*

Proof. We divide the analysis into the following three regions:

1. $s < \log(2R_{out})$
2. $\log(R_{out}) < s < \log(3R_{ext}/2)$
3. $\log(R_{ext}) < s$.

For Regions 1 and 2, we apply appropriate cut-off functions to our solution u and note that the error terms arising from commuting our operators with the cut-off functions are lower order and supported in the other regions. Hence, these error terms can be controlled with the methods from the other regions. We will elaborate more on this momentarily. The bound in Region 1 follows immediately from Proposition 5.1.

The proof for the bound of Region 2 is similar to the proof of Proposition 5.1. Instead of working with $\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + 2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2} + \langle(\frac{1}{2}\frac{\phi''}{r^2} - 4\frac{\phi'}{r^2})v, P^{self}v\rangle_{L^2L^2}$, we bound $\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + 2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2} - \langle \frac{\phi'}{r^2}v, P^{self}v\rangle_{L^2L^2}$ from below. We have essentially made all of the necessary computations in the proof of Proposition 5.1. Indeed, applying the methods from (5.14) and (5.15), we see

$$\begin{aligned}(5.26)\quad -\langle\frac{\phi'}{r^2}v, P^{self}v\rangle_{L^2L^2} \ge &-\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}g^{ij}\partial_i v\partial_j v\,dxdt + \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_t v)^2\,dxdt \\ &-\int_0^T\int_{\mathbb{R}^3}\partial_i\frac{\phi'}{r^2}g^{ij}\partial_j v v\,dxdt + \int_0^T\int_{\mathbb{R}^3}g^{ij}\frac{\phi'^3}{r^4}\frac{x_i}{r}\frac{x_j}{r}v^2\,dxdt + \tilde{E}_2[v]\Big|_0^T + \tilde{H}_2[v],\end{aligned}$$

where

$$\Big|\tilde{E}_2[v]\Big|_0^T\Big| + \Big|\tilde{H}_2[v]\Big| \lesssim C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).$$

The remaining terms $\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + 2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2}$ are still controlled by (5.12) and (5.13), up to h-dependent terms and time boundary terms controlled by $C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big)$, as usual. Hence, performing nearly identical analysis to that in (5.16), (5.17), (5.18), (5.20), and recalling we are now allowing for a lower order error term, we obtain the bound:

$$
\begin{aligned}
&\langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + 2\|(\phi')^{-1/2}P^{skew}v\|^2_{L^2L^2} - \langle \frac{\phi'}{r^2}v, P^{self}v\rangle_{L^2L^2} \\
&\quad \geq \int_0^T \int_{\mathbb{R}^3} \frac{\phi'}{r^2}(\partial_t v)^2\, dxdt + 4\int_0^T\int_{\mathbb{R}^3} \partial_j\big(\frac{\phi'}{r}\frac{x_k}{r}\big)\partial_k v \partial_j v\, dxdt + 8\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_r v)^2\, dxdt \\
&\qquad\quad - \int_0^T\int_{\mathbb{R}^3}\big(\frac{\phi'}{r^2} + \frac{11}{2}\frac{1}{r^2}\big)|\nabla_x v|^2\, dxdt - \|\!|g|\!\|_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\, dxdt \\
&\qquad\quad + \tilde{E}_3[v]\Big|_0^T + \tilde{H}_3[v] - C\|\langle r\rangle^{-2}(1+\phi')^{3/2}v\|^2_{L^2L^2(R_{out}<r<2R_{ext})} \\
&\quad = \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_t v)^2\, dxdt + \int_0^T\int_{\mathbb{R}^3}\big(3\frac{\phi'}{r^2} - \frac{11}{2}\frac{1}{r^2}\big)|\nabla_x v|^2\, dxdt + 4\int_0^T\int_{\mathbb{R}^3}\frac{\phi''}{r^2}(\partial_r v)^2\, dxdt \\
&\qquad\quad - \|\!|g|\!\|_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\, dxdt + \tilde{E}_3[v]\Big|_0^T + \tilde{H}_3[v] \\
&\qquad\quad - C\|\langle r\rangle^{-2}(1+\phi')^{3/2}v\|^2_{L^2L^2(R_{out}<r<2R_{ext})} \\
&\quad \gtrsim \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}(\partial_t v)^2\, dxdt + \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\, dxdt \\
&\qquad\quad - \|\!|g|\!\|_{>R_{m1}}\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r^2}|\nabla_x v|^2\, dxdt + \tilde{E}_3[v]\Big|_0^T + \tilde{H}_3[v] \\
&\qquad\quad - \|\langle r\rangle^{-2}(1+\phi')^{3/2}v\|^2_{L^2L^2(R_{out}<r<2R_{ext})},
\end{aligned}
$$

where C is a constant that does not depend on any important parameters and

$$\Big|\tilde{E}_3[v]\big|_0^T\Big| + \Big|\tilde{H}_3[v]\Big| \lesssim C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).$$

This yields the desired bound for Region 2.

The bound for Region 3 is perturbative off the exterior estimate, Theorem 3.2. Indeed, we are in the regime $s > \log(R_{ext})$, where $|\phi'| \ll 1$. Further for $r \geq 2R_{ext}$, ϕ is constant. An application of the Mean Value Theorem and the triangle inequality therefore shows $\phi(s) < \phi(\log(2R_{ext})) + 1$ and $\phi(\log(2R_{ext})) < \phi(s) + 1$ in this regime. Hence in Region 3, $\exp(\phi(s)) \approx \exp(\phi(\log(2R_{ext})))$. Multiplying both sides of (3.13) by $R_{ext}^{-1}\exp(\phi(\log(2R_{ext})))$, we obtain

$$R_{ext}^{-1}\exp(\phi(\log(2R_{ext})))\|u\|^2_{LE^1_{>R_{ext}}} \lesssim R_{ext}^{-1}\exp(\phi(\log(2R_{ext})))\|\partial u\|^2_{L^\infty L^2}$$
$$+ R_{ext}^{-3}\exp(\phi(\log(2R_{ext})))\|u\|^2_{LE_{R_{ext}}} + R_{ext}^{-1}\exp(\phi(\log(2R_{ext})))\int_0^T\int_{\mathbb{R}^3}|P_h u|\Big(|\partial u| + \frac{|u|}{\langle r\rangle}\Big)\,dxdt$$
$$+ R_{ext}^{-1}\exp(\phi(\log(2R_{ext})))\delta\log(2+T)\|u\|^2_{LE^1}.$$

Since $\exp(\phi(\log(2R_{ext}))) \approx \exp(\phi(s))$ in this region, we can move the exponential function inside of the integrals to obtain

$$(5.27)\quad R_{ext}^{-1}\|e^\phi u\|^2_{LE^1_{>R_{ext}}} \lesssim R_{ext}^{-3}\|e^\phi u\|^2_{LE_{R_{ext}}} + R_{ext}^{-1}\int_0^T\int_{\mathbb{R}^3} e^\phi|P_h u|\Big(|\partial u| + \frac{|u|}{\langle r\rangle}\Big)\,dxdt$$
$$+ C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big),$$

which is the desired bound for Region 3.

We now piece together our bounds in Regions 1, 2, and 3 with cut-off functions. For Region 1, we apply a smooth, monotonically decreasing, cut-off function $\beta_1(r)$ that is identically one for $r \le 3R_{out}/2$ and 0 for $r \ge 2R_{out}$. This introduces the following error term $\|e^\phi[P_h,\beta_1(r)]u\|^2_{L^2L^2}$.

Similarly, for Region 2, we apply a cut-off function $\beta_2(r)$ to u, where $\beta_2(r) \equiv 1$ for $3R_{out}/2 \le r \le R_{ext}$ and 0 for $R_{out} \le r$ and $r \ge 3R_{ext}/2$. Hence we need to bound the following commutator term:

$$\|e^\phi[P_h,\beta_2(r)]u\|^2_{L^2L^2} = \|e^\phi[P_h,\beta_2(r)]u\|^2_{L^2L^2_{R_{out}}} + \|e^\phi[P_h,\beta_2(r)]u\|^2_{L^2L^2_{R_{ext}}}.$$

Observe that the bound for Region 3 already has a spatial cut-off built in to the estimate. Therefore, combining our bounds in Regions 1, 2, and 3 along with the error terms, we have shown

$$(5.28)\quad \|r^{-1}(1+\phi''_+)^{1/2}e^\phi(r^{-1}(1+\phi')u,\nabla_x u)\|^2_{L^2L^2_{<2R_{ext}}} + \|r^{-1}(1+\phi')^{1/2}e^\phi\partial_t u\|^2_{L^2L^2_{<2R_{ext}}}$$
$$+ R_{ext}^{-1}\|e^\phi u\|^2_{LE^1_{>R_{ext}}} \lesssim \|e^\phi P_h u\|^2_{L^2L^2_{<2R_{ext}}} + R_{ext}^{-1}\int_0^T\int_{\mathbb{R}^3} e^\phi|P_h u|\Big(|\partial u| + \frac{|u|}{\langle r\rangle}\Big)\,dxdt$$
$$+ \|\langle r\rangle^{-2}(1+\phi')^{3/2}e^\phi u\|^2_{L^2L^2(R_{out}<r<2R_{ext})}$$
$$+ C_{R_{out},R_{ext},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big)$$
$$- \|e^\phi[P_h,\beta_1(r)]u\|^2_{L^2L^2} - \|e^\phi[P_h,\beta_2(r)]u\|^2_{L^2L^2_{R_{out}}} - \|e^\phi[P_h,\beta_2(r)]u\|^2_{L^2L^2_{R_{ext}}}.$$

Returning to the first error term in (5.28):

$$\|e^{\phi}[P_h, \beta_1(r)]u\|^2_{L^2L^2} \lesssim R_{out}^{-2}\|e^{\phi}\partial u\|^2_{L^2L^2_{R_{out}}} + R_{out}^{-4}\|e^{\phi}u\|^2_{L^2L^2_{R_{out}}}. \tag{5.29}$$

The last term on the right-hand side of the above equation is controlled by $\|\langle r\rangle^{-2}(1+\phi')^{3/2}e^{\phi}u\|^2_{L^2L^2(R_{out}<r<2R_{ext})}$ and so this is a harmless error term. The first term on the right-hand side of the equation, $R_{out}^{-2}\|e^{\phi}\partial u\|^2_{L^2L^2_{R_{out}}}$, can be bootstrapped into the left-hand side of (5.28) by using the largeness of ϕ'' in this regime.

The second error term in (5.28), $\|e^{\phi}[P_h, \beta_2(r)]u\|^2_{L^2L^2_{R_{out}}}$, can be bound in the exact same manner. The final error term can be controlled by observing:

$$\|e^{\phi}[P_h, \beta_2(r)]u\|^2_{L^2L^2_{R_{ext}}} \lesssim R_{ext}^{-2}\|e^{\phi}\partial u\|^2_{L^2L^2_{R_{ext}}} + R_{ext}^{-4}\|e^{\phi}u\|^2_{L^2L^2_{R_{ext}}}.$$

Both terms on the right-hand side of the above equation are immediately controlled by applying our exterior estimate, (5.27). An application of Cauchy's inequality on the $R_{ext}^{-1}\int_0^T\int_{\mathbb{R}^3} e^{\phi}|P_h u|\Big(|\partial u| + \frac{|u|}{\langle r\rangle}\Big)\,dxdt$ term in (5.28) completes the proof of Proposition 5.2. □

We need to pair Proposition 5.2 with an interior estimate. To this end, we consider the following Carleman estimate, motivated by the prior work of [29].

Proposition 5.3. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix R_{AF} sufficiently large and fix $R_{in} > R_{AF}$. Let $\phi = \phi(r)$ be the weight constructed in Appendix B. Observe ϕ satisfies*

$$\begin{aligned} &\phi'(0) = 0, \qquad \phi'' \approx \lambda + \sigma\phi', \qquad |\phi'''| \lesssim \sigma^2\phi', \qquad \lambda, \sigma \gg 1 \qquad \text{and} \\ &0 \le \phi'' - \frac{\phi'}{r} \lesssim_{\sigma} \phi' \qquad \forall\, r \qquad \text{while} \qquad \frac{\phi'}{r} \approx \phi'' \qquad \text{for } r \ll_{\sigma} 1. \end{aligned} \tag{5.30}$$

Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T]\times\mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$. Further assume u is supported in $\{r < R_{in}\}$. Then the following bound holds:

$$\|(\phi'/r)^{1/2}e^{\phi}\partial u\|^2_{L^2L^2} + \|(\phi'')^{1/2}\phi' e^{\phi} u\|^2_{L^2L^2} + \|(\phi'/r)e^{\phi}u\|^2_{L^2L^2}$$
$$\lesssim \|e^{\phi}P_h u\|^2_{L^2L^2} + \|(\phi'')^{1/2}e^{\phi}\partial_t u\|^2_{L^2L^2(|x|\gtrsim 1)}$$
$$+ C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta \log(2+T)\|u\|^2_{LE^1}\Big). \tag{5.31}$$

Here the implicit constant is independent of T and δ.

Remark. *Note that the assumptions on ϕ ensure that ϕ is increasing and that $\phi'/r \gtrsim \lambda$. Near the origin, morally, ϕ' acts as a linear function. Please see Appendix B for an explicit construction of a function satisfying these properties.*

Proof. The proof is similar in flavor to the proof of Proposition 5.1 and hence it suffices to work in conjugated coordinates and prove

$$\|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2} + \|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} + \|(\phi'/r)v\|^2_{L^2L^2}$$
$$\lesssim \|P_\phi v\|^2_{L^2L^2} + \|(\phi'')^{1/2}\partial_t v\|^2_{L^2L^2(|x|\gtrsim 1)} + \tilde{E}_4[v]\Big|_0^T + \tilde{H}_4[v], \tag{5.32}$$

where $\tilde{E}_4[v]\Big|_0^T$ and $\tilde{H}_4[v]$ are controlled by

$$\tilde{E}_4[v]\Big|_0^T \lesssim C_{R_{in},\phi}\|\partial u\|^2_{L^\infty L^2},$$

and

$$\tilde{H}_4[v] \lesssim C_{R_{in},\phi}\delta\Big(\|\partial u\|^2_{L^\infty L^2} + \log(2+T)\|u\|^2_{LE^1}\Big).$$

We use a positive commutator argument centered around the operator $[P^{self}, P^{skew}]$, as before. We can reuse many of our calculations provided we are careful by replacing ϕ'/r with ϕ' and $\phi''/r^2 - \phi'/r^2$ with ϕ'', as we are now working with a weight that is directly a function of r as opposed to a function of $\log(r)$. As in the proof of Proposition 5.1, we immediately bound

$$\langle [D_\alpha h^{\alpha\beta}D_\beta - h^{ij}(\phi')^2\frac{x_i}{r}\frac{x_j}{r}, P^{skew}]v, v\rangle_{L^2L^2}$$
$$+ \langle [-D_t^2 + D_i g^{ij} D_j - g^{ij}(\phi')^2\frac{x_i}{r}\frac{x_j}{r}, -\frac{1}{i}\phi'\frac{x_i}{r}h^{i\beta}D_\beta - \frac{1}{i}D_\beta\phi'\frac{x_i}{r}h^{i\beta}]v, v\rangle_{L^2L^2}$$
$$\lesssim C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \int_0^T\int_{\mathbb{R}^3}\big(|\partial^2 h| + |\partial h| + |h|\big)\big(|\partial u| + \frac{|u|}{\langle r\rangle}\big)^2\,dxdt\Big)$$
$$\lesssim C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big),$$

where we have applied (2.3) in the last line by choosing δ sufficiently small. So it suffices to consider

$$\int_0^T\int_{\mathbb{R}^3}[D_i g^{ij}D_j - g^{ij}(\phi')^2\frac{x_i}{r}\frac{x_j}{r}, -\frac{1}{i}\phi'\frac{x_l}{r}g^{lk}D_k - \frac{1}{i}D_k\phi'\frac{x_l}{r}g^{lk}]v\overline{v}\,dxdt,$$

which, from our prior calculations, we immediately see is equal to

$$\begin{aligned}(5.33)\quad & 4\int_0^T\int_{\mathbb{R}^3} g^{ij}\partial_j(\phi'\frac{x_l}{r}g^{lk})\partial_i v\partial_k v\,dxdt - 2\int_0^T\int_{\mathbb{R}^3}\phi'\frac{x_l}{r}g^{lk}\partial_k g^{ij}\partial_i v\partial_j v\,dxdt\\ &+2\int_0^T\int_{\mathbb{R}^3}\partial_k g^{ij}(\phi')^3\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}g^{lk}v^2\,dxdt + 4\int_0^T\int_{\mathbb{R}^3}(\phi')^2\phi''(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2v^2\,dxdt\\ &-4\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^3}{r}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2v^2\,dxdt + 4\int_0^T\int_{\mathbb{R}^3}g^{ik}g^{lk}\frac{(\phi')^3}{r}\frac{x_i}{r}\frac{x_l}{r}v^2\,dxdt\\ &+2\int_0^T\int_{\mathbb{R}^3}g^{ij}\partial_j\partial_k(\phi'\frac{x_l}{r}g^{lk})\partial_i vv\,dxdt.\end{aligned}$$

We break our analysis into two regimes. The first region is for $r \ll 1$ and the second region is for $r \gtrsim 1$.

1. Region 1: $r \ll 1$. We note that in this regime $\phi'/r \approx \phi''$. Also, recall the useful bound $\phi'/r \gtrsim \lambda$. We begin by examining the highest order terms in (5.33):

$$\begin{aligned}(5.34)\quad & 4\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}g^{ij}g^{jk}\partial_i v\partial_k v\,dxdt + 4\int_0^T\int_{\mathbb{R}^3}(\phi'' - \frac{\phi'}{r})g^{ij}g^{lk}\frac{x_j}{r}\frac{x_l}{r}\partial_i v\partial_k v\,dxdt\\ &+4\int_0^T\int_{\mathbb{R}^3}g^{ij}\phi'\frac{x_l}{r}\partial_j g^{lk}\partial_i v\partial_k v\,dxdt - 2\int_0^T\int_{\mathbb{R}^3}g^{lk}\phi'\frac{x_l}{r}\partial_k g^{ij}\partial_i v\partial_j v\,dxdt\\ &\gtrsim \sum_{i=1}^3\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}|g^{ij}\partial_j v|^2\,dxdt \gtrsim \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}|\nabla_x v|^2\,dxdt.\end{aligned}$$

We elaborate on the bound from below in (5.34). As g^{ij} is strictly elliptic, $\sum_i |g^{ij}\partial_j v|^2 \gtrsim |\nabla_x v|^2$. Indeed, $|\nabla_x v|^2 \lesssim g^{ij}\partial_j v\partial_i v$, and so the bound follows by applying Cauchy's inequality. Therefore, for $r \ll 1$, we can absorb the last two terms into the first term. Observing that the second term is nonnegative completes the bound.

We now examine the lower order terms in (5.33):

$$\begin{aligned}(5.35)\quad & 2\int_0^T\int_{\mathbb{R}^3}\partial_k g^{ij}(\phi')^3\frac{x_i}{r}\frac{x_j}{r}\frac{x_l}{r}g^{lk}v^2\,dxdt + 4\int_0^T\int_{\mathbb{R}^3}(\phi')^2\phi''(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2v^2\,dxdt\\ &-4\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^3}{r}(g^{ij}\frac{x_i}{r}\frac{x_j}{r})^2v^2\,dxdt + 4\int_0^T\int_{\mathbb{R}^3}g^{ik}g^{lk}\frac{(\phi')^3}{r}\frac{x_i}{r}\frac{x_l}{r}v^2\,dxdt\\ &\gtrsim \int_0^T\int_{\mathbb{R}^3}(\phi')^2\phi''v^2\,dxdt,\end{aligned}$$

where we have used $(\phi')^2\phi'' - (\phi')^3/r \geq 0$, $\phi'' \approx \phi'/r$, and that g is strictly elliptic so that $g^{ik}g^{lk}\frac{x_i}{r}\frac{x_l}{r} \gtrsim 1$. We now examine the remaining term in (5.33):

$$2\int_0^T \int_{\mathbb{R}^3} g^{ij}\partial_j\partial_k(\phi'\frac{x_l}{r}g^{lk})\partial_i v v\, dxdt.$$

Taking derivatives and grouping like terms, the above integral becomes

(5.36)

$$\begin{aligned}
2\int_0^T \int_{\mathbb{R}^3} g^{ij}(\phi'' - \frac{\phi'}{r})\Big(-\frac{3}{r}\frac{x_j}{r}\frac{x_l}{r}\frac{x_k}{r}g^{lk} + \frac{1}{r}\frac{x_j}{r}\delta_{lk}g^{lk} + \frac{1}{r}\frac{x_l}{r}\delta_{jk}g^{lk} + \frac{1}{r}\frac{x_k}{r}\delta_{jl}g^{lk} + \frac{1}{r}\frac{x_k}{r}\frac{x_l}{r}\partial_j g^{lk}\Big)\partial_i v v\, dxdt \\
+ 2\int_0^T \int_{\mathbb{R}^3} g^{ij}\Big(\phi'''\frac{x_k}{r}\frac{x_j}{r}\frac{x_l}{r}g^{lk} + \phi''\frac{x_j}{r}\frac{x_l}{r}\partial_k g^{lk} + \frac{\phi'}{r}\delta_{jl}\partial_k g^{lk} - \frac{\phi'}{r}\frac{x_l}{r}\frac{x_k}{r}\partial_j g^{lk} \\
+ \frac{\phi'}{r}\delta_{kl}\partial_j g^{lk} + \frac{\phi'}{r}\frac{x_l}{r}\partial^2_{jk}g^{lk}\Big)\partial_i v v\, dxdt.
\end{aligned}$$

Utilizing (5.30) and Cauchy's inequality, we find that the above line is bounded by

$$\|(\phi'/r)^{3/4}v\|^2_{L^2L^2} + \|(\phi'/r)^{1/4}\partial v\|^2_{L^2L^2}. \tag{5.37}$$

Using the largeness of ϕ'/r, we will be able to absorb $\|(\phi'/r)^{1/4}\partial v\|^2_{L^2L^2}$ into (5.34). We will see that we can absorb $\|(\phi'/r)^{3/4}v\|^2_{L^2L^2}$ later.

In order to get useful bounds on terms involving time derivatives, we consider $-\int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}v\overline{P^{self}v}\,dxdt$ and note that as usual

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}^3} \frac{\phi'}{r}v\overline{D_\alpha h^{\alpha\beta}D_\beta v}\, dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{\phi'}{r}\overline{v h^{ij}(\phi')^2\frac{x_i}{r}\frac{x_j}{r}v}\, dxdt \\
\lesssim C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \int_0^T\int_{\mathbb{R}^3}\big(|\partial^2 h| + |\partial h| + |h|\big)\big(\|\partial u| + \frac{|u|}{\langle r\rangle}\big)^2\, dxdt\Big) \\
\lesssim C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).
\end{aligned}$$

Therefore it suffices to consider

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}^3} \frac{\phi'}{r}(P^{self}v - D^\alpha h^{\alpha\beta}D_\beta v + h^{ij}(\phi')^2\frac{x_i}{r}\frac{x_j}{r}v)\overline{v}\, dxdt \\
= \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}(\partial_t v)^2\, dxdt - \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}g^{ij}\partial_i v\partial_j v\, dxdt \\
+ \int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^3}{r}g^{ij}\frac{x_i}{r}\frac{x_j}{r}v^2\, dxdt + \int_0^T\int_{\mathbb{R}^3}(\frac{\phi'}{r^2} - \frac{\phi''}{r})g^{ij}\frac{x_i}{r}\partial_j v v\, dxdt - \int_{\mathbb{R}^3}\frac{\phi'}{r}\partial_t v v\, dx\Big|_0^T.
\end{aligned} \tag{5.38}$$

The time boundary term is controlled by $C_{R_{in},\phi}\|\partial u\|^2_{L^\infty L^2}$.

Combining (5.34), (5.35), (5.37), and (5.38), we see for $c > 0$ chosen sufficiently small,

$$\|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} + \|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2} \lesssim \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} - c\langle \frac{\phi'}{r}v, P^{self}v\rangle_{L^2L^2} + \tilde{E}_5[v]\Big|_0^T + \tilde{H}_5[v] + \|(\phi'/r)^{3/4}v\|^2_{L^2L^2}, \tag{5.39}$$

where $\tilde{E}_5[v]\Big|_0^T + \tilde{H}_5[v]$ are controlled by

$$C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta \log(2+T)\|u\|^2_{LE^1}\Big).$$

One can add $\|r^{-1}\phi' v\|^2_{L^2L^2}$ to the left-hand side of (5.39). Recall, we are in the regime $r \ll 1$. Utilizing $\phi'/r \approx \phi''$ in this regime, it suffices to show $\|\phi'' v\|^2_{L^2L^2} \lesssim \|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} + \|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2}$. Making use of polar coordinates, we have

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^3}(\phi'')^2v^2\,dxdt =&\frac{1}{3}\int_0^T\int_{\mathbb{S}^2}\int_0^\infty(\phi'')^2v^2\partial_r(r^3)\,drd\sigma(\omega)dt\\
=&-\frac{2}{3}\int_0^T\int_{\mathbb{R}^3}\phi''\phi'''rv^2\,dxdt - \frac{2}{3}\int_0^T\int_{\mathbb{R}^3}(\phi'')^2rv\partial_r v\,dxdt\\
\lesssim_\sigma&\int_0^T\int_{\mathbb{R}^3}\phi'\phi''rv^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}(\phi'')^2r|v||\partial_r v|\,dxdt\\
\lesssim_\sigma&\int_0^T\int_{\mathbb{R}^3}\frac{(\phi')^2}{\lambda}\phi''v^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}(\phi'')^2r|v||\partial_r v|\,dxdt\\
\lesssim_\sigma&\|(\phi'')^{1/2}\phi' v\|^2_{L^2} + \int_0^T\int_{\mathbb{R}^3}\phi''\phi'|v||\partial_r v|\,dxdt.
\end{aligned}$$

So it suffices to bound $\int_0^T\int_{\mathbb{R}^3}\phi''\phi'|v||\partial_r v|\,dxdt$:

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^3}\phi''\phi'|v||\partial_r v|\,dxdt &\lesssim_\sigma \int_0^T\int_{\mathbb{R}^3}(\phi'')^2r\phi'|v|^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}|\partial v|^2\,dxdt\\
&\lesssim_\sigma \int_0^T\int_{\mathbb{R}^3}\phi''(\phi')^2|v|^2\,dxdt + \int_0^T\int_{\mathbb{R}^3}\frac{\phi'}{r}|\partial v|^2\,dxdt,
\end{aligned}$$

as desired. Hence, we have shown

$$\|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} + \|(\phi'/r)v\|^2_{L^2L^2} + \|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2} \lesssim \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} - c\langle P^{self}v, \frac{\phi'}{r}v\rangle_{L^2L^2} + \tilde{E}_5[v]\Big|_0^T + \tilde{H}_5[v] + \|(\phi'/r)^{3/4}v\|^2_{L^2L^2}. \tag{5.40}$$

Using the largeness of ϕ'/r, we can bootstrap the error term $\|(\phi'/r)^{3/4}v\|^2_{L^2L^2}$ into the second term on the right-hand side of the equation. Applying Cauchy's inequality, conjugating back to our regular coordinates, and recalling (5.24) proves Proposition 5.3 for $r \ll 1$.

2. Region II: $r \gtrsim 1$. The result in this region is almost immediate due to our prior calculations. Indeed, the left-hand side of (5.34) is bounded from below by $-\|(\phi'/r)^{1/2}\nabla_x v\|^2_{L^2}$. The bound in (5.35) still holds which is clear by using $\phi'' \approx \lambda + \sigma\phi'$ to absorb poorly signed terms into the second term of the left-hand side of the equation.

The terms in (5.36) are now controlled by $\|\phi' v\|^2_{L^2L^2} + \|\lambda^{1/2}v\|^2_{L^2L^2} + \|\lambda^{1/2}\partial v\|^2_{L^2L^2}$, which is clear for all terms by utilizing our assumptions in (5.30) and Cauchy's inequality, with perhaps the exception of the second term on the second line of the equation. We control this term via

$$\int_0^T \int_{\mathbb{R}^3} g^{ij}\phi''\frac{x_j}{r}\frac{x_l}{r}\partial_k g^{lk}\partial_i v v\, dxdt \lesssim \int_0^T \int_{\mathbb{R}^3} \lambda|\nabla_x v||v|\, dxdt + \sigma\int_0^T \int_{\mathbb{R}^3} \phi'|\nabla_x v||v|\, dxdt,$$

and then apply Cauchy's inequality. Hence,

$$\begin{aligned}\|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} &\lesssim \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + \|\phi' v\|^2_{L^2L^2} + \|\lambda^{1/2}v\|^2_{L^2L^2} \\ &\qquad + \|\lambda^{1/2}\partial v\|^2_{L^2L^2} + \tilde{E}_6[v]\Big|_0^T + \tilde{H}_6[v] \\ &\lesssim \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + \|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2} + \tilde{E}_6[v]\Big|_0^T + \tilde{H}_6[v],\end{aligned}$$

where we have used the largeness of ϕ'' and ϕ' to absorb the second and third terms on the right-hand side of the equation into the left-hand side and made use of the fact $\phi'/r \gtrsim \lambda$ in the last line. Here $\tilde{E}_6[v]\Big|_0^T + \tilde{H}_6[v]$ are controlled by

$$C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).$$

Since we computed $-\langle(\phi'/r)v, P^{self}v\rangle_{L^2L^2}$ in (5.38), we reuse this computation and Cauchy's inequality to see

$$\begin{aligned}\|(\phi'')^{1/2}\phi' v\|^2_{L^2L^2} + \|(\phi'/r)^{1/2}\partial v\|^2_{L^2L^2} &\lesssim \langle [P^{self}, P^{skew}]v, v\rangle_{L^2L^2} + C_1\langle(\phi'/r)v, P^{self}v\rangle_{L^2L^2} \\ &\qquad + C_2\|(\phi'/r)^{1/2}\partial_t v\|^2_{L^2L^2} + \tilde{E}_7[v]\Big|_0^T + \tilde{H}_7[v],\end{aligned} \tag{5.41}$$

where $C_2 > C_1 \gg 1$ are chosen significantly large, but are independent of λ, and $\tilde{E}_7[v]\Big|_0^T + \tilde{H}_7[v]$ is controlled by

$$C_{R_{in},\phi}\Big(\|\partial u\|^2_{L^\infty L^2} + \delta\log(2+T)\|u\|^2_{LE^1}\Big).$$

An application of Cauchy's inequality and conjugating back to our original coordinates now yields Proposition 5.3 in this regime. This completes the proof of Proposition 5.3. □

We are now able to prove Theorem 5.1.

Proof of Theorem 5.1. We prove Theorem 5.1 by combining Proposition 5.2 and Proposition 5.3. We fix R_{in} sufficiently large so that Proposition 5.3 holds for u supported in $\{r \leq R_{in}\}$ and Proposition 5.2 holds for u supported in $\{r \geq R_{in}/4\}$. We define ϕ_{in} from Proposition 5.3 by integrating ϕ'_{in} beginning at $R_{in}/2$ and ϕ_{out} from Proposition 5.2 by integrating ϕ'_{out} also beginning at $R_{in}/2$. Further, we ensure $\phi'_{out} > \phi'_{in}$ for $R_{in}/4 \leq r \leq R_{in}$ by multiplying ϕ'_{out} by an appropriate constant if needed. Observe this ensures $\phi_{out} > \phi_{in}$ for $R_{in}/2 < \phi \leq R_{in}$ and $\phi_{in} > \phi_{out}$ for $R_{in}/4 \leq r < R_{in}/2$. To be precise, we define $\phi_{out}(r) = C\int_{R_{in}/2}^{r} \phi'_{out}(\log(u))\,du$ and $\phi_{in}(r) = \int_{R_{in}/2}^{r} \phi'_{in}(u)\,du$. Here ϕ'_{out} and ϕ'_{in} are as in Appendix B.

Applying Proposition 5.3 to $\chi_{R_{in}/2}(|x|)u$ and Proposition 5.2 to $(1-\chi_{R_{in}/4})(|x|)u$ yields Theorem 5.1 provided we can bound the following commutator terms: $\|e^{\phi_{in}}[P_h, \chi_{R_{in}/2}(|x|)]u\|^2_{L^2L^2}$ and $\|e^{\phi_{out}}[P_h, \chi_{R_{in}/4}(|x|)]u\|^2_{L^2L^2}$. We control the first term via:

$$\|e^{\phi_{in}}[P_h, \chi_{R_{in}/2}(|x|)]u\|^2_{L^2L^2} \lesssim R_{in}^{-2}\|e^{\phi_{in}}\partial u\|^2_{L^2L^2_{R_{in}/2}} + R_{in}^{-4}\|e^{\phi_{in}}u\|^2_{L^2L^2_{R_{in}/2}}.$$

Both of these terms are supported where Proposition 5.2 holds and $\phi_{out} > \phi_{in}$. Hence, they can be absorbed into the left-hand side of Proposition 5.2 using the largeness of ϕ' and ϕ'' in this regime.

The bound for $\|e^{\phi_{out}}[P_h, \chi_{R_{in}/4}(|x|)]u\|^2_{L^2L^2}$ is similar. Indeed,

$$\|e^{\phi_{out}}[P_h, \chi_{R_{in}/4}(|x|)]u\|^2_{L^2L^2} \lesssim R_{in}^{-2}\|e^{\phi_{out}}\partial u\|^2_{L^2L^2_{R_{in}/4}} + R_{in}^{-4}\|e^{\phi_{out}}u\|^2_{L^2L^2_{R_{in}/4}}.$$

Both of these terms are supported where Proposition 5.3 hold and $\phi_{in} > \phi_{out}$. Hence, they can be absorbed into the left-hand side of Proposition 5.3 using the largeness of ϕ' and ϕ'' in this regime. This completes the proof of Theorem 5.1. □

Chapter 6: Low Frequency Analysis

We now turn to developing a local energy estimate that admits an error term that can be absorbed for small time frequencies. These methods are similar to those developed in [29]. We now state the main theorem of this section.

Theorem 6.1. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix R_{AF} sufficiently large. Further, fix R_1 such that $R_1 \gg R_{AF}$ and fix $T > 0$. Suppose $u(t,x) \in C^2([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ such that $|\partial^{\leq 2} u(t,x)| \to 0$ as $|x| \to \infty$ for all $t \in [0,T]$. Then*

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + C_{R_1}\Big(\|P_h u\|_{LE^*}^2 + \|\partial_t u\|_{LE^1_{<R_1}}^2\Big) + \delta \log(2+T)\|u\|_{LE^1}^2. \tag{6.1}$$

Here the implicit constant is independent of T and δ.

We prove Theorem 6.1 by utilizing a series of reductions based on the properties of the fundamental solution to Poisson's equation. Throughout the rest of this section, we will make use of the results found in the low frequency section of [29, Section 6], often directly expositing their results. The next lemma is a restatement of a lemma found in [29].

Lemma 6.1 ([29, Lemma 6.4])**.** *For $|s| = 0,1,2$, we have the following estimates for $D_x^s \Delta^{-1}$:*

$$\|\langle x\rangle^{-1} u\|_{\mathcal{LE}} \lesssim \|\Delta u\|_{\mathcal{LE}^*} \tag{6.2}$$

$$\|\langle x\rangle^{-2+|s|} D_x^s u\|_{\mathcal{LE}^*} \lesssim \|\Delta u\|_{\mathcal{LE}^*} \quad |s| = 1,2 \tag{6.3}$$

$$\|\langle x\rangle^{|s|} D_x^s u\|_{\mathcal{LE}} \lesssim \|\langle x\rangle \Delta u\|_{\mathcal{LE}^*} \quad |s| = 0,1,2. \tag{6.4}$$

Proof. We will prove (6.3) for $|s| = 1$ since the other proofs are very similar. The proof of this lemma hinges on the decay rate of the fundamental solution to Poisson's equation. Hence we consider $\Delta u(x) = f(x)$ and write $f = \sum_k f_k$, where each f_k is supported on a single dyadic region $|x| \approx 2^k$. This follows from choosing an appropriate partition of unity. Let u_k denote a solution to $\Delta u_k = f_k$. Appealing to the fundamental solution to Poisson's equation and the support of f_k, we see

$$u_k(x) = \frac{1}{4\pi}\int_{|y|\approx 2^k} \frac{1}{|x-y|} f_k(y)\, dy,$$

from which it follows

$$|\nabla u_k| \approx \Big|\int_{|y|\approx 2^k} \frac{1}{|x-y|^2} f_k(y)\, dy\Big|. \tag{6.5}$$

We can apply another partition of unity to write each u_k as $\sum_j u_{kj}$, where u_{kj} is supported where $|x| \approx 2^j$. Now if $k-1 \le j \le k+1$, (6.5) can be written as

$$|\nabla u_{kj}| \approx \Big|\int_{|y|\approx 2^k} \frac{\beta_0(|x-y|)}{|x-y|^2} f_k(y)\, dy\Big|,$$

where $\beta_0(|x|)$ is a cut-off function that is identically 1 for $0 \le |x| \le 2^{j+2}$ and is supported on a slightly fattened interval. An application of Young's convolution inequality shows

$$\|\nabla u_{kj}\|_{L^2} \lesssim \|\beta_0(|x|)|x|^{-2}\|_{L^1}\|f_k\|_{L^2} \lesssim 2^j\|f_k\|_{L^2}. \tag{6.6}$$

Now if $k \ge j+2$ we find (6.5) becomes

$$\nabla u_{kj} \approx 2^{-2k}\int_{|y|\approx 2^k} \frac{1}{|\frac{x}{2^k}-1|^2} f_k(y)\, dy \lesssim 2^{-2k}\int_{|y|\approx 2^k} f_k(y)\, dy \lesssim 2^{-k/2}\|f_k\|_{L^2}.$$

Hence,

$$2^{-j/2}\|\nabla u_{kj}\|_{L^2} \lesssim 2^j 2^{-k/2}\|f_k\|_{L^2}. \tag{6.7}$$

Lastly, for $k \le j-2$, (6.5) becomes

$$\nabla u_{kj} \approx 2^{-2j}\int_{|y|\approx 2^k} \frac{1}{|1-\frac{y}{2^j}|^2} f_k(y)\, dy \lesssim 2^{-2j}\int_{|y|\approx 2^k} f_k(y)\, dy \lesssim 2^{-2j}2^{3k/2}\|f_k\|_{L^2}.$$

Hence,

$$2^{-j/2}\|\nabla u_{kj}\|_{L^2} \lesssim 2^{-j}2^{3k/2}\|f_k\|_{L^2}. \tag{6.8}$$

We now must combine our bounds in (6.6), (6.7), and (6.8). We begin by observing

$$\begin{aligned}\|\langle x\rangle^{-1/2}\nabla u\|_{L^2} &\lesssim \sum_k \sum_j 2^{-j/2}\|\nabla u_{kj}\|_{L^2} \\ &\lesssim \sum_{k=0}^{\infty}\sum_{j=0}^{k-2} 2^{-j/2}\|\nabla u_{kj}\|_{L^2} + \sum_{k=0}^{\infty}\sum_{j=k-1}^{k+1} 2^{-j/2}\|\nabla u_{kj}\|_{L^2} \\ &\qquad + \sum_{k=0}^{\infty}\sum_{j=k+2}^{\infty} 2^{-j/2}\|\nabla u_{kj}\|_{L^2}.\end{aligned}$$

We now bound each term in the above equation. For the first term, we apply (6.7) to see

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k-2} 2^{-j/2}\|\nabla u_{kj}\|_{L^2} \lesssim \sum_{k=0}^{\infty}\sum_{j=0}^{k-2} 2^{j}2^{-k/2}\|f_k\|_{L^2} \le \sum_{k=0}^{\infty} 2^{-k/2}(2^{k-1}-1)\|f_k\|_{L^2} \lesssim \|f\|_{\mathcal{LE}^*}.$$

For the second term, we use (6.6) to observe

$$\sum_{k=0}^{\infty}\sum_{j=k-1}^{k+1} 2^{-j/2}\|\nabla u_{kj}\|_{L^2} \lesssim \sum_{k=0}^{\infty}\sum_{j=k-1}^{k+1} 2^{j/2}\|f_k\|_{L^2} \approx \sum_{k=0}^{\infty} 2^{k/2}\|f_k\|_{L^2} \approx \|f\|_{\mathcal{LE}^*}.$$

For the third term, an application of (6.8) yields the desired bound and completes the proof:

$$\sum_{k=0}^{\infty}\sum_{j=k+2}^{\infty} 2^{-j/2}\|\nabla u_{kj}\|_{L^2} \lesssim \sum_{k=0}^{\infty}\sum_{j=k+2}^{\infty} 2^{-j}2^{3k/2}\|f_k\|_{L^2} = \sum_{k=0}^{\infty} 2^{-2-k}2^{3k/2}\|f_k\|_{L^2} \lesssim \|f\|_{\mathcal{LE}^*}.$$

□

Our next lemma is essentially a restatement of a result from [29] applied to a "zero frequency" version of our operator P_h. The operator $P_{h,0}$ is obtained from P_h by setting instances of $D_t = 0$. Specifically, we will investigate the operator:

$$P_{h,0} = D_i g^{ij} D_j + h^{ij} D_i D_j, \tag{6.9}$$

where $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of (1.4), and asymptotically Euclidean in the sense of (1.6). Further h^{ij} is symmetric and sufficiently small in the sense that (1.11) holds for some $\delta > 0$ sufficiently small.

Lemma 6.2. *Let $P_{h,0}$ be as in* (6.9)*, where h^{ij} is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix R_{AF} sufficiently large. Further, fix R_1 such that $R_1 \gg R_{AF}$ and fix $T > 0$. Suppose*

$u(t,x) \in C^2([0,T]\times\mathbb{R}^3)$ *such that* $|\partial^{\leq 2}u(t,x)| \to 0$ *as* $|x|\to\infty$ *for all* $t\in[0,T]$. *Then*

$$\|\langle x\rangle u\|_{\mathcal{LE}^1} \lesssim \|\langle x\rangle P_{h,0}u\|_{\mathcal{LE}^*}, \tag{6.10}$$

$$\|u\|_{\mathcal{LE}^1} + \|\langle x\rangle^{-1}\nabla_x u\|_{\mathcal{LE}^*} \lesssim \|P_{h,0}u\|_{\mathcal{LE}^*}, \tag{6.11}$$

$$\|u\|_{\mathcal{LE}^1_{<R_1}} + \|\langle x\rangle^{-1}\nabla_x u\|_{\mathcal{LE}^*_{<R_1}} \lesssim \|P_{h,0}u\|_{\mathcal{LE}^*_{<R_1}} + \|u\|_{\mathcal{LE}^1_{R_1}}. \tag{6.12}$$

Proof. This proof is essentially an argument in elliptic regularity and appeals to Lemma 6.1. In fact, if we replace $P_{h,0}$ by Δ, then (6.10) and (6.11) immediately follow from Lemma 6.1. Observe, one can write the Laplacian as

$$-\Delta = P_{h,0} - \tilde{g}^{ij}D_iD_j - \frac{1}{i}\partial_i\tilde{g}^{ij}D_j - h^{ij}D_iD_j,$$

where $\tilde{g}^{ij} = g^{ij} - \delta^{ij}$.

Now consider an operator $\tilde{P}_{h,0}$ which agrees with $P_{h,0}$ for $r > R_{AF}$ and is a small, asymptotically Euclidean perturbation of $\Box$ when $h=0$. Observe that the Laplacian is related to $\tilde{P}_{h,0}$ via:

$$-\Delta = \tilde{P}_{h,0} - \phi_2^{ij}(x)D_iD_j - \frac{1}{i}\phi_1^j(x)D_j - h^{ij}D_iD_j, \tag{6.13}$$

where $\|\phi_2\|_{\ell^1_l L^\infty(A_l)} + \|\langle x\rangle\phi_1\|_{\ell^1_l L^\infty(A_l)} < \varepsilon \ll 1$. Now (6.10) and (6.11) holds for $\tilde{P}_{h,0}$ as we can directly see via the triangle inequality and Lemma 6.1:

$$\begin{aligned}\|\langle x\rangle\Delta u\|_{\mathcal{LE}^*} \lesssim \|\langle x\rangle\tilde{P}_{h,0}u\|_{\mathcal{LE}^*} + \|\phi_2\|_{\ell^1L^\infty(A_l)}\|\langle x\rangle^2\partial_x^2u\|_{\mathcal{LE}} + \|\langle x\rangle\phi_1\|_{\ell^1L^\infty(A_l)}\|\langle x\rangle\partial_x u\|_{\mathcal{LE}}\\ + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)}\|\langle x\rangle^2\partial_x^2u\|_{\mathcal{LE}}.\end{aligned} \tag{6.14}$$

Here we are using ∂_x^2 to denote $\sum_{|\mu|=2}\partial_x^\mu$. By fixing R_{AF} sufficiently large (which forces ε to be small) and δ sufficiently small, we find ϕ_1, ϕ_2, and h are small enough to absorb the error terms using Lemma 6.1. Indeed, the error terms are controlled by

$$\begin{aligned}\|\phi_2\|_{\ell^1L^\infty}\|\langle x\rangle^2\partial_x^2u\|_{\mathcal{LE}} + \|\langle x\rangle\phi_1\|_{\ell^1L^\infty(A_l)}\|\langle x\rangle\partial_x u\|_{\mathcal{LE}} + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)}\|\langle x\rangle^2\partial_x^2u\|_{\mathcal{LE}}\\ < (\varepsilon+2\delta)\|\langle x\rangle^2\partial_x^2u\|_{\mathcal{LE}} + \varepsilon\|\langle x\rangle\partial_x u\|_{\mathcal{LE}}\\ < 2(\varepsilon+\delta)\|\langle x\rangle\Delta u\|_{\mathcal{LE}^*},\end{aligned} \tag{6.15}$$

where we have applied (6.4) in the last line. We note that the smallness assumption in (1.11) implies $\|h\|_{L^\infty L^\infty([0,T]\times A_l)} < \delta 2^{-l}$, so that $\|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)} < 2\delta$. By choosing ε and δ sufficiently small, these terms can be bootstrapped into the left-hand side of (6.14). This proves the bound for (6.10) for small perturbations. The proof for (6.11) is similar.

Now we write $u = \tilde{P}_{h,0}^{-1} P_{h,0} u + \tilde{u}$, noting $\tilde{P}_{h,0}^{-1}$ exists as $\tilde{P}_{h,0}$ is a small perturbation of the Laplacian. Applying (6.10) and (6.11) for the operator $\tilde{P}_{h,0}$ to $\tilde{P}_{h,0}^{-1} P_{h,0} u$, we obtain the desirable bounds:

$$\|\langle x\rangle \tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^1} \lesssim \|\langle x\rangle P_{h,0} u\|_{\mathcal{LE}^*}$$

and

$$\|\tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^1} + \|\langle x\rangle^{-1} \nabla_x \tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^*} \lesssim \|P_{h,0} u\|_{\mathcal{LE}^*}.$$

We now obtain similar bounds for $\tilde{u}$. Note that $P_{h,0}\tilde{u}$ solves the following equation:

$$P_{h,0}\tilde{u} = \left(\tilde{P}_{h,0} - P_{h,0}\right) \tilde{P}_{h,0}^{-1} P_{h,0} u$$

and is supported in $\{|x| < R_{AF}\}$. Hence

$$|P_{h,0}\tilde{u}| \lesssim |D_x^2 \tilde{P}_{h,0}^{-1} P_{h,0} u| + R_{AF} \langle x\rangle^{-1} |D_x \tilde{P}_{h,0}^{-1} P_{h,0} u|.$$

An application of (6.3) to $\tilde{P}_{h,0}$ yields

$$\begin{aligned} \|P_{h,0}\tilde{u}\|_{L^2} \lesssim \|P_{h,0}\tilde{u}\|_{\mathcal{LE}^*} &\lesssim \|D_x^2 \tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^*} + R_{AF} \|\langle x\rangle^{-1} D_x \tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^*} \\ &\lesssim R_{AF} \|\tilde{P}_{h,0} \tilde{P}_{h,0}^{-1} P_{h,0} u\|_{\mathcal{LE}^*} \approx R_{AF} \|P_{h,0} u\|_{\mathcal{LE}^*}. \end{aligned} \tag{6.16}$$

As we allow constants to depend on R_{AF} in this chapter, we will no longer track instances of R_{AF} in the remaining analysis. Combining the above line with the strict ellipticity assumption on $D_i g^{ij} D_j$, our smallness assumption on h, and the compact support of $P_{h,0}\tilde{u}$, we obtain

$$\begin{aligned} \|\nabla_x \tilde{u}\|_{L^2}^2 &\lesssim \int_{\mathbb{R}^3} P_{h,0}\tilde{u}\overline{\tilde{u}}\, dx - \int_{\mathbb{R}^3} h^{ij} D_i D_j \tilde{u}\overline{\tilde{u}}\, dx \\ &= \int_{\mathbb{R}^3} \chi_{R_{AF}}(|x|) P_{h,0}\tilde{u}\overline{\tilde{u}}\, dx - \int_{\mathbb{R}^3} h^{ij} \partial_j \tilde{u} \overline{\partial_i \tilde{u}}\, dx - \int_{\mathbb{R}^3} \partial_i h^{ij} \partial_j \tilde{u}\overline{\tilde{u}}\, dx \\ &\lesssim \frac{1}{\tilde{\varepsilon}} \|P_{h,0}\tilde{u}\|_{L^2}^2 + \tilde{\varepsilon} \|\langle x\rangle^{-1} \tilde{u}\|_{L^2}^2 - \int_{\mathbb{R}^3} h^{ij} \partial_j \tilde{u} \partial_i \tilde{u}\, dx - \int_{\mathbb{R}^3} \partial_i h^{ij} \partial_j \tilde{u}\tilde{u}\, dx \\ &\lesssim \frac{1}{\tilde{\varepsilon}} \|P_{h,0}\tilde{u}\|_{L^2}^2 + \tilde{\varepsilon} \|\langle x\rangle^{-1} \tilde{u}\|_{L^2}^2 + \|h\|_{L^\infty L^\infty} \|\nabla_x \tilde{u}\|_{L^2}^2 - \int_{\mathbb{R}^3} \partial_i h^{ij} \partial_j \tilde{u}\tilde{u}\, dx \end{aligned}$$

$$\lesssim \frac{1}{\tilde{\varepsilon}}\|P_{h,0}\tilde{u}\|_{L^2}^2 + \tilde{\varepsilon}\|\langle x\rangle^{-1}\tilde{u}\|_{L^2}^2 + \|\langle x\rangle\partial^{\le 1}h\|_{L^\infty L^\infty}\|\nabla_x\tilde{u}\|_{L^2}^2 + \|\langle x\rangle\partial h\|_{L^\infty L^\infty}\|\langle x\rangle^{-1}\tilde{u}\|_{L^2}^2.$$

By applying a Hardy inequality (2.4) on the lower order terms and choosing $\tilde{\varepsilon}$ and δ sufficiently small to bootstrap the second, third, and fourth terms on the right-hand side, we obtain the bound:

$$\|\nabla_x\tilde{u}\|_{L^2}^2 \lesssim \|P_{h,0}\tilde{u}\|_{L^2}^2 \lesssim \|P_{h,0}u\|_{\mathcal{LE}^*}^2. \tag{6.17}$$

Note that we have applied (6.16) in the above line. On the compact set $|x| < 2R_{AF}$, the weights are not important and so we have shown

$$\|\langle x\rangle\tilde{u}\|_{\mathcal{LE}^1_{<2R_{AF}}} + \|\nabla_x\tilde{u}\|_{L^2} \lesssim \|P_{h,0}u\|_{\mathcal{LE}^*}. \tag{6.18}$$

For larger $|x|$, we utilize a cut-off function to see

$$\begin{aligned}\|\tilde{P}_{h,0}(1-\chi_{R_{AF}})(|x|)\tilde{u}\|_{L^2} &= \|P_{h,0}(1-\chi_{R_{AF}})(|x|)\tilde{u}\|_{L^2}\\ &\lesssim \|P_{h,0}u\|_{\mathcal{LE}^*} + \|[P_{h,0},(1-\chi_{R_{AF}})(|x|)]\tilde{u}\|_{L^2}.\end{aligned}$$

Now

$$|[P_{h,0},(1-\chi_{R_{AF}})(|x|)]\tilde{u}| \lesssim R_{AF}^{-2}|\chi''(|x|/R_{AF})\tilde{u}| + R_{AF}^{-2}|\chi'(|x|/R_{AF})\tilde{u}| + R_{AF}^{-1}|\chi'(|x|/R_{AF})D_x\tilde{u}|.$$

Using this computation and a Hardy inequality (2.4), applying (6.10) and (6.11) for $\tilde{P}_{h,0}$, and utilizing the support of $P_{h,0}\tilde{u}$ and derivatives of $\chi_{R_{AF}}(|x|)$, we see

$$\begin{aligned}\|\langle x\rangle(1-\chi_{R_{AF}})(|x|)\tilde{u}\|_{\mathcal{LE}^1} &+ \|\langle x\rangle^{-1}(1-\chi_{R_{AF}})(|x|)\nabla\tilde{u}\|_{\mathcal{LE}^*}\\ &\lesssim \|\langle x\rangle\tilde{P}_{h,0}(1-\chi_{R_{AF}})(|x|)\tilde{u}\|_{\mathcal{LE}^*}\\ &\lesssim \|\langle x\rangle(1-\chi_{R_{AF}})(|x|)\tilde{P}_{h,0}\tilde{u}\|_{L^2} + \|\langle x\rangle[\tilde{P}_{h,0},(1-\chi_{R_{AF}})]\tilde{u}\|_{L^2}\\ &\lesssim \|P_{h,0}\tilde{u}\|_{L^2} + \|\nabla_x\tilde{u}\|_{L^2}\\ &\lesssim \|P_{h,0}u\|_{\mathcal{LE}^*}.\end{aligned}$$

Here we have used (6.17) in the last line. Combining the above line with (6.18) yields the bound

$$\|\langle x\rangle\tilde{u}\|_{\mathcal{LE}^1} + +\|\langle x\rangle^{-1}\nabla\tilde{u}\|_{\mathcal{LE}^*} \lesssim \|P_{h,0}u\|_{\mathcal{LE}^*},$$

which completes the proof of (6.10) and (6.11) for the operator $P_{h,0}$.

We now prove (6.12) by applying (6.11) to the following function:

$$u_1 = \chi_{R_1}(|x|)u + r^{-1}(1-\chi_{R_1})(|x|)(ru)_{R_1}, \tag{6.19}$$

where $(ru)_{R_1}$ is the average of ru over the R_1 annulus:

$$(ru)_{R_1} = \frac{1}{R_1^3}\int_{r\approx R_1} ru\,dx. \tag{6.20}$$

As in [29], by the Poincaré inequality, we have

$$\|\nabla(u - r^{-1}(ru)_{R_1})\|_{L^2_{R_1}} + R_1^{-1}\|u - r^{-1}(ru)_{R_1}\|_{L^2_{R_1}} \lesssim \|r^{-1}\nabla(ru)\|_{L^2_{R_1}}. \tag{6.21}$$

A direct computation shows

$$P_{h,0}u_1 = \chi_{R_1}(|x|)P_{h,0}u + [P_{h,0}, \chi_{R_1}(|x|)]\left(u - r^{-1}(ru)_{R_1}\right) + (1-\chi_{R_1})(|x|)\,(ru)_{R_1}\,(P_{h,0}-\Delta)\,r^{-1}, \tag{6.22}$$

where we have utilized that, away from the origin, $\Delta r^{-1} = 0$ since r^{-1} is proportional to the fundamental solution to Laplace's equation. Now

$$\|(1-\chi_{R_1})(|x|)\,(ru)_{R_1}\,(P_{h,0}-\Delta)\,r^{-1}\|_{\mathcal{LE}^*} \lesssim (|\!|\!|g|\!|\!|_{>R_1} + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)})R_1^{-1}|(ru)_{R_1}|. \tag{6.23}$$

Utilizing this bound, the triangle inequality, and (6.21), we find

$$\begin{aligned}\|P_{h,0}u_1\|_{\mathcal{LE}^*} \lesssim \|\chi_{R_1}(|x|)P_{h,0}u\|_{\mathcal{LE}^*} + R_1^{-1/2}\|r^{-1}\nabla(ru)\|_{L^2_{R_1}} \\ + (|\!|\!|g|\!|\!|_{>R_1} + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)})R_1^{-1}|(ru)_{R_1}|.\end{aligned} \tag{6.24}$$

Observe:

$$\|u_1\|_{\mathcal{LE}^1} + \|\langle x\rangle^{-1}\nabla u_1\|_{\mathcal{LE}^*} \gtrsim \|u\|_{\mathcal{LE}^1_{<R_1}} + \|\langle x\rangle^{-1}\nabla u\|_{\mathcal{LE}^*_{<R_1}} + R_1^{-1}|(ru)_{R_1}|. \tag{6.25}$$

Combining (6.11), (6.24), and (6.25), we find

$$\begin{aligned}\|u\|_{\mathcal{LE}^1_{<R_1}} + \|\langle x\rangle^{-1}\nabla u\|_{\mathcal{LE}^*_{<R_1}} + R_1^{-1}|(ru)_{R_1}| &\lesssim \|u_1\|_{\mathcal{LE}^1} + \|\langle x\rangle^{-1}\nabla u_1\|_{\mathcal{LE}^*} \\ &\lesssim \|P_{h,0}u\|_{\mathcal{LE}^*} + R_1^{-1/2}\|r^{-1}\nabla(ru)\|_{L^2_{R_1}} \\ &\quad + (|||g|||_{>R_1} + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)})R_1^{-1}|(ru)_{R_1}|.\end{aligned}$$

By choosing R_1 sufficiently large and δ sufficiently small, we can bootstrap $(|||g|||_{>R_1} + \|h\|_{\ell^1_l L^\infty L^\infty([0,T]\times A_l)})R_1^{-1}|(ru)_{R_1}|$ into the left-hand side of the above equation. Recognizing $R_1^{-1/2}\|r^{-1}\nabla(ru)\|_{L^2_{R_1}} \approx \|u\|_{\mathcal{LE}^1_{R_1}}$ completes the proof of (6.12) and hence the proof of Lemma 6.2. □

We now prove Theorem 6.1 and note that all implicit constants are independent of R_1 unless explicitly stated.

Proof of Theorem 6.1. There exists an $R \in [2R_{AF}, R_1]$ such that

$$\|\nabla u\|_{\mathcal{LE}_R} \leq \min\{\|\nabla u\|_{\mathcal{LE}_{2R_{AF}}}, ..., \|\nabla u\|_{\mathcal{LE}_{R_1}}\}. \tag{6.26}$$

Equations (6.26) and (6.12) imply

$$\|\nabla u\|_{\mathcal{LE}_R} \lesssim \|P_{h,0}\|_{\mathcal{LE}^*_{<R_1}} + \Big(\log(R_1/R_{AF})\Big)^{-1}\|u\|_{\mathcal{LE}^1_{R_1}}. \tag{6.27}$$

We integrate (6.27) L^2 in time to obtain

$$\|\nabla u\|_{LE_R} \lesssim C_{R_1}\|P_{h,0}\|_{LE^*_{<R_1}} + \Big(\log(R_1/R_{AF})\Big)^{-1}\|u\|_{LE^1_{R_1}}. \tag{6.28}$$

Similarly, we integrate (6.12) L^2 in time to find

$$\|\langle x\rangle^{-1}u\|_{LE_{<R_1}} + \|\nabla_x u\|_{LE_{<R_1}} \lesssim \|P_{h,0}u\|_{LE^*_{<R_1}} + \|u\|_{LE^1{}_{R_1}}. \tag{6.29}$$

By our exterior estimate (3.23), we have

$$\|u\|^2_{LE^1_{>R}} \lesssim \|\partial u\|^2_{L^\infty L^2} + \|P_h u\|^2_{LE^*} + \|\partial u\|^2_{LE_R} + \delta\log(2+T)\|u\|^2_{LE^1}. \tag{6.30}$$

Combining (6.28), (6.29), and (6.30), we find

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + \|\partial_t u\|_{LE_{<R_1}}^2 + \|P_h u\|_{LE^*}^2 + C_{R_1}\|P_{h,0}u\|_{LE^*_{<R_1}}^2 + \Big(\log(R_1/R_{AF})\Big)^{-1}\|u\|_{LE^1_{R_1}}^2 + \delta\log(2+T)\|u\|_{LE^1}^2.$$

Choosing R_1 sufficiently large to bootstrap $\Big(\log(R_1/R_{AF})\Big)^{-1}\|u\|_{LE^1_{R_1}}$, we find

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + \|\partial_t u\|_{LE_{<R_1}}^2 + \|P_h u\|_{LE^*}^2 + C_{R_1}\|P_{h,0}u\|_{LE^*_{<R_1}}^2 + \delta\log(2+T)\|u\|_{LE^1}^2.$$

Recognizing that $P_h = P_{h,0} - D_t^2 + 2h^{0j}D_tD_j + h^{00}D_t^2$, upon applying the triangle inequality, we obtain

$$\|u\|_{LE^1}^2 \lesssim \|\partial u\|_{L^\infty L^2}^2 + C_{R_1}\Big(\|\partial_t u\|_{LE^1_{<R_1}}^2 + \|P_h u\|_{LE^*}^2\Big) + \delta\log(2+T)\|u\|_{LE^1}^2,$$

which completes the proof of Theorem 6.1. □

Chapter 7: Proof of Theorem 1.1

We are now in a position to prove our main local energy estimate, Theorem 1.1, by piecing together our high frequency, medium frequency, and low frequency estimates: Theorem 4.2, Corollary 5.1, and Theorem 6.1, respectively and then applying our uniform energy bound Proposition 2.2. Fix $0 < \tau_{low} \ll_{R_1} 1$ sufficiently small such that for u with time frequency support $\{|\tau| \leq \tau_{low}\}$, we can absorb the error term in Theorem 6.1: $C_{R_1}\|\partial_t u\|^2_{LE^1_{<R_1}}$ into the $\|u\|^2_{LE^1}$ term on the left-hand side of (6.1). Similarly, fix $\tau_{high} \gg_{R_1} 1$ such that for u with time frequency support $\{|\tau| \geq \tau_{high}\}$, the error term $\|u\|^2_{L^2 L^2_{<2R_1}}$ can be absorbed into the $\|u\|^2_{LE^1}$ term on the left-hand side of (4.19). Recall Corollary 5.1 implies Theorem 1.1 for u with any bounded time-frequency support away from 0. So this estimate fills the gap between Theorem 6.1 and Theorem 4.2.

Now we write $u = Q_l(|D_t|)u + Q_m(|D_t|)u + Q_h(|D_t|)u$, where $Q_l(|D_t|) = \chi_{\tau_{low}/2}(|D_t|)$, $Q_h = (1 - \chi_{\tau_{high}})(|D_t|)$, and $Q_m(|D_t|) = 1 - Q_l(|D_t|) - Q_h(|D_t|)$. Here χ is the cut-off function in (1.10) and "h" stands for high,"m" for medium, and "l" for low. We estimate $Q_l u$ with Theorem 6.1, $Q_m u$ with Corollary 5.1, and $Q_h u$ with Theorem 4.2, absorbing the errors as described above. So, we have shown

$$\begin{aligned}
\|u\|_{LE^1} &\leq \sum_{i=l,m,h} \|Q_i u\|_{LE^1} \\
&\lesssim \sum_{i=l,m,h} \Big(\|Q_i \partial u\|_{L^\infty L^2} + \delta^{1/2} \log^{1/2}(2+T) \|Q_i u\|_{LE^1} + \|P_h Q_i u\|_{LE^*} \Big) + \sum_{i=1}^{3} \|[P_h, Q_i]u\|_{LE^*} \\
&\lesssim \|\partial u\|_{L^\infty L^2} + \delta^{1/2} \log^{1/2}(2+T) \|u\|_{LE^1} + \|P_h u\|_{LE^*} + \sum_{i=l,m,h} \|[P_h, Q_i]u\|_{LE^*} \\
&\lesssim \|\partial u\|_{L^\infty L^2} + \delta \log(2+T) \|u\|_{LE^1} + \|P_h u\|_{LE^*} + \sum_{i=l,m,h} \|[P_h, Q_i]u\|_{LE^*},
\end{aligned}$$

where we have applied Young's convolution inequality in the second to last line and Cauchy's inequality in the last line. So if we can show

$$\sum_{i=l,m,h} \|[P_h, Q_i]u\|_{LE^*} \lesssim \|\partial u\|_{L^\infty L^2} + \delta \log(2+T) \|u\|_{LE^1} + \|P_h u\|_{LE^*}, \tag{7.1}$$

an application of Proposition 2.2 completes the proof of Theorem 1.1. Indeed, the following stronger bound holds, which immediately implies (7.1).

Lemma 7.1. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is symmetric and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and asymptotically Euclidean in the sense of* (1.6)*. Fix $T > 0$. Suppose $u(t,x) \in C^2([0,T]\times\mathbb{R}^3)$ solves $P_h u = F \in LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^2(\mathbb{R}^3)\times C^1(\mathbb{R}^3)$ such that $|\partial^{\le 2} u(t,x)| \to 0$ as $|x|\to\infty$ for all $t \in [0,T]$. Then*

$$\|[P_h, Q_i]u\|_{LE^*} \lesssim \delta\big(\|P_h u\|_{LE^*} + \|\partial u\|_{L^\infty L^2} + \log(2+T)\|u\|_{LE^1}\big), \tag{7.2}$$

where i is a place holder for l, m, h and Q_i is as above. Here the implicit constant is independent of T and δ.

Proof. As g is stationary and the exact support of each Q_i is unimportant for the proof, we need only demonstrate the bound

$$\|[h^{\alpha\beta} D_\alpha D_\beta, Q_{\le 1}]u\|_{LE^*} \lesssim \delta\big(\|P_h u\|_{LE^*} + \|\partial u\|_{L^\infty L^2} + \log(2+T)\|u\|_{LE^1}\big), \tag{7.3}$$

where $Q_{\le 1} = \chi(|D_t|)$. We will estimate each term $\|h^{\alpha\beta} D_\alpha D_\beta Q_{\le 1} u\|_{LE^*}$ and $\|Q_{\le 1} h^{\alpha\beta} D_\alpha D_\beta u\|_{LE^*}$ separately.

We begin by estimating $\|h^{\alpha\beta} D_\alpha D_\beta Q_{\le 1} u\|_{LE^*}$, first working with the $h^{00} D_t^2$ term. We see

$$\begin{aligned} \|h^{00} D_t^2 Q_{\le 1} u\|_{LE^*} &\lesssim \|\partial_t^2 Q_{\le 1} u\|_{LE} \sum_{l=0}^{\log(2+T)} \|\langle x\rangle h\|_{L^\infty L^\infty([0,T]\times A_l)} \\ &\quad + \|\langle x\rangle h\|^{1/2}_{\ell_l^\infty L^\infty L^\infty([0,T]\times A_l)} \|\partial_t^2 Q_{\le 1} u\|_{L^2L^2} \sum_{l=\log(2+T)}^{\infty} \|h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)} \\ &\lesssim \|u\|_{LE^1} \sum_{l=0}^{\log(2+T)} \|\langle x\rangle h\|_{L^\infty L^\infty([0,T]\times A_l)} \\ &\quad + \|\langle x\rangle h\|^{1/2}_{\ell_l^\infty L^\infty L^\infty([0,T]\times A_l)} \|\partial_t u\|_{L^2L^2} \sum_{l=\log(2+T)}^{\infty} \|h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)} \\ &\lesssim \delta \log(2+T)\|u\|_{LE^1} \\ &\quad + T^{1/2}\|\langle x\rangle h\|^{1/2}_{\ell_l^\infty L^\infty L^\infty([0,T]\times A_l)} \|\partial u\|_{L^\infty L^2} \sum_{l=\log(2+T)}^{\infty} \|h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)}. \end{aligned} \tag{7.4}$$

Now (1.11) implies $\|\partial^{\le 1} h\|_{L^\infty L^\infty([0,T]\times A_l)} \lesssim 2^{-l}\delta$, and so

$$T^{1/2} \sum_{l=\log(2+T)}^{\infty} \|\partial^{\le 1} h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)} \lesssim \delta^{1/2} T^{1/2} \sum_{l=\log(2+T)} \sqrt{2^{-l}} \lesssim \delta^{1/2}.$$

Therefore

$$\|\langle x\rangle h\|^{1/2}_{\ell^\infty_l L^\infty L^\infty([0,T]\times A_l)}\|\partial u\|_{L^\infty L^2}T^{1/2}\sum_{l=\log(2+T)}^{\infty}\|\partial^{\le 1}h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)}\lesssim \delta\|\partial u\|_{L^\infty L^2},$$

and so we have shown

$$\|h^{00}D_t^2Q_{\le 1}u\|_{LE^*}\lesssim \delta\log(2+T)\|u\|_{LE^1}+\delta\|\partial u\|_{L^\infty L^2}.$$

A nearly identical analysis demonstrates

$$2\|h^{0j}D_tD_jQ_{\le 1}u\|_{LE^*}\lesssim \delta\log(2+T)\|u\|_{LE^1}+\delta\|\partial u\|_{L^\infty L^2}.$$

Before investigating $\|h^{ij}D_iD_jQ_{\le 1}u\|_{LE^*}$, we consider $\|Q_{\le 1}h^{00}D_t^2\|_{LE^*}$ and $\|Q_{\le 1}h^{0j}D_tD_j\|_{LE^*}$. We begin with $\|Q_{\le 1}h^{00}D_t^2u\|_{LE^*}$ and write $Q_{\le 1}h^{00}D_t^2 = Q_{\le 1}D_th^{00}D_t + iQ_{\le 1}\partial_th^{00}D_t$ to see

$$\begin{aligned}(7.5)\quad \|Q_{\le 1}h^{00}D_t^2u\|_{LE^*} &\le \|Q_{\le 1}D_th^{00}D_tu\|_{LE^*}+\|Q_{\le 1}\partial_th^{00}D_tu\|_{LE^*}\\ &\lesssim \|h^{00}\partial_tu\|_{LE^*}+\|\partial_th^{00}\partial_tu\|_{LE^*}\\ &\lesssim \|u\|_{LE^1}\sum_{l=0}^{\log(2+T)}\|\langle x\rangle\partial^{\le 1}h\|_{L^\infty L^\infty([0,T]\times A_l)}\\ &\quad+\|\langle x\rangle\partial^{\le 1}h\|^{1/2}_{\ell^\infty_l L^\infty L^\infty([0,T]\times A_l)}\|\partial u\|_{L^\infty L^2}T^{1/2}\sum_{l=\log(2+T)}^{\infty}\|\partial^{\le 1}h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)}\\ &\lesssim \delta\log(2+T)\|u\|_{LE^1}+\delta\|\partial u\|_{L^\infty L^2}.\end{aligned}$$

A nearly identical analysis shows

$$\|Q_{\le 1}h^{0j}D_tD_ju\|_{LE^*}\lesssim \delta\log(2+T)\|u\|_{LE^1}+\delta\|\partial u\|_{L^\infty L^2}.$$

We now turn to bounding terms with spatial derivatives only. We begin by investigating $\|h^{ij}D_iD_jQ_{\le 1}u\|_{LE^*}$ and first consider a single dyadic interval. To simplify notation, we set $v=\beta(|x|/2^l)Q_{\le 1}u$, where $\beta(|x|)$ is a cut-off function that is identically 1 for $1\le |x|\le 2$ and 0 for $|x|\ge 4$ or $|x|\le 1/2$. It suffices to bound:

$$\int_0^T\int_{\mathbb{R}^3}2^l|h|^2(\partial_x^2v)^2\,dxdt,$$

where $\partial_x^2 = \sum_{|\mu|=2} \partial_x^\mu$. Using that $D_i g^{ij} D_j$ is strictly elliptic and integrating by parts, we find

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 (\partial_x^2 v)^2 \, dxdt &\lesssim \int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 g^{ij} \partial_j \partial_x v \partial_i \partial_x v \, dxdt \\
&\leq - \int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 \partial_x g^{ij} \partial_j v \partial_i \partial_x v \, dxdt \\
&\quad + 2 \int_0^T \int_{\mathbb{R}^3} 2^l |\partial h| |h| |g^{ij} \partial_j v \partial_i \partial_x v| \, dxdt \\
&\quad - \int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 g^{ij} \partial_j v \partial_i \partial_x^2 v \, dxdt.
\end{aligned} \tag{7.6}$$

Integrating by parts, we see (7.6) can be controlled by

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 \partial_x g^{ij} \partial_j v \partial_i \partial_x v \, dxdt + 2 \int_0^T \int_{\mathbb{R}^3} 2^l |\partial h| |h| |g^{ij} \partial_j v \partial_i \partial_x v| \, dxdt \\
&\quad + 2 \int_0^T \int_{\mathbb{R}^3} 2^l |\partial h| |h| |g^{ij} \partial_j v \partial_x^2 v| \, dxdt + \int_0^T \int_{\mathbb{R}^3} 2^l |h|^2 \partial_i (g^{ij} \partial_j v) \partial_x^2 v \, dxdt.
\end{aligned} \tag{7.7}$$

Combining (7.6), (7.7), and applying Cauchy's inequality, we have shown

$$\begin{aligned}
2^l \int_0^T \int_{\mathbb{R}^3} |h|^2 (\partial_x^2 v)^2 \, dxdt &\lesssim 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} \partial_x \big(\beta(|x|/2^l) u \big) \Big)^2 \, dxdt \\
&\quad + 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} \partial_i \big(g^{ij} \partial_j (\beta(|x|/2^l) u) \big) \Big)^2 \, dxdt.
\end{aligned} \tag{7.8}$$

We will be able to bound the first term on the right-hand side of (7.8) using our prior methods. For the second term, we introduce the operator P_h via the following:

$$\begin{aligned}
&2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} \partial_i \big(g^{ij} \partial_j (\beta(|x|/2^l) u) \big) \Big)^2 \, dxdt \\
&\lesssim 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} P_h \beta(|x|/2^l) u \Big)^2 \, dxdt \\
&+ 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} D_t^2 \beta(|x|/2^l) u \Big)^2 \, dxdt \\
&+ 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} h^{00} D_t^2 \beta(|x|/2^l) u \Big)^2 \, dxdt \\
&+ 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} h^{0j} D_t D_j \beta(|x|/2^l) u \Big)^2 \, dxdt \\
&+ 2^l \|h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\leq 1} h^{ij} D_i D_j \beta(|x|/2^l) u \Big)^2 \, dxdt.
\end{aligned} \tag{7.9}$$

Combining (7.8), (7.9), and summing over dyadic regions, we have shown

$$\|h^{ij}D_iD_jQ_{\le 1}u\|_{LE^*} \lesssim \delta\Big(\|P_h u\|_{LE^*} + \|\partial u\|_{L^\infty L^2} + \log(2+T)\|u\|_{LE^1}\Big) + \delta\|Q_{\le 1}h^{ij}D_iD_ju\|_{LE^*}. \tag{7.10}$$

Here we are applying the method used in (7.5) to control the second, third, and fourth terms on the right-hand side of (7.9). The first term on the right-hand side of (7.8) is controlled by observing

$$\begin{aligned}\sum_l 2^{l/2}\|\partial^{\le 1}h\|_{L^\infty L^\infty([0,T]\times A_l)}\|Q_{\le 1}\partial_x\big(\beta(|x|/2^l)u\big)\|_{L^2L^2}& \\ \lesssim \|u\|_{LE^1}\sum_{l=0}^{\log(2+T)}\|\langle x\rangle\partial^{\le 1}h\|_{L^\infty L^\infty([0,T]\times A_l)}& \\ + \|\langle x\rangle\partial^{\le 1}h\|^{1/2}_{\ell_l^\infty L^\infty L^\infty([0,T]\times A_l)}\|\partial u\|_{L^2L^2}\sum_{l=\log(2+T)}^{\infty}\|\partial^{\le 1}h\|^{1/2}_{L^\infty L^\infty([0,T]\times A_l)},&\end{aligned} \tag{7.11}$$

which is nearly identical to terms in (7.4) and so this is controlled via $\delta\log(2+T)\|u\|_{LE^1} + \delta\|\partial u\|_{L^\infty L^2}$. Lastly, we note commuting P_h with the cutoff functions, as in the first term on the right-hand side of (7.9) poses no problems as it results in terms that are controlled by $\|h\|_{\ell_l^1 L^\infty L^\infty([0,T]\times A_l)}\|u\|_{LE^1} \lesssim \delta\|u\|_{LE^1}$.

We now turn to $\|Q_{\le 1}h^{ij}D_iD_ju\|_{LE^*}$ and observe by the triangle inequality $\|Q_{\le 1}h^{ij}D_iD_ju\|_{LE^*} \le \|Q_{\le 1}h^{ij}D_iD_ju_{\ge 1}\|_{LE^*} + \|Q_{\le 1}h^{ij}D_iD_ju_{\le 1}\|_{LE^*}$. Here $u_{\le 1} = Q_{\le 1}u$ and $u_{\ge 1} = (1-Q_{\le 1})u$. By Plancherel's Theorem, we have already controlled $\|Q_{\le 1}h^{ij}D_iD_ju_{\le 1}\|_{LE^*}$ in (7.10). Therefore it suffices to bound $\|Q_{\le 1}h^{ij}D_iD_ju_{\ge 1}\|_{LE^*}$. We demonstrate the following bound:

$$\|Q_{\le 1}h^{ij}D_iD_ju_{\ge 1}\|_{LE^*} \lesssim \delta\Big(\|P_h u\|_{LE^*} + \|\partial u\|_{L^\infty L^2} + \log(2+T)\|u\|_{LE^1}\Big) + \delta\|h^{ij}D_iD_ju_{\le 1}\|_{LE^*}. \tag{7.12}$$

We first observe

$$\begin{aligned}\mathcal{F}\big(Q_{\le 1}h^{ij}\partial_i\partial_ju_{\ge 1}\big)(\tau) &= \int_{-\infty}^{\infty}\chi(|\tau|)\hat{h}^{ij}(s)\partial_i\partial_j\hat{u}_{\ge 1}(\tau-s)\,ds \\ &= \int_{-\infty}^{\infty}\chi(|\tau|)\hat{h}^{ij}(s)\partial_i\partial_j\frac{\tau-s}{\tau-s}\hat{u}_{\ge 1}(\tau-s)\,ds \\ &= \int_{-\infty}^{\infty}\tau\chi(|\tau|)\hat{h}^{ij}(s)\partial_i\partial_j\frac{1}{\tau-s}\hat{u}_{\ge 1}(\tau-s)\,ds \\ &\qquad - \int_{-\infty}^{\infty}\chi(|\tau|)s\hat{h}^{ij}(s)\partial_i\partial_j\frac{1}{\tau-s}\hat{u}_{\ge 1}(\tau-s)\,ds.\end{aligned} \tag{7.13}$$

By Plancherel's theorem and the triangle inequality, we have shown

$$\|Q_{\le 1}h^{ij}\partial_i\partial_ju_{\ge 1}\|_{LE^*} \lesssim \||\partial^{\le 1}h|\partial_x^2(|D_t|^{-1}u_{\ge 1})\|_{LE^*}. \tag{7.14}$$

We begin by working on a single dyadic annulus. We set $w = |D_t|^{-1}\beta(|x|/2^l)u_{\geq 1}$ and proceed nearly identically to the work in (7.6) and (7.7) to obtain the analog of (7.8):

$$\begin{aligned}(7.15)\quad 2^l \int_0^T \int_{\mathbb{R}^3} |\partial^{\leq 1} h|^2 (\partial_x^2 w)^2\, dxdt \\ \lesssim 2^l \|\partial^{\leq 2} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1}\partial_x\big(\beta(|x|/2^l)u\big)\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1}\partial_i\big(g^{ij}\partial_j(\beta(|x|/2^l)u)\big)\Big)^2\, dxdt.\end{aligned}$$

We will be able to bound the first term on the right-hand side of (7.15) using our prior methods. For the second term, we introduce the operator P_h via the following:

$$\begin{aligned}(7.16)\quad 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1}\partial_i\big(g^{ij}\partial_j(\beta(|x|/2^l)u)\big)\Big)^2\, dxdt \\ \lesssim 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} P_h \beta(|x|/2^l)u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} D_t^2 \beta(|x|/2^l)u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} h^{00} D_t^2 \beta(|x|/2^l)u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} h^{0j} D_t D_j \beta(|x|/2^l)u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} h^{ij} D_i D_j \beta(|x|/2^l)u\Big)^2\, dxdt.\end{aligned}$$

Writing $h^{00}D_t^2 + h^{0j}D_tD_j$ as $D_t h^{00} D_t + D_t h^{0j} D_j + i\partial_t h^{00} D_t + i\partial_t h^{0j} D_j$, we find (7.16) is controlled by

$$\begin{aligned}(7.17)\quad 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(P_h \beta(|x|/2^l)u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(\beta(|x|/2^l)\partial_t u\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(|\partial^{\leq 1} h|\partial\big(\beta(|x|/2^l)u\big)\Big)^2\, dxdt \\ + 2^l \|\partial^{\leq 1} h\|^2_{L^\infty L^\infty([0,T]\times A_l)} \int_0^T \int_{\mathbb{R}^3} \Big(Q_{\geq 1}|D_t|^{-1} h^{ij} D_i D_j \beta(|x|/2^l)u\Big)^2\, dxdt.\end{aligned}$$

For the last term in (7.17), we use the triangle inequality to bound this by

$$
\begin{aligned}
(7.18)\qquad & 2^l\|\partial^{\le 1}h\|^2_{L^\infty L^\infty([0,T]\times A_l)}\int_0^T\int_{\mathbb{R}^3}\Big(Q_{\ge 1}|D_t|^{-1}h^{ij}D_iD_j\beta(|x|/2^l)u\Big)^2\,dxdt\\
&\lesssim 2^l\|\partial^{\le 1}h\|^2_{L^\infty L^\infty([0,T]\times A_l)}\int_0^T\int_{\mathbb{R}^3}\Big(h^{ij}\partial_i\partial_j\big(\beta(|x|/2^l)u_{\le 1}\big)\Big)^2\,dxdt\\
&\quad+2^l\|\partial^{\le 1}h\|^2_{L^\infty L^\infty([0,T]\times A_l)}\int_0^T\int_{\mathbb{R}^3}\Big(Q_{\ge 1}|D_t|^{-1}h^{ij}\partial_i\partial_j\big(\beta(|x|/2^l)u_{\ge 1}\big)\Big)^2\,dxdt\\
&\lesssim 2^l\|\partial^{\le 1}h\|^2_{L^\infty L^\infty([0,T]\times A_l)}\int_0^T\int_{\mathbb{R}^3}\Big(h^{ij}\partial_i\partial_j\big(\beta(|x|/2^l)u_{\le 1}\big)\Big)^2\,dxdt\\
&\quad+2^l\|\partial^{\le 1}h\|^2_{L^\infty L^\infty([0,T]\times A_l)}\int_0^T\int_{\mathbb{R}^3}\Big(|\partial^{\le 1}h|\partial_x^2\big(\beta(|x|/2^l)|D_t|^{-1}u_{\ge 1}\big)\Big)^2\,dxdt,
\end{aligned}
$$

where we have used work similar to that in (7.13) in the last line. Specifically, setting $v_{\ge 1}=\beta(|x|/2^l)u_{\ge 1}$, we have used

$$
\begin{aligned}
\mathcal{F}\big(Q_{\ge 1}|D_t|^{-1}h^{ij}\partial_i\partial_j v_{\ge 1}\big)(\tau)&=\int_{-\infty}^{\infty}(1-\chi(|\tau|))|\tau|^{-1}\hat{h}^{ij}(s)\partial_i\partial_j\hat{v}_{\ge 1}(\tau-s)\,ds\\
&=\int_{-\infty}^{\infty}(1-\chi(|\tau|))|\tau|^{-1}\hat{h}^{ij}(s)\partial_i\partial_j\frac{\tau-s}{\tau-s}\hat{v}_{\ge 1}(\tau-s)\,ds\\
&=\int_{-\infty}^{\infty}(1-\chi(|\tau|))|\tau|^{-1}\tau\hat{h}^{ij}(s)\partial_i\partial_j\frac{1}{\tau-s}\hat{v}_{\ge 1}(\tau-s)\,ds\\
&\qquad-\int_{-\infty}^{\infty}(1-\chi(|\tau|))|\tau|^{-1}s\hat{h}^{ij}(s)\partial_i\partial_j\frac{1}{\tau-s}\hat{v}_{\ge 1}(\tau-s)\,ds,
\end{aligned}
$$

combined with Plancherel's Theorem.

Combining (7.14), (7.15), (7.16), (7.17), (7.18), summing over dyadic regions, and utilizing work leading to the bound in (7.10), we have shown

$$
\|Q_{\le 1}h^{ij}D_iD_ju_{\ge 1}\|_{LE^*}\lesssim\delta\Big(\|P_hu\|_{LE^*}+\|\partial u\|_{L^\infty L^2}+\log(2+T)\|u\|_{LE^1}\Big)+\delta\|h^{ij}D_iD_ju_{\le 1}\|_{LE^*}.
$$

Specifically, we control the first term on the right-hand side of (7.15) as well as the second and third terms on the right-hand side of (7.17) via the work in (7.11). We have chosen δ sufficiently small to absorb the last term on the right-hand side of (7.18) into the left-hand side of (7.15).

Combining (7.10) and (7.12) completes the proof of Lemma 7.1. □

Chapter 8: Energy Estimates and Vector Fields

In this chapter, we record several lemmas describing energy estimates with vector fields applied to our solution u to $P_h u = F$. These are rooted in the work of [15]. We begin with a local energy estimate involving the generators of translations applied to u.

Lemma 8.1. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is smooth, symmetric, and satisfies* (1.11) *for some $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is smooth, symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and can be written as in* (1.15) *where g_r, g_ω, and g_{sr}^{ij} satisfy* (1.16)*. Fix $T > 0$. Suppose $u(t,x) \in C^\infty([0,T] \times \mathbb{R}^3)$ solves $P_h u = F \in C^\infty([0,T] \times \mathbb{R}^3) \cap LE^*$, with initial data $(u, \partial_t u)(0,\cdot) = (f_1, f_2) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$ such that u vanishes for large $|x|$ for all $t \in [0,T]$. Then for fixed $N = 0, 1, 2, ...15$:*

$$(8.1) \quad \sum_{|\mu| \le N} \|\partial^\mu u\|^2_{LE^1} + \sum_{|\mu| \le N} \|\partial^\mu \partial u\|^2_{L^\infty L^2} \lesssim \sum_{|\mu| \le N} \|\partial^\mu \partial u(0)\|^2_{L^2} + \sum_{k \le N} \|\partial_t^k P_h u\|^2_{LE^*} + \sum_{k \le N} \|[h^{\alpha\beta}\partial_\alpha\partial_\beta, \partial_t^k]u\|^2_{LE^*} + \sum_{|\mu| \le N} \delta^2 \log^2(2+T)\|\partial^\mu u\|^2_{LE^1} + \sum_{|\mu| \le N-1} \|\partial^\mu P_g u\|^2_{LE} + \sum_{|\mu| \le N-1} \|\partial^\mu P_g u\|^2_{L^\infty L^2}.$$

Here the implicit constant is independent of T and δ.

We first show the following bound:

$$(8.2) \quad \sum_{|\mu| \le N} \|\partial^\mu \partial u\|^2_{LE^1} + \sum_{|\mu| \le N} \|\partial^\mu \partial u\|^2_{L^\infty L^2} \lesssim \sum_{k \le N} \|\partial_t^k \partial u(0,\cdot)\|^2_{L^2} + \sum_{k \le N} \|P_h \partial_t^k u\|^2_{LE^*} + \sum_{|\mu| \le N} \delta^2 \log^2(2+T)\|\partial^\mu u\|^2_{LE^1} + \sum_{|\mu| \le N-1} \|P_g \partial^\mu u\|^2_{L^\infty L^2} + \sum_{\mu \le N-1} \|P_g \partial^\mu u\|^2_{LE}$$

via an induction argument.

Proof of (8.2). Observe (8.2) follows immediately when ∂^μ is replaced by ∂_t^k (all derivatives are taken with respect to time), by applying Theorem 1.1 to $\partial_t^k u$. We now use an induction argument to complete the bound. Observe the case $N = 0$ follows from Theorem 1.1. Assume (8.2) holds when N is replaced by $N - 1$. We show that this implies (8.2) for a full N derivatives applied to u.

Observe

$$\sum_{|\mu|\le N}\|\partial^\mu\partial u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N}\|\partial^\mu u\|^2_{LE^1}\lesssim \sum_{|\mu|\le N-1}\sum_{|\nu|=2}\|\partial^\mu\nabla_x^\nu u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N-1}\sum_{|\nu|=2}\|\partial^\mu\nabla_x^\nu u\|^2_{LE}$$
$$+\sum_{|\mu|\le N-1}\|\partial^\mu\partial_t\partial u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N-1}\|\partial^\mu\partial_t\partial u\|^2_{LE}+\|\partial u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N-1}\|\partial^\mu u\|^2_{LE^1}. \tag{8.3}$$

The bound for the last two terms on the right-hand side of the equation follows from the inductive hypothesis. The bound for the third and fourth terms on the right-hand side of (8.3) follows by applying the inductive hypothesis to $\partial_t u$. For the first term on the right-hand side of (8.3), we use the fact that $D_i g^{ij} D_j$ is strictly elliptic and integrate by parts to observe

$$\begin{aligned}\int_{\mathbb{R}^3}(\partial_x^2\partial^\mu u)^2\,dx &\lesssim \int_{\mathbb{R}^3} g^{ij}\partial_j\partial_x\partial^\mu u\partial_i\partial_x\partial^\mu u\,dx\\ &= -\int_{\mathbb{R}^3}(P_g\partial^\mu u)\partial^\mu\partial_x^2 u\,dx+\int_0^T\int_{\mathbb{R}^3}\partial^\mu\partial_t^2 u\partial^\mu\partial_x^2 u\,dxdt\\ &\qquad -\int_0^T\int_{\mathbb{R}^3}\partial_x g^{ij}\partial_j\partial^\mu u\partial_i\partial^\mu\partial_x u\,dxdt\\ &\lesssim \frac{1}{\varepsilon}\|P_g\partial^\mu u\|^2_{L^2}+\varepsilon\|\partial_x^2\partial^\mu u\|^2_{L^2}+\frac{1}{\varepsilon}\|\partial^\mu\partial_t^2 u\|^2_{L^2}+\frac{1}{\varepsilon}\|\partial^\mu\partial u\|^2_{L^2}.\end{aligned}$$

Here $\partial_x^2=\sum_{|\mu|=2}\partial_x^\mu$. We choose $\varepsilon>0$ sufficiently small to bootstrap the $\varepsilon\|\partial_x^2\partial^\mu u\|^2_{L^2}$ term into the left-hand side of the above equation. Therefore, we have shown

$$\sum_{|\mu|\le N-1}\sum_{|\nu|=2}\|\partial^\mu\nabla_x^\nu u\|^2_{L^\infty L^2}\lesssim \sum_{|\mu|\le N-1}\|P_g\partial^\mu u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N-1}\|\partial^\mu\partial_t^2 u\|^2_{L^\infty L^2}+\sum_{|\mu|\le N-1}\|\partial^\mu\partial u\|^2_{L^\infty L^2}.$$

The last term on the right-hand side of the above equation is bound using the inductive hypothesis. Similarly, the second term on the right-hand side of the above equation is bound by applying the inductive hypothesis to $\partial_t u$.

We use a similar argument for the second term on the right-hand side of (8.3). We begin by considering a single dyadic interval. To simplify notation, we set $v=\beta(|x|/2^l)u$, where $\beta(|x|)$ is a cut-off function that is identically 1 for $1\le|x|\le 2$ and 0 for $|x|\ge 4$ or $|x|\le 1/2$. Using that $D_i g^{ij} D_j$ is strictly elliptic and integrating by parts, we find

$$\begin{aligned}2^{-l}\int_0^T\int_{\mathbb{R}^3}(\partial_x^2\partial^\mu v)^2\,dxdt &\lesssim 2^{-l}\int_0^T\int_{\mathbb{R}^3} g^{ij}\partial_j\partial_x\partial^\mu v\partial_i\partial_x\partial^\mu v\,dxdt\\ &= -2^{-l}\int_0^T\int_{\mathbb{R}^3}(P_g\partial^\mu v)\partial^\mu\partial_x^2 v\,dxdt+2^{-l}\int_0^T\int_{\mathbb{R}^3}\partial^\mu\partial_t^2 v\partial^\mu\partial_x^2 v\,dxdt\\ &\qquad -2^{-l}\int_0^T\int_{\mathbb{R}^3}\partial_x g^{ij}\partial_j\partial^\mu v\partial_i\partial^\mu\partial_x v\,dxdt\end{aligned}$$

$$\lesssim \frac{1}{\varepsilon}2^{-l}\|P_g\partial^\mu v\|^2_{L^2L^2} + \varepsilon 2^{-l}\|\partial_x^2\partial^\mu v\|^2_{L^2L^2} + \frac{1}{\varepsilon}2^{-l}\|\partial^\mu\partial_t^2 v\|^2_{L^2L^2} + \frac{1}{\varepsilon}2^{-l}\|\partial^\mu\partial v\|^2_{L^2L^2}.$$

We choose $\varepsilon > 0$ small enough to absorb the error term $\varepsilon 2^{-l}\|\partial_x^2\partial^\mu v\|^2_{L^2}$ into the left-hand side of the above equation.

Commuting with our cut-off function, β, we have shown

$$\sum_{|\mu|\le N-1}\sum_{|\nu|=2}\|\partial^\mu\nabla_x^\nu u\|^2_{LE} \lesssim \sum_{|\mu|\le N-1}\|P_g\partial^\mu v\|^2_{LE} + \sum_{|\mu|\le N-1}\|\partial^\mu\partial_t^2 u\|^2_{LE} + \sum_{|\mu|\le N-1}\|\partial^\mu u\|^2_{LE^1}.$$

The third term on the right hand side of the above equation is controlled using the inductive hypothesis and the second term is controlled by applying the inductive hypothesis to $\partial_t u$. We have therefore demonstrated (8.2). □

Proof of Lemma 8.1. We now obtain (8.1) with another induction argument. Equation (8.1) holds when $N = 0$ from Theorem 1.1. Now assume (8.1) is true when N is replaced by $N - 1$. We will show that this implies the bound for N. Applying Cauchy's inequality and the triangle inequality to the following terms on the right-hand side of (8.2) demonstrates

$$\sum_{k\le N}\|P_h\partial_t^k u\|^2_{LE^*} + \sum_{|\mu|\le N-1}\|P_g\partial^\mu u\|^2_{LE} + \sum_{|\mu|\le N-1}\|P_g\partial^\mu u\|^2_{L^\infty L^2}$$
$$\lesssim \sum_{k\le N}\|\partial_t^k P_h u\|^2_{LE^*} + \sum_{k\le N}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta, \partial_t^k]u\|^2_{LE^*} + \sum_{|\mu|\le N-1}\|\partial^\mu P_g u\|^2_{LE} + \sum_{|\mu|\le N-1}\|\partial^\mu P_g u\|^2_{L^\infty L^2}$$
$$+ \sum_{|\mu|\le N-1}\|[P_g, \partial^\mu]u\|^2_{LE} + \sum_{|\mu|\le N-1}\|[P_g, \partial^\mu]u\|^2_{L^\infty L^2}.$$

Observe

$$\sum_{|\mu|\le N-1}\|[P_g, \partial^\mu]u\|^2_{LE} + \sum_{|\mu|\le N-1}\|[P_g, \partial^\mu]u\|^2_{L^\infty L^2} \lesssim \sum_{|\mu|\le N-1}\|\partial^\mu\partial u\|^2_{LE} + \sum_{|\mu|\le N-1}\|\partial^\mu\partial u\|^2_{L^\infty L^2},$$

and so these terms are bound via our induction hypothesis. This completes the proof of Lemma 8.1. □

We state a second local energy estimate for $Z^\mu u$, again motivated by the prior work of [15].

Lemma 8.2. *Let P_h be as in* (1.2)*, where $h^{\alpha\beta}$ is smooth, symmetric, and satisfies* (1.11) *for $\delta > 0$ sufficiently small. Further $D_i g^{ij} D_j$ is smooth, symmetric, stationary, strictly elliptic in the sense of* (1.4)*, and can be written as in* (1.15) *where g_r, g_ω, and g_{sr}^{ij} satisfy* (1.16)*. Fix $T > 0$. Suppose $u(t, x) \in C^\infty([0, T] \times \mathbb{R}^3)$ solves $P_h u = F \in C^\infty([0, T] \times \mathbb{R}^3) \cap LE^*$, with initial data $(u, \partial_t u)(0, \cdot) = (f_1, f_2) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$ such that u vanishes for large $|x|$ for all $t \in [0, T]$. Then for fixed $N = 0, 1, 2, ..., 14$:*

$$\sum_{|\mu|\le N}\|Z^\mu u\|^2_{LE^1}+\sum_{|\mu|\le N}\|Z^\mu\partial u\|^2_{L^\infty L^2}\lesssim\sum_{|\mu|\le N}\|Z^\mu\partial u(0)\|^2_{L^2}+\sum_{|\mu|\le N}\|Z^\mu P_h u\|^2_{LE^*}$$
$$+\sum_{|\mu|\le N}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta,Z^\mu]u\|^2_{LE^*}+\sum_{|\mu|\le N}\delta^2\log^2(2+T)\|Z^\mu u\|^2_{LE^1}$$
$$+\sum_{|\mu|\le N-1}\|Z^\mu P_g u\|^2_{LE}+\sum_{|\mu|\le N-1}\|Z^\mu P_g u\|^2_{L^\infty L^2}. \tag{8.4}$$

Here the implicit constant is independent of T and δ.

Proof. This lemma follows from Lemma 8.1 if $Z^\mu=\partial^\mu$. We now assume (8.4) holds for $Z^\mu=\partial^\gamma\Omega^{\nu-1}$ and show that (8.4) still holds for $Z^\mu=\partial^\gamma\Omega^\nu$. We are using multi-index notation for μ, γ, and ν. We can consider this operator ordering since $[\partial_k,x_i\partial_j-x_j\partial_j]=\delta_{ik}\partial_j-\delta_{jk}\partial_i$. To be precise $Z^\mu=\partial^\gamma\Omega^\nu$ denotes $\sum_{|\gamma|+|\nu|=|\mu|}\partial^\gamma\Omega^\nu$, while $Z^\mu=\partial^\gamma\Omega^{\nu-1}$ denotes $\sum_{|\gamma|+|\nu-1|=|\mu|,\,|\gamma|\le|\mu|}\partial^\gamma\Omega^\nu$. Applying Lemma 8.1 to $\partial^\gamma\Omega^\nu u$, we find

$$\|\partial^\gamma\Omega^\nu u\|^2_{LE^1}+\|\partial^\gamma\Omega^\nu\partial u\|^2_{L^\infty L^2}\lesssim\sum_{|\mu|\le|\gamma|+|\nu|}\|Z^\mu\partial u(0)\|^2_{L^2}+\sum_{k\le|\gamma|}\|\partial_t^k P_h\Omega^\nu u\|^2_{LE^*}$$
$$+\sum_{k\le|\gamma|}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta,\partial_t^k]\Omega^\nu u\|^2_{LE^*}+\sum_{|\mu|\le|\gamma|+|\nu|}\delta^2\log^2(2+T)\|Z^\mu u\|^2_{LE^1}$$
$$+\sum_{|\mu|\le|\gamma|-1}\|\partial^\mu P_g\Omega^\nu u\|^2_{LE}+\sum_{|\mu|\le|\gamma|-1}\|\partial^\mu P_g\Omega^\nu u\|^2_{L^\infty L^2}.$$

Utilizing the long-range spherical symmetry assumption on $D_ig^{ij}D_j$ in (1.15) and recognizing $[P_h,\Omega^\nu]=[h^{\alpha\beta}\partial_\alpha\partial_\beta,\Omega^\nu]+[D_ig^{ij}_{sr}D_j,\Omega^\nu]$, we find

$$\|\partial^\gamma\Omega^\nu u\|^2_{LE^1}+\|\partial^\gamma\Omega^\nu\partial u\|^2_{L^\infty L^2}\lesssim\sum_{|\mu|\le|\gamma|+|\nu|}\|Z^\mu\partial u(0)\|^2_{L^2}+\sum_{|\mu|\le|\gamma|+|\nu|}\|Z^\mu P_h u\|^2_{LE^*}$$
$$+\sum_{|\mu|\le|\gamma|+|\nu|}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta,Z^\mu]u\|^2_{LE^*}+\sum_{|\mu|\le|\gamma|+|\nu|}\delta^2\log^2(2+T)\|Z^\mu u\|^2_{LE^1}$$
$$+\sum_{|\mu|\le|\gamma|+|\nu|-1}\|Z^\mu P_g u\|^2_{LE}+\sum_{|\mu|\le|\gamma|+|\nu|-1}\|Z^\mu P_g u\|^2_{L^\infty L^2}+\sum_{|\mu|\le|\gamma|-1}\|\partial^\mu[D_ig^{ij}_{sr}D_j,\Omega^\nu]u\|^2_{LE}$$
$$+\sum_{|\mu|\le|\gamma|-1}\|\partial^\mu[D_ig^{ij}_{sr}D_j,\Omega^\nu]u\|^2_{L^\infty L^2}+\sum_{k\le|\gamma|}\|\partial_t^k[D_ig^{ij}_{sr}D_j,\Omega^\nu]u\|^2_{LE^*}.$$

Utilizing the decay properties of g^{ij}_{sr} in (1.16), we see the last three terms in the above equation are controlled via

$$\sum_{|\mu|\le|\gamma|-1} \|\partial^\mu [D_i g_{sr}^{ij} D_j, \Omega^\nu] u\|_{LE}^2 + \sum_{|\mu|\le|\gamma|-1} \|\partial^\mu [D_i g_{sr}^{ij} D_j, \Omega^\nu] u\|_{L^\infty L^2}^2 + \sum_{k\le|\gamma|} \|\partial_t^k [D_i g_{sr}^{ij} D_j, \Omega^\nu] u\|_{LE^*}^2$$
$$\lesssim \sum_{|\mu|\le|\gamma|} \sum_{|\eta|\le|\nu|-1} \|\partial^\mu \Omega^\eta \partial u\|_{LE}^2 + \sum_{|\mu|\le|\gamma|} \sum_{|\eta|\le|\nu|-1} \|\partial^\mu \Omega^\eta \partial u\|_{L^\infty L^2}^2 + \sum_{|\mu|\le|\gamma|} \sum_{|\eta|\le|\nu|-1} \|\partial^\mu \Omega^\eta \partial \partial_t u\|_{LE}^2.$$

The first two terms on the right-hand side of the above equation are bound using the inductive hypothesis. The third term on the right-hand side of the above equation is also bound by applying the inductive hypothesis to $\partial_t u$. This completes the proof. □

Chapter 9: Proof of Theorem 1.2

In this chapter, we prove Theorem 1.2 via an iteration argument. Given the energy estimates in the prior chapter, we can apply the argument of [27], rooted in the previous argument of [15]. We first record a now standard weighted Sobolev inequality as proved in [19].

Lemma 9.1. *Let $f(x) \in C^\infty(\mathbb{R}^3)$ and fix $R > 1$. Then*

$$\|f\|_{L^\infty(\{R/2\le|x|\le R\})} \lesssim R^{-1} \sum_{|\mu|+|\nu|\le 2} \|\partial_x^\mu \Omega^\nu f\|_{L^2(\{R/4\le|x|\le 2R\})}. \tag{9.1}$$

Remark. *The proof follows by applying Sobolev embeddings on $\mathbb{R}\times\mathbb{S}^2$.*

Remark. *A similar bound holds for $|x| \le 1$ by standard Sobolev embeddings.*

We set $u_{-1} \equiv 0$ and define a sequence u_k, $k = 0, 1, 2, \ldots$ to solve the following linearized problem:

$$\begin{cases} P_g u_k = Q(\partial u_{k-1}, \partial^2 u_k) \quad (t,x) \in [0, T_\varepsilon] \times \mathbb{R}^3 \\ u_k(0,\cdot) = f_1 \in C^\infty(\mathbb{R}^3), \quad \partial_t u_k(0,\cdot) = f_2 \in C^\infty(\mathbb{R}^3), \end{cases} \tag{9.2}$$

where f_1 and f_2 are small in the following sense:

$$\sum_{|k|+|\gamma|\le 15} \|\partial_x^k \Omega^\gamma \nabla_x f_1\|_{L^2} + \sum_{|k|+|\gamma|\le 15} \|\partial_x^k \Omega^\gamma f_2\|_{L^2} \le \varepsilon.$$

We now show that a solution to (9.2) satisfies

$$\sum_{|\mu|\le 15} \Big(\|\partial^\mu \partial u_k\|_{L^\infty L^2} + \|\partial^\mu u_k\|_{LE^1} \Big) + \sum_{|\mu|\le 14} \Big(\|Z^\mu \partial u_k\|_{L^\infty L^2} + \|Z^\mu u_k\|_{LE^1} \Big) \le C\varepsilon, \tag{9.3}$$

for $0 \le t \le T_\varepsilon$, where T_ε is as in (1.20). Here C is a uniform constant in k. We will apply Lemma 8.1 and Lemma 8.2 by setting $h^{\alpha\beta} = -B_\gamma^{\alpha\beta}\partial_\gamma u_{k-1}$. We define

$$M_k(T_\varepsilon) = \sum_{|\mu|\le 15} \Big(\|\partial^\mu \partial u_k\|_{L^\infty L^2} + \|\partial^\mu u_k\|_{LE^1} \Big) + \sum_{|\mu|\le 14} \Big(\|Z^\mu \partial u_k\|_{L^\infty L^2} + \|Z^\mu u_k\|_{LE^1} \Big), \tag{9.4}$$

where all norms in time are taken over $[0, T_\varepsilon]$. Observe by (8.1) and (8.4), $M_0(T_\varepsilon) \le C_0\varepsilon$, where C_0 is the sum of the implicit constants in (8.1) and (8.4). We will show inductively that for $\varepsilon < \varepsilon_0$, where ε_0 is sufficiently small and $\kappa > 0$ is sufficiently small, where κ is as in (1.20), that

$$M_k(T_\varepsilon) \le 10C_0\varepsilon. \tag{9.5}$$

Proof of (9.5). Applying (8.1) and (8.4) with $\delta = C_1\varepsilon$, we see

$$\begin{aligned} M_k(T_\varepsilon) \le C_0\varepsilon + C_0\Bigg(&\sum_{j\le 15} \|\partial_t^j P_h u_k\|_{LE^*}^2 + \sum_{|\mu|\le 14} \|Z^\mu P_h u_k\|_{LE^*}^2 + \sum_{j\le 15} \|[h^{\alpha\beta}\partial_\alpha\partial_\beta, \partial_t^j]u_k\|_{LE^*} \\ &+ \sum_{|\mu|\le 14} \|[h^{\alpha\beta}\partial_\alpha\partial_\beta, Z^\mu]u_k\|_{LE^*} + \sum_{|\mu|\le 14} \|\partial^\mu P_g u_k\|_{LE} + \sum_{|\mu|\le 14} \|\partial^\mu P_g u_k\|_{L^\infty L^2} \\ &+ \sum_{|\mu|\le 13} \|Z^\mu P_g u_k\|_{LE} + \sum_{|\mu|\le 13} \|Z^\mu P_g u_k\|_{L^\infty L^2}\Bigg) \\ &+ \sum_{|\mu|\le 15} C_1^2\varepsilon^2 \log^2(2+T_\varepsilon)\|\partial^\mu u_k\|_{LE^1} + \sum_{|\mu|\le 14} C_1^2\varepsilon^2 \log^2(2+T_\varepsilon)\|Z^\mu u_k\|_{LE^1}. \end{aligned} \tag{9.6}$$

It is permissible to take $\delta = C_1\varepsilon$ by our inductive hypothesis and (9.1). Indeed, for $0 \le t \le T_\varepsilon$,

$$\|\langle x\rangle \partial^{\le 2} h\|_{L^\infty L^\infty([0,T]\times A_l)} \approx \|\langle x\rangle \partial^{\le 2}\partial u_{k-1}\|_{L^\infty L^\infty([0,T]\times A_l)} \lesssim \sum_{|\mu|+|\nu|\le 4} \|\partial_x^\mu \Omega^\nu \partial u_{k-1}\|_{L^\infty L^2([0,T]\times \tilde{A}_l)},$$

where we have applied (9.1) and $\tilde{A}_l$ denotes a slightly fattened dyadic annulus. Hence,

$$\|\langle x\rangle \partial^{\le 2} h\|_{\ell_l^\infty L^\infty L^\infty([0,T]\times A_l)} \lesssim M_{k-1}(T_\varepsilon) \le C_1\varepsilon,$$

where C_1 is a constant that depends on C_0, the implicit constant in (9.1), and the collection of constants in $B_\gamma^{\alpha\beta}$.

Observe that

$$\begin{aligned} \sum_{|\mu|\le 15} \Big(|\partial^\mu P_h u_k| + |[\partial^\mu, h^{\alpha\beta}\partial_\alpha\partial_\beta]u_k|\Big) \lesssim &\sum_{|\mu|\le 7} |\partial^\mu \partial u_{k-1}| \sum_{|\nu|\le 15} |\partial^\nu \partial u_k| \\ &+ \sum_{|\mu|\le 8} |\partial^\mu \partial u_k| \sum_{|\nu|\le 15} |\partial^\nu \partial u_{k-1}| + \sum_{|\mu|\le 7} |\partial^\mu \partial u_{k-1}| \sum_{|\nu|\le 15} |\partial^\nu \partial u_{k-1}| \end{aligned} \tag{9.7}$$

and

$$\sum_{|\mu|\le 14}\Big(|Z^\mu P_h u_k| + |[Z^\mu, h^{\alpha\beta}\partial_\alpha\partial_\beta]u_k|\Big) \lesssim \sum_{|\mu|\le 7}|Z^\mu \partial u_{k-1}|\sum_{|\nu|\le 14}|Z^\nu\partial u_k|$$
$$+\sum_{|\mu|\le 8}|Z^\mu\partial u_k|\sum_{|\nu|\le 14}|Z^\nu\partial u_{k-1}| + \sum_{|\mu|\le 7}|Z^\mu\partial u_{k-1}|\sum_{|\nu|\le 14}|Z^\nu\partial u_{k-1}|. \tag{9.8}$$

Therefore the second, third, fourth, and fifth terms on the right-hand side of (9.6) are controlled by the following:

$$\sum_{j\le 15}\|\partial_t^j P_h u_k\|_{LE^*} + \sum_{|\mu|\le 14}\|Z^\mu P_h u_k\|_{LE^*} + \sum_{j\le 15}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta, \partial_t^j]u_k\|_{LE^*} + \sum_{|\mu|\le 14}\|[h^{\alpha\beta}\partial_\alpha\partial_\beta, Z^\mu]u_k\|_{LE^*}$$
$$\lesssim \sum_l 2^{l/2}\sum_{|\mu|\le 7}\|\partial^\mu\partial u_{k-1}\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_k\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{l/2}\sum_{|\mu|\le 8}\|\partial^\mu\partial u_k\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{l/2}\sum_{|\mu|\le 7}\|\partial^\mu\partial u_{k-1}\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$\sum_l 2^{l/2}\sum_{|\mu|\le 7}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_k\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{l/2}\sum_{|\mu|\le 8}\|Z^\mu\partial u_k\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{l/2}\sum_{|\mu|\le 7}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^\infty([0,T_\varepsilon]\times A_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}.$$

Applying (9.1), we find the above is controlled by

$$\sum_l 2^{-l/2}\sum_{|\mu|\le 9}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_k\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{-l/2}\sum_{|\mu|\le 10}\|Z^\mu\partial u_k\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{-l/2}\sum_{|\mu|\le 9}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{-l/2}\sum_{|\mu|\le 9}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_k\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{-l/2}\sum_{|\mu|\le 10}\|Z^\mu\partial u_k\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)}$$
$$+\sum_l 2^{-l/2}\sum_{|\mu|\le 9}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^2([0,T_\varepsilon]\times\tilde{A}_l)}\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^2L^2([0,T_\varepsilon]\times A_l)},$$

where $\tilde{A}_l$ is a slightly fattened annulus. The above line, in turn, is bounded by

$$\sum_{|\mu|\le 9} \|Z^\mu \partial u_{k-1}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 15} \log(2+T_\varepsilon)\|\partial^\nu u_k\|_{LE^1} + \sum_{|\nu|\le 15}\|\partial^\nu u_k\|_{L^\infty L^2}\Big)$$
$$+ \sum_{|\mu|\le 10} \|Z^\mu \partial u_{k}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 15} \log(2+T_\varepsilon)\|\partial^\nu u_{k-1}\|_{LE^1} + \sum_{|\nu|\le 15}\|\partial^\nu u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+ \sum_{|\mu|\le 9} \|Z^\mu \partial u_{k-1}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 15} \log(2+T_\varepsilon)\|\partial^\nu u_{k-1}\|_{LE^1} + \sum_{|\nu|\le 15}\|\partial^\nu u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+ \sum_{|\mu|\le 9} \|Z^\mu \partial u_{k-1}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 14} \log(2+T_\varepsilon)\|Z^\nu u_k\|_{LE^1} + \sum_{|\nu|\le 14}\|Z^\nu u_k\|_{L^\infty L^2}\Big)$$
$$+ \sum_{|\mu|\le 10} \|Z^\mu \partial u_{k}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 14} \log(2+T_\varepsilon)\|Z^\nu u_{k-1}\|_{LE^1} + \sum_{|\nu|\le 14}\|Z^\nu u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+ \sum_{|\mu|\le 9} \|Z^\mu \partial u_{k-1}\|_{L^\infty L^2}\Big(\sum_{|\nu|\le 14} \log(2+T_\varepsilon)\|Z^\nu u_{k-1}\|_{LE^1} + \sum_{|\nu|\le 14}\|Z^\nu u_{k-1}\|_{L^\infty L^2}\Big),$$

where we have used methods as in (2.3). Using the inductive hypothesis, this is bounded by

$$\Big(1+\log(2+T_\varepsilon)\Big)\Big(\varepsilon M_k(T_\varepsilon)+\varepsilon^2\Big). \tag{9.9}$$

The sixth, seventh, eighth, and ninth terms in (9.6) are controlled via

$$\sum_{|\mu|\le 14}\|\partial^\mu P_g u_k\|_{LE} + \sum_{|\mu|\le 14}\|\partial^\mu P_g u_k\|_{L^\infty L^2} + \sum_{|\mu|\le 13}\|Z^\mu P_g u_k\|_{LE} + \sum_{|\mu|\le 13}\|Z^\mu P_g u_k\|_{L^\infty L^2}$$
$$\lesssim \sum_{|\mu|\le 14}\|\partial^\mu (\partial u_{k-1})^2\|_{LE} + \sum_{|\mu|\le 14}\|\partial^\mu (\partial u_{k-1})^2\|_{L^\infty L^2} + \sum_{|\mu|\le 13}\|Z^\mu (\partial u_{k-1})^2\|_{LE}$$
$$+ \sum_{|\mu|\le 13}\|Z^\mu (\partial u_{k-1})^2\|_{L^\infty L^2} + \sum_{|\mu|\le 14}\|\partial^\mu (\partial u_{k-1}\partial^2 u_k)\|_{LE} + \sum_{|\mu|\le 14}\|\partial^\mu (\partial u_{k-1}\partial^2 u_k)\|_{L^\infty L^2}$$
$$+ \sum_{|\mu|\le 13}\|Z^\mu (\partial u_{k-1}\partial^2 u_k)\|_{LE} + \sum_{|\mu|\le 13}\|Z^\mu (\partial u_{k-1}\partial^2 u_k)\|_{L^\infty L^2}.$$

Applying nearly identical analysis to bounding the second, third, fourth, and fifth terms, we find that this is controlled by

$$\sum_{|\mu|\le 7}\|\partial^\mu\partial u_{k-1}\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 15}\|\partial^\nu\partial u_k\|_{LE}+\sum_{|\nu|\le 15}\|\partial^\nu\partial u_k\|_{L^\infty L^2}\Big)$$
$$+\sum_{|\mu|\le 8}\|\partial^\mu\partial u_k\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{LE}+\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+\sum_{|\mu|\le 7}\|\partial^\mu\partial u_{k-1}\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{LE}+\sum_{|\nu|\le 15}\|\partial^\nu\partial u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+\sum_{|\mu|\le 7}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 14}\|Z^\nu\partial u_k\|_{LE}+\sum_{|\nu|\le 14}\|Z^\nu\partial u_k\|_{L^\infty L^2}\Big)$$
$$+\sum_{|\mu|\le 8}\|Z^\mu\partial u_k\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{LE}+\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^\infty L^2}\Big)$$
$$+\sum_{|\mu|\le 7}\|Z^\mu\partial u_{k-1}\|_{L^\infty L^\infty}\Big(\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{LE}+\sum_{|\nu|\le 14}\|Z^\nu\partial u_{k-1}\|_{L^\infty L^2}\Big).$$

Applying Sobolev embeddings (9.1) and our inductive hypothesis, we find that this is controlled by

$$\varepsilon M_k(T_\varepsilon)+\varepsilon^2. \tag{9.10}$$

Combing (9.6), (9.9), and (9.10), we have proved the following bound

$$M_k(T_\varepsilon)\le C_0\varepsilon + C_2\log(2+T_\varepsilon)\Big(\varepsilon M_k(T_\varepsilon)+\varepsilon^2\Big)+C_3\Big(\varepsilon M_k(T_\varepsilon)+\varepsilon^2\Big)+C_1^2\varepsilon^2\log^2(2+T_\varepsilon)M_k(T_\varepsilon),$$

where C_2 and C_3 can depend on C_0. Choosing $\kappa \ll 1$ sufficiently small compared to C_1, C_2, and C_3, and then $\varepsilon > 0$ sufficiently small, we obtain (9.5) as desired. □

To show that u_k converges to a solution of (1.13), we define

$$A_k(T_\varepsilon)=\sum_{|\mu|\le 14}\Big(\|\partial^\mu\partial(u_k-u_{k-1})\|_{L^\infty L^2}+\|\partial^\mu(u_k-u_{k-1})\|_{LE^1}\Big)$$
$$+\sum_{|\mu|\le 13}\Big(\|Z^\mu\partial(u_k-u_{k-1})\|_{L^\infty L^2}+\|Z^\mu(u_k-u_{k-1})\|_{LE^1}\Big), \tag{9.11}$$

and observe that it suffices to show $A_k(T_\varepsilon)$ is Cauchy. Specifically, we will prove

$$A_k(T_\varepsilon)\le \frac{1}{2}A_{k-1}(T_\varepsilon). \tag{9.12}$$

Proof of (9.12). This is very similar to the proof of (9.5). We set $h^{\alpha\beta}=-B^{\alpha\beta}_\gamma\partial_\gamma u_{k-1}$ and observe

$$P_h(u_k-u_{k-1})=Q(\partial u_{k-1})-Q(\partial u_{k-2})+B^{\alpha\beta}_\gamma\partial_\alpha\partial_\beta u_{k-1}(\partial^\gamma u_{k-1}-\partial^\gamma u_{k-2}). \tag{9.13}$$

Applying Lemma 8.1 and Lemma 8.2 to $u_k - u_{k-1}$, one obtains

$$
\begin{aligned}
A_k(T_\varepsilon) \lesssim & \sum_{j\le 14} \|\partial_t^j P_h(u_k-u_{k-1})\|_{LE^*} + \sum_{|\mu|\le 13} \|Z^\mu P_h(u_k-u_{k-1})\|_{LE^*} \\
& + \sum_{j\le 14} \|[h^{\alpha\beta}\partial_\alpha\partial_\beta, \partial_t^j](u_k-u_{k-1})\|_{LE^*} + \sum_{|\mu|\le 13} \|[h^{\alpha\beta}\partial_\alpha\partial_\beta, Z^\mu](u_k-u_{k-1})\|_{LE^*} \\
& + \sum_{|\mu|\le 13} \|\partial^\mu P_g(u_k-u_{k-1})\|_{LE} + \sum_{|\mu|\le 13} \|\partial^\mu P_g(u_k-u_{k-1})\|_{L^\infty L^2} \\
& + \sum_{|\mu|\le 12} \|Z^\mu P_g(u_k-u_{k-1})\|_{LE} + \sum_{|\mu|\le 12} \|Z^\mu P_g(u_k-u_{k-1})\|_{L^\infty L^2} \\
& + \sum_{|\mu|\le 14} \varepsilon^2 \log^2(2+T_\varepsilon)\|\partial^\mu(u_k-u_{k-1})\|_{LE^1} + \sum_{|\mu|\le 13} \varepsilon^2 \log^2(2+T_\varepsilon)\|Z^\mu(u_k-u_{k-1})\|_{LE^1}.
\end{aligned}
\tag{9.14}
$$

Since our nonlinearity is quadratic, we have the following useful bounds:

$$
\begin{aligned}
& \sum_{|\mu|\le 14} \Big(|\partial^\mu P_h(u_k-u_{k-1})| + |[\partial^\mu, h^{\alpha\beta}\partial_\alpha\partial_\beta](u_k-u_{k-1})|\Big) \\
& \lesssim \sum_{|\mu|\le 7} |\partial^\mu \partial u_{k-1}| \sum_{|\nu|\le 14} |\partial^\nu\partial(u_k-u_{k-1})| + \sum_{|\mu|\le 14} |\partial^\mu \partial u_{k-1}| \sum_{|\nu|\le 7} |\partial^\nu\partial(u_k-u_{k-1})| \\
& \quad + \Big(\sum_{|\mu|\le 8} |\partial^\mu\partial u_{k-1}| + \sum_{|\mu|\le 7} |\partial^\mu\partial u_{k-2}|\Big) \sum_{|\nu|\le 14} |\partial^\nu\partial(u_{k-1}-u_{k-2})| \\
& \quad + \Big(\sum_{|\mu|\le 15} |\partial^\mu\partial u_{k-1}| + \sum_{|\mu|\le 14} |\partial^\mu\partial u_{k-2}|\Big) \sum_{|\nu|\le 7} |\partial^\nu\partial(u_{k-1}-u_{k-2})|
\end{aligned}
\tag{9.15}
$$

and

$$
\begin{aligned}
& \sum_{|\mu|\le 13} \Big(|Z^\mu P_h(u_k-u_{k-1})| + |[Z^\mu, h^{\alpha\beta}\partial_\alpha\partial_\beta](u_k-u_{k-1})|\Big) \\
& \lesssim \sum_{|\mu|\le 7} |Z^\mu \partial u_{k-1}| \sum_{|\nu|\le 13} |Z^\nu\partial(u_k-u_{k-1})| + \sum_{|\mu|\le 13} |Z^\mu \partial u_{k-1}| \sum_{|\nu|\le 7} |Z^\nu\partial(u_k-u_{k-1})| \\
& \quad + \Big(\sum_{|\mu|\le 8} |Z^\mu\partial u_{k-1}| + \sum_{|\mu|\le 7} |Z^\mu\partial u_{k-2}|\Big) \sum_{|\nu|\le 13} |Z^\nu\partial(u_{k-1}-u_{k-2})| \\
& \quad + \Big(\sum_{|\mu|\le 14} |Z^\mu\partial u_{k-1}| + \sum_{|\mu|\le 13} |Z^\mu\partial u_{k-2}|\Big) \sum_{|\nu|\le 7} |Z^\nu\partial(u_{k-1}-u_{k-2})|.
\end{aligned}
\tag{9.16}
$$

Hence applying (9.5) and a nearly identical analysis to our bound for (9.6), we find the first four terms on the right-hand side of (9.14) are controlled by

$$M_{k-1}(T_\varepsilon)\Big(1+\log(2+T_\varepsilon)\Big)A_k(T_\varepsilon)+\Big(M_{k-1}(T_\varepsilon)+M_{k-2}(T_\varepsilon)\Big)\Big(1+\log(2+T_\varepsilon)\Big)A_{k-1}(T_\varepsilon). \tag{9.17}$$

Similarly, the fifth, sixth, seventh, and eighth terms on the right-hand side of (9.14) are bounded from above by

$$M_{k-1}(T_\varepsilon)A_k(T_\varepsilon)+\Big(M_{k-1}(T_\varepsilon)+M_{k-2}(T_\varepsilon)\Big)A_{k-1}(T_\varepsilon). \tag{9.18}$$

Combining (9.14), (9.17), and (9.18) shows

$$\begin{aligned}A_k(T_\varepsilon)\lesssim M_{k-1}(T_\varepsilon)\Big(1+\log(2+T_\varepsilon)\Big)A_k(T_\varepsilon)+\Big(M_{k-1}(T_\varepsilon)+M_{k-2}(T_\varepsilon)\Big)\Big(1+\log(2+T_\varepsilon)\Big)A_{k-1}(T_\varepsilon)\\ +\varepsilon^2\log^2(2+T)A_k(T_\varepsilon).\end{aligned}$$

Utilizing (9.5), defining T_ε as in (1.20), fixing $\kappa>0$ sufficiently small compared to our implicit constant, and $\varepsilon>0$ sufficiently small completes the proof of (9.12). □

Proof of uniqueness. To demonstrate uniqueness, let u and v be two solutions to (1.13). We define T_1 to be the first time that u and v differ via $T_1=\inf_t\{t:|u(t)-v(t)|>0\}$. Uniqueness now follows from running a standard local argument beginning at time T_1 [46, Theorem 4.1]. Indeed, if $u\neq v$ for $t>T_1$, this will contradict local uniqueness of solutions. □

Proof of smoothness. Smoothness on the interval $[0,T_\varepsilon]$ is proven similarly. Indeed, we proceed by contradiction and define T_2 to be the first time that u is not smooth via $T_2=\inf_t\{t:u\notin C^\infty([0,T_\varepsilon]\times\mathbb{R}^3)\}$. Smoothness now follows from running a standard local argument beginning at time T_2 [46, Theorem 4.3]. Indeed, if $u\notin C^\infty([0,T_\varepsilon]\times\mathbb{R}^3)\}$ for $t>T_2$, this will contradict local smoothness of solutions. □

Appendix A: Some Microlocal Analysis

We record some standard definitions and theorems from microlocal analysis. There are many excellent texts on the subject. We primarily use [10] and [53]. Another excellent reference is [60], although its perspective is that of semiclassical analysis instead of traditional microlocal analysis.

We begin by defining the standard symbol class S^m for symbols $\sigma(x,\xi) \in \mathbb{R}^{2n}$:

Definition A.1 Symbol Classes. [53, (1.4)]. The symbol class, S^m, consists of the following elements:

$$S^m = \{\sigma \in C^\infty(\mathbb{R}^{2n}) : |D_\xi^\alpha D_x^\beta \sigma(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}\}, \tag{A.1}$$

where $m \in \mathbb{R}$.

Definition A.2 Weyl Quantization. [10, (2.3)]. We use the notation OPS^m to denote the operator class corresponding to the Weyl quantization of symbols $\sigma(x,\xi) \in S^m$ obtained via the formula:

$$\sigma^w(x, D_x)u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(\frac{x+y}{2},\xi)u(y)\, dyd\xi. \tag{A.2}$$

Definition A.3 Kohn-Nirenberg Quantization. [10, (2.31)]. We use $\sigma(x, D_x)_{KN}$ to denote the operator corresponding to the so-called standard quantization or Kohn-Nirenberg quantization:

$$\sigma(x, D_x)_{KN}u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)u(y)\, dyd\xi. \tag{A.3}$$

The following formula is a useful way of describing the Kohn-Nirenberg quantization (see [60], (4.1.5)):

$$\sigma(x, D_x)_{KN}u(x) = \mathcal{F}^{-1}a(x,\xi)\mathcal{F}u(x), \tag{A.4}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse Fourier transform, respectively. We now list several standard theorems.

Theorem A.1 (Schwartz Kernel Representation. [53, (2.1)], [10, (2.2)]). *To an operator $\sigma^w(x, D_x) \in OPS^m$, there corresponds a Schwartz kernel $K(x,y) \in D'(\mathbb{R}^n, \mathbb{R}^n)$, satisfying*

$$\langle v(x)u(y), K(x,y)\rangle_{L^2} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v(x)e^{i(x-y)\cdot\xi}\sigma(\frac{x+y}{2},\xi)u(y)\, dyd\xi dx. \tag{A.5}$$

Therefore $K(x,y)$ is given by the following oscillatory integral (in the sense of distributions)

$$K(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(\frac{x+y}{2},\xi)\,d\xi. \tag{A.6}$$

Remark. *The above theorem can be found in [10] for the Weyl quantization. The same statement holds for the Kohn-Nirenberg quantization and can be found in [53]. The results and method of proof are the same for both quantizations. We note that [53] provides slightly more details and our statement more closely resembles this framework.*

We note that the integral kernel defined above has good spatial decay away from the diagonal. In particular, the following theorem holds:

Theorem A.2 ([10, Theorem 2.53])**.** *Suppose $\sigma \in S^m$ and let $K(x,y)$ be the integral kernel of $\sigma^w(x,D_x)$. Then $(x-y)^j K(x,y)$ is of class C^k, and its derivatives of order $\leq k$ are bounded, provided $j > m+n+k$. In particular, $K(x,y)$ is C^∞ off the diagonal and is rapidly decreasing as $x-y \to \infty$.*

Remark. *Theorem A.2 is also true for the Kohn-Nirenberg quantization. For more on this, see the discussion under Proposition 5.1 in [53]. The exposition in [53] clarifies the meaning of rapidly decreasing as $|x-y| \to \infty$ via the following useful estimate:*

$$|K(x,y)| \leq C_N|x-y|^{-N} \qquad |x-y| \geq 1, \tag{A.7}$$

which is true for each $N \in \mathbb{N}$ where C_N is a positive constant.

Theorem A.3 ([60, Theorem 4.1])**.** *Let $\sigma(x,\xi) \in S^m$ be a real valued symbol. Then, $\sigma^w(x,D_x)$ is a self-adjoint operator.*

It is sometimes useful to switch between the Weyl and the Kohn-Nirenberg quantizations. The following theorem allows us to do so, at the expense of a lower order operator.

Theorem A.4 ([10, Corollary 2.43])**.** *Let $\sigma(x,\xi) \in S^m$. Then $\sigma(x,D_x)_{KN} - \sigma^w(x,D_x) \in OPS^{m-1}$.*

Theorem A.5 ([10, Corollary 2.51])**.** *Let $\sigma_1(x,\xi) \in S^{m_1}$ and $\sigma_2(x,\xi) \in S^{m_2}$. Then $[\sigma_1^w(x,\xi), \sigma_2^w(x,\xi)] \in OPS^{m_1+m_2-1}$. Note that the same conclusion is true when considering the Kohn-Nirenberg quantization (see [53, (3.24)]).*

Theorem A.6 ([10, Corollary 2.61])**.** *Let $\sigma(x,\xi) \in S^m$, with $m, s \in \mathbb{R}$. Then $\sigma^w(x,D_x)$ is a bounded linear operator from $H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$. The same is true for the Kohn-Nirenberg quantization (see [53, (3.24)]) .*

The next theorem yields a useful asymptotic expansion for the composition of operators in the Weyl calculus.

Theorem A.7 ([10, Theorem 2.47, Theorem 2.49]). *Let* $a(x,\xi) \in S^{m_1}$ *and* $b(x,\xi) \in S^{m_2}$. *Then* $a^w(x,D_x)b^w(x,D_x) \equiv (a\#b)^w(x,D_x) \in OPS^{m_1+m_2}$ *and we have the following asymptotic expansion for such an operator:*

$$a^w b^w \equiv (a\#b)^w = \Big(\sum_{|\alpha|+|\beta|=0}^{N} \frac{2^{-|\alpha|-|\beta|}}{|\alpha|!|\beta|!}\frac{(-1)^{|\alpha|}}{(i)^{|\alpha|+|\beta|}}\partial_\xi^\beta\partial_x^\alpha a(x,\xi)\partial_\xi^\alpha\partial_x^\beta b(x,\xi)\Big)^w + R_N(x,\xi), \tag{A.8}$$

where $N \in \mathbb{N}$ *and* $R_N(x,\xi) \in OPS^{m_1+m_2-N-1}$. *We note that truncating* (A.8) *after* $N = 1$ *and* $N = 2$ *(respectively) immediately implies the following two equations, which will be primarily what we utilize:*

$$a^w b^w = (ab)^w + \frac{1}{2i}\{a,b\}^w + R_2(x,\xi)$$

and

$$[a^w, b^w] = \frac{1}{i}\{a,b\}^w + \tilde{R}(x,\xi),$$

where $\tilde{R}(x,\xi) \in OPS^{m_1+m_2-3}$. *Similar results hold for the Kohn-Nirenberg quantization, except now* $\tilde{R}(x,\xi) \in OPS^{m_1+m_2-2}$ *(see [53, Proposition 3.3],).*

Theorem A.8 ([10, Theorem 2.63]). *The Gårding Inequality.*

Suppose $\sigma \in S^m$ *and for some* $A, B > 0$,

$$\operatorname{Re}\sigma(x,\xi) \geq A\langle\xi\rangle^m \quad for \quad \langle\xi\rangle \geq B.$$

Then for any $\varepsilon > 0$ *and* $a > 0$, *there is a constant* $C_{a\varepsilon}$ *such that*

$$\operatorname{Re}\langle\sigma^w(x,D_x)f(x), f(x)\rangle_{L^2} \geq (A-\varepsilon)\|f\|^2_{H^{m/2}} - C_{a\varepsilon}\|f\|^2_{H^{(m-a)/2}},$$

where $f \in H^{m/2}(\mathbb{R}^n)$. *The same is true for the Kohn-Nirenberg quantization ([53, Theorem 6.1]).*

Appendix B: Construction of the Weight Functions from Proposition 5.2 and Proposition 5.3

In this appendix, we construct the weight functions used in the proofs of Proposition 5.2 and Proposition 5.3. For the remainder of this appendix, we fix R_{AF} sufficiently large and further fix $R_{ext} > R_{out} > R_{m1} > R_{AF}$, where R_{out} and R_{ext} will be given somewhat explicitly when necessary.

Constructing the Weight from Proposition 5.2

We begin with the weight from Proposition 5.2. Recall, we need a function $\phi = \phi(s)$, where $s = \log(r)$ such that

$$\lambda \lesssim \phi''(s) \leq \phi'(s)/2 \lesssim \phi''(s), \qquad |\phi'''(s)| \ll \phi'(s), \qquad \lambda \gg 1 \tag{B.1}$$

for $r \leq 2R_{out}$ and is constant for $r \geq 2R_{ext}$. In the transition region, we may break the conditions on ϕ' and ϕ'', but will still have good bounds for ϕ' and ϕ''. In particular for $R_{out} \leq r \leq 3R_{ext}/2$, $\phi' \gtrsim 1$, but ϕ'' will be in part negative, with the following bound from below $|\phi''| \leq \phi'/2$. For $r \geq R_{ext}$, $|\phi'| + |\phi''| + |\phi'''| \ll 1$.

Remark. *We note that this weight satisfies all the requirements for Proposition 5.1 on the support of u.*

We define ϕ via:

$$\phi(s) = \int_{R_{AF}}^{s} \beta(q) \frac{\lambda R_0 e^{q/2}}{R_0 + \varepsilon_0 e^q}\, dq, \tag{B.2}$$

where $\beta(s)$ is a smooth cut-off function that is identically 1 for $s \leq \log(3R_{ext}/2)$ and 0 for $s \geq \log(2R_{ext})$. Here $R_0 \gg R_{AF}$ is chosen sufficiently large and $\varepsilon_0 > 0$ is chosen sufficiently small. Again, $\lambda \gg 1$. We define $R_{out} = R_0/(4\varepsilon_0)$ and $R_{ext} = 100\lambda^2 R_0^2/\varepsilon_0^2$.

Observe, for $s < \log(3R_{ext}/2)$,

$$\phi' = \frac{\lambda R_0 e^{s/2}}{R_0 + \varepsilon_0 e^s}, \qquad \phi'' = \frac{\lambda R_0 e^{s/2}(R_0 - \varepsilon_0 e^s)}{2(R_0 + \varepsilon_0 e^s)^2}, \qquad \phi''' = \frac{\lambda R_0 e^{s/2}(R_0^2 - 6R_0\varepsilon e^s + \varepsilon^2 e^{2s})}{4(R_0 + \varepsilon_0 e^s)^3}, \tag{B.3}$$

which shows $|\phi''| \leq \phi'/2$ and $|\phi'''| \lesssim \phi'$, as required. Further, a direct computation shows for $s \leq \log(2R_{out})$, $\lambda \lesssim \phi'(s) \approx \phi''(s)$, while for $\log(R_{out}) \leq s \leq \log(3R_{ext}/2)$, $\phi' \gtrsim 1$. Also for $r \geq \log(R_{ext})$, $|\phi'| + |\phi''| + |\phi'''| \ll 1$. Therefore, this construction meets all the desired requirements for Proposition 5.2.

Constructing the Weight from Proposition 5.3

We now construct a function that satisfies the hypothesis of Proposition 5.3 and verify that it satisfies all desired requirements. We need a radial function $\phi = \phi(r)$ that satisfies

$$\phi'(0) = 0, \qquad \phi'' \approx \lambda + \sigma\phi', \qquad |\phi'''| \lesssim \sigma^2\phi', \qquad \lambda, \sigma \gg 1 \qquad \text{and}$$
$$0 \le \phi'' - \frac{\phi'}{r} \lesssim_\sigma \phi' \qquad \forall\, r \qquad \text{while} \qquad \frac{\phi'}{r} \approx \phi'' \qquad \text{for } r \ll_\sigma 1. \tag{B.4}$$

We also note the useful properties that $\phi' \ge 0$ and $\phi'/r \ge \lambda$. We claim that

$$\phi(r) = \frac{\lambda}{\sigma^2}e^{\sigma r} - \frac{\lambda}{\sigma}r - \lambda\sigma\frac{r^3}{6} - \lambda\sigma^2\frac{r^4}{24} \tag{B.5}$$

satisfies all desired assumptions. Indeed

$$\phi'(r) = \frac{\lambda}{\sigma}e^{\sigma r} - \frac{\lambda}{\sigma} - \lambda\sigma\frac{r^2}{2} - \lambda\sigma^2\frac{r^3}{6},$$

and so $\phi'(0) = 0$. Taking a Taylor series expansion, we see

$$\phi'(r) = \lambda r + \frac{\lambda}{\sigma}\sum_{n=4}^{\infty}\frac{(\sigma r)^n}{n!},$$

and so $\phi'(r) \ge 0$ and $\phi'(r)/r \ge \lambda$ for all r. Further,

$$\phi''(r) = \lambda + \lambda\sum_{n=3}^{\infty}\frac{(\sigma r)^n}{n!}.$$

Hence,

$$\phi'' = \lambda + \sigma\phi' - \sigma\lambda r + \lambda\sigma^3 r^3/6.$$

This shows $\phi'' \approx \lambda + \sigma\phi'$ for $r \lesssim_\sigma 1$ and for $r \ge 1$. To investigate other values of r, we take a derivative of the function $g(r) = \lambda - \lambda\sigma r + \lambda\sigma^3 r^3/6$ to find $g'(r) = -\lambda\sigma + \lambda\sigma^3 r^2/2$. Since $r \ge 0$, this has one critical point: $r = \sqrt{2}/\sigma$. Since $g(\sqrt{2}/\sigma) \approx \lambda$, we see that $\phi'' \approx \lambda + \sigma\phi'$, as claimed.

Clearly, for $r \ll_\sigma 1$, $\phi'/r \approx \phi''$ as λ dominates all other terms. In addition,

$$\phi'' - \phi'/r = \lambda\sum_{n=3}^{\infty}\frac{(\sigma r)^n}{n!}\frac{n}{n+1} < \lambda\sum_{n=3}^{\infty}\frac{(\sigma r)^n}{n!} = \sigma\phi' - \lambda\sigma r + \lambda\sigma^3 r^3/6 \le \sigma\phi' + \lambda\sigma^3 r^3/6.$$

Observing $\lambda\sigma^3r^3/6 \le \lambda\sigma r + \lambda\sigma^4r^4/24$ (the first two terms of $\sigma\phi'$), shows that $\phi'' - \phi'/r \lesssim_\sigma \phi'$. Note this is true since the polynomial $x + x^4/24 - x^3/6$ has no real, positive roots.

Finally, direct computations show $\phi''' = \sigma^2\phi' - \lambda\sigma^2 r + \lambda\sigma^3r^2/2 + \lambda\sigma^4r^3/6 \lesssim \sigma^2\phi'$, as desired.

REFERENCES

[1] S. Alinhac. On the Morawetz-Keel-Smith-Sogge inequality for the wave equation on a curved background. *Publ. Res. Inst. Math. Sci.*, 42(3):705–720, 2006.

[2] J.-F. Bony and D. Häfner. The semilinear wave equation on asymptotically Euclidean manifolds. *Comm. Partial Differential Equations*, 35(1):23–67, 2010.

[3] R. Booth. Energy estimates on asymptotically flat surfaces of revolution. Master's Project, University of North Carolina at Chapel Hill, 2011.

[4] R. Booth, H. Christianson, J. Metcalfe, and J. Perry. Localized energy for wave equations with degenerate trapping. *Math. Res. Lett., to appear*, 2018.

[5] N. Burq. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.*, 180(1):1–29, 1998.

[6] N. Burq. Global Strichartz estimates for nontrapping geometries: about an article by H. F. Smith and C. D. Sogge: "Global Strichartz estimates for nontrapping perturbations of the Laplacian". *Comm. Partial Differential Equations*, 28(9-10):1675–1683, 2003.

[7] N. Burq, F. Planchon, J. G. Stalker, and A. S. Tahvildar-Zadeh. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay. *Indiana Univ. Math. J.*, 53(6):1665–1680, 2004.

[8] S.-i. Doi. Remarks on the Cauchy problem for Schrödinger-type equations. *Comm. Partial Differential Equations*, 21(1-2):163–178, 1996.

[9] L. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.

[10] G. B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[11] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[12] K. Hidano, J. Metcalfe, H. F. Smith, C. D. Sogge, and Y. Zhou. On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles. *Trans. Amer. Math. Soc.*, 362(5):2789–2809, 2010.

[13] K. Hidano and K. Yokoyama. A remark on the almost global existence theorems of Keel, Smith and Sogge. *Funkcial. Ekvac.*, 48(1):1–34, 2005.

[14] F. John. Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.*, 34(1):29–51, 1981.

[15] M. Keel, H. F. Smith, and C. D. Sogge. Almost global existence for some semilinear wave equations. *J. Anal. Math.*, 87:265–279, 2002. Dedicated to the memory of Thomas H. Wolff.

[16] M. Keel, H. F. Smith, and C. D. Sogge. Almost global existence for quasilinear wave equations in three space dimensions. *J. Amer. Math. Soc.*, 17(1):109–153, 2004.

[17] C. E. Kenig, G. Ponce, C. Rolvung, and L. Vega. Variable coefficient Schrödinger flows for ultrahyperbolic operators. *Adv. Math.*, 196(2):373–486, 2005.

[18] C. E. Kenig, G. Ponce, and L. Vega. On the Zakharov and Zakharov-Schulman systems. *J. Funct. Anal.*, 127(1):204–234, 1995.

[19] S. Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Comm. Pure Appl. Math.*, 38(3):321–332, 1985.

[20] S. Klainerman and T. C. Sideris. On almost global existence for nonrelativistic wave equations in 3D. *Comm. Pure Appl. Math.*, 49(3):307–321, 1996.

[21] H. Koch and D. Tataru. Carleman estimates and absence of embedded eigenvalues. *Comm. Math. Phys.*, 267(2):419–449, 2006.

[22] H. Lindblad and M. Tohaneanu. Global existence for quasilinear wave equations close to schwarzschild. *arXiv preprint arXiv:1610.00674*, 2016.

[23] J. Luk. The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes. *J. Eur. Math. Soc. (JEMS)*, 15(5):1629–1700, 2013.

[24] J. Marzuola, J. Metcalfe, and D. Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.

[25] J. Marzuola, J. Metcalfe, and D. Tataru. Wave packet parametrices for evolutions governed by PDO's with rough symbols. *Proc. Amer. Math. Soc.*, 136(2):597–604, 2008.

[26] J. Marzuola, J. Metcalfe, D. Tataru, and M. Tohaneanu. Strichartz estimates on Schwarzschild black hole backgrounds. *Comm. Math. Phys.*, 293(1):37–83, 2010.

[27] J. Metcalfe and C. D. Sogge. Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods. *SIAM J. Math. Anal.*, 38(1):188–209, 2006.

[28] J. Metcalfe and C. D. Sogge. Global existence of null-form wave equations in exterior domains. *Math. Z.*, 256(3):521–549, 2007.

[29] J. Metcalfe, J. Sterbenz, and D. Tataru. Local energy decay for scalar fields on time dependent non-trapping backgrounds. *Amer. J. Math., to appear*, 2017.

[30] J. Metcalfe and D. Tataru. Decay estimates for variable coefficient wave equations in exterior domains. In *Advances in phase space analysis of partial differential equations*, volume 78 of *Progr. Nonlinear Differential Equations Appl.*, pages 201–216. Birkhäuser Boston, Inc., Boston, MA, 2009.

[31] J. Metcalfe and D. Tataru. Global parametrices and dispersive estimates for variable coefficient wave equations. *Math. Ann.*, 353(4):1183–1237, 2012.

[32] J. Metcalfe, D. Tataru, and M. Tohaneanu. Price's law on nonstationary space-times. *Adv. Math.*, 230(3):995–1028, 2012.

[33] J. Metcalfe, D. Tataru, and M. Tohaneanu. Pointwise decay for the Maxwell field on black hole space-times. *Adv. Math.*, 316:53–93, 2017.

[34] J. L. Metcalfe. Global existence for semilinear wave equations exterior to nontrapping obstacles. *Houston J. Math.*, 30(1):259–281, 2004.

[35] C. S. Morawetz. Exponential decay of solutions of the wave equation. *Comm. Pure Appl. Math.*, 19:439–444, 1966.

[36] C. S. Morawetz. Time decay for the nonlinear Klein-Gordon equations. *Proc. Roy. Soc. Ser. A*, 306:291–296, 1968.

[37] C. S. Morawetz. Decay for solutions of the exterior problem for the wave equation. *Comm. Pure Appl. Math.*, 28:229–264, 1975.

[38] C. S. Morawetz, J. V. Ralston, and W. A. Strauss. Decay of solutions of the wave equation outside nontrapping obstacles. *Comm. Pure Appl. Math.*, 30(4):447–508, 1977.

[39] R. H. Price. Nonspherical perturbations of relativistic gravitational collapse. i. scalar and gravitational perturbations. *Phys. Rev. D*, 5:2419–2438, May 1972.

[40] R. H. Price. Nonspherical perturbations of relativistic gravitational collapse. ii. integer-spin, zero-rest-mass fields. *Phys. Rev. D*, 5:2439–2454, May 1972.

[41] T. C. Sideris. Global behavior of solutions to nonlinear wave equations in three dimensions. *Comm. Partial Differential Equations*, 8(12):1291–1323, 1983.

[42] T. C. Sideris. The null condition and global existence of nonlinear elastic waves. *Invent. Math.*, 123(2):323–342, 1996.

[43] T. C. Sideris. Nonresonance and global existence of prestressed nonlinear elastic waves. *Ann. of Math. (2)*, 151(2):849–874, 2000.

[44] T. C. Sideris and S.-Y. Tu. Global existence for systems of nonlinear wave equations in 3D with multiple speeds. *SIAM J. Math. Anal.*, 33(2):477–488, 2001.

[45] H. F. Smith and C. D. Sogge. Global Strichartz estimates for nontrapping perturbations of the Laplacian. *Comm. Partial Differential Equations*, 25(11-12):2171–2183, 2000.

[46] C. D. Sogge. *Lectures on non-linear wave equations*. International Press, Boston, MA, second edition, 2008.

[47] C. D. Sogge and C. Wang. Concerning the wave equation on asymptotically Euclidean manifolds. *J. Anal. Math.*, 112:1–32, 2010.

[48] G. Staffilani and D. Tataru. Strichartz estimates for a Schrödinger operator with nonsmooth coefficients. *Comm. Partial Differential Equations*, 27(7-8):1337–1372, 2002.

[49] J. Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *Int. Math. Res. Not.*, (4):187–231, 2005. With an appendix by Igor Rodnianski.

[50] W. A. Strauss. Dispersal of waves vanishing on the boundary of an exterior domain. *Comm. Pure Appl. Math.*, 28:265–278, 1975.

[51] D. Tataru. Parametrices and dispersive estimates for Schrödinger operators with variable coefficients. *Amer. J. Math.*, 130(3):571–634, 2008.

[52] D. Tataru. Local decay of waves on asymptotically flat stationary space-times. *Amer. J. Math.*, 135(2):361–401, 2013.

[53] M. E. Taylor. *Partial Differential Equations II*. Springer, 2014.

[54] M. Tohaneanu. Strichartz estimates on Kerr black hole backgrounds. *Trans. Amer. Math. Soc.*, 364(2):689–702, 2012.

[55] C. Wang and X. Yu. Global existence of null-form wave equations on small asymptotically Euclidean manifolds. *J. Funct. Anal.*, 266(9):5676–5708, 2014.

[56] S. Yang. Global solutions of nonlinear wave equations in time dependent inhomogeneous media. *Arch. Ration. Mech. Anal.*, 209(2):683–728, 2013.

[57] S. Yang. Global solutions of nonlinear wave equations with large data. *Selecta Math. (N.S.)*, 21(4):1405–1427, 2015.

[58] S. Yang. Global stability of solutions to nonlinear wave equations. *Selecta Math. (N.S.)*, 21(3):833–881, 2015.

[59] S. Yang. On the quasilinear wave equations in time dependent inhomogeneous media. *J. Hyperbolic Differ. Equ.*, 13(2):273–330, 2016.

[60] M. Zworski. *Semiclassical analysis*, volume 138. American Mathematical Society Providence, RI, 2012.