BRIDGELAND STABILITY AND NON-COMMUTATIVE TORI

SAMUEL DEHORITY

1. Introduction

The goal of this paper is to look at strong theorems relating the differential geometry of vector bundles to their stability in an algebraic sense and start to see how these theorems might be extended to more refined notions of stability. Bridgeland stability is in particular defined for objects in the derived category $D(X) = D^b(Coh(X))$ for a variety X. We will show in Corollary 9.1 that, at least when X is an elliptic curve, there is a heart of a bounded t-structure in $D(X)$ which is equivalent to a category of vector bundles on a non-commutative torus related to X based on a description of this latter category by Polishchuk and Schwarz [17]. Under this equivalence, Bridgeland stable objects correspond to bundles with special connections.

This paper is outlined as follows: Section 2 recalls the Hermitian-Yang-Mills equation, and Section 3 describes the relation between this equation and Mumford stability. Section 4 recounts the extension of this to Gieseker stability. Then Section 5 gives background on t-structures and tilting, which is used in Section 6 to describe Bridgeland stability in general and Section 7 which specializes to Bridgeland stability on an elliptic curve. Using a rapid introduction to complex non-commutative geometry in Section 8, the aforementioned equivalence is described in Section 9.

2. Hermitian-Yang-Mills Equation

To introduce the topic, it is useful to recall the workings of Hodge theory, as there are sharp analogies between Hodge theory and the types of questions with which we are presently concerned. Let X be a complex Kahler manifold of real dimension n and let $[\alpha] \in H^{p,q}_{\bar{\partial}}(X)$ be a Dolbeault cohomology class. We define a norm on the space $A^{p,q}(X)$ of (p,q)-forms by first defining an inner product on $\Lambda^k V$ for a vector space V as $\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle)$ and extending sesquilinearly. Then define, by integrating over X, the norm

$$|\phi|^2 = \int_X \langle \phi(x), \phi(x) \rangle dx. \tag{2.1}$$

The Hodge theorem in particular states that there is a unique representative $\alpha_0 \in \{\phi | [\phi] = [\alpha]\}$ which minimizes $|\phi|^2$. We find that α_0 is the unique solution to the Euler-Lagrange equation of the action (2.1), which is $\overline{\partial}^* \phi = 0$, where $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ with respect to this inner product. Since $\overline{\partial} \phi = 0$ by assumption, as we are looking at representatives of cohomology classes, this turns out to be equivalent to $\Delta \phi = 0$ where $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}$ is the Hodge Laplacian. Thus we get some topological conditions giving a minimum possible value among all representatives of a topological class, and a differential equation which picks out the unique representative of this class which satisfies this bound.

A similar thing happens with the curvature of some vector bundles. Let E be a vector bundle over X. Given a hermitian metric h on E then there is a unique connection ∇_E compatible with this metric and the complex structure on X and ∇_E has curvature R_∇ which is a $(1,1)$-form with values in $\mathrm{End}(E)$, i.e. a section of $\Lambda^2 T^* X \otimes \mathrm{End}(E)$. Using the previously given metric on $(1,1)$-forms along with an invariant metric (in the Lie algebra sense) on $\mathrm{End}(E)$ we define an action functional on such curvatures:

$$S_{\mathrm{YM}}(R_E) = \int_X \langle R_E, R_E \rangle dx$$

Given a complex vector bundle E, is there a metric h on E whose compatible connection minimizes this functional? The Euler-Lagrange equation in this case is

$$\sqrt{-1} \Lambda R_E - c \, \mathrm{Id} = 0 \tag{2.2}$$

where $\Lambda = (\omega \wedge)^*$ is the adjoint of wedging with ω, the Kahler form, and where c is some constant and Id is the identity endomorphism of E (for this computation, see [12]). Equation (2.2) is called the Hermitian-Yang-Mills equation. Such connections are called Hermitian-Yang-Mills connections and vector bundles which admit such connections are called Hermitian-Einstein bundles.

3. Mumford Stability and Hermitian-Yang-Mills Connections

There is a beautiful result, due to Narasimhan and Seshadri in dimension 1, due to Donaldson in dimension 2, and Uhlenbeck and Yau in higher dimensions, which states that Hermitian-Einstein bundles are precisely those which are *semistable*. Semistability is a simple algebraic condition on vector bundles due to Mumford which arises when one is trying to construct moduli spaces of bundles by forming moduli spaces of marked bundles and then forming a quotient by some group through a general process called forming the GIT (Geometric Invariant Theory) quotient. Semistability has a natural definition in terms of GIT and it is usually required to restrict to semistable bundles in order to produce separated moduli spaces. If ω is the Kahler form on X and $\deg_\omega(E) = \int_X c_1(E) \wedge \omega^{n-1}$ then we define

$$\mu(E) = \frac{\deg_\omega(E)}{\mathrm{rk}(E)} \tag{3.1}$$

and finally say that

Definition 3.2. the bundle E is semistable if and only if for every subbundle $F \subset E$ we have $\mu(F) \leq \mu(E)$. Further E is stable if and only if for every $F \subset E$ we have $\mu(F) < \mu(E)$.

To see the usefulness of this condition for the construction of moduli spaces, consider the family of rank 2 bundles over $\mathbb{P}^1$ parametrized by $\lambda \in \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$, which is one dimensional, such that E_λ is the corresponding extension

$$0 \to \mathcal{O}(-1) \to E_\lambda \to \mathcal{O}(1) \to 0$$

. For $\lambda \neq 0$ we have that $E_\lambda \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ but the trivial extension is of course the direct sum $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. So if we wanted to represent the functor $Y \mapsto \{\text{vector bundles over } \mathbb{P}^1 \text{ parametrized by Y}\}$, the representing space couldn't be separated. The issue here is that $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ is not semistable because $\mu(\mathcal{O}(1)) > \mu(E)$.

Then we have the following theorem:

Theorem 3.3 ((Donaldson-Uhlenbeck-Yau) [8, 19])**.** *A vector bundle E is semistable if and only if it admits a metric whose compatible connection is Hermititan-Yang-Mills.*

It should be noted that in this case the value of c in equation (2.2) is determined by the topology of the vector bundle by taking the trace of (2.2) and integrating. Normalizing the volume to 1 and using the fact that $c_1(E) = [\mathrm{tr}(\frac{\sqrt{-1}}{2\pi}R_E)]$ we get

$$2\pi \int_X \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}(\Lambda R_E)\wedge\omega^n = \int_X c\,\mathrm{tr}(\mathrm{Id}_E)\wedge\omega^n$$
$$2\pi\int_X c_1(E)\wedge\omega^{n-1} = \mathrm{rk}(E)c$$
$$c = 2\pi\frac{\deg_\omega(E)}{\mathrm{rk}(E)}.$$

One key feature of the theory of stability is the fact that every vector bundle can be formed by a finite number of extensions of semistable bundles, and we can break any bundle into its constituent semistable factors in a unique way. This is contained in the following theorem:

Theorem 3.4 (Harder-Narasimhan filtration)**.** *Let E be any vector bundle, then there is a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the composition factors $E_1/E_0, E_2/E_1, \ldots, E_n/E_{n-1}$ of this filtration are all semistable and $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ for all i.

4. Gieseker Stability and Almost Hermitian-Yang-Mills Connections

Although Mumford stability of vector bundles over higher dimensional varieties is enshrined in the illustrious Donaldson-Uhlenbeck-Yau theorem, there are several ways in which there is a better notion of stability for such vector bundles, called Gieseker stability, which arises in studying moduli spaces of vector bundles over higher dimensional varieties using GIT. Mumford stability will turn out to be a sort of linearization of Giesker stability. Luckily, there is a version of Theorem 3.3 in this case.

To give the proper definition, first recall that given a coherent sheaf E on X and an ample line bundle L, the Hilbert polynomial of E is defined to be

$$P_E(n) = \chi(E\otimes L^n). \tag{4.1}$$

Using this, we define

Definition 4.2 (Gieseker Stability)**.** A coherent torsion free sheaf E is called Gieseker semistable if for any subsheaf F of rank $0 < \mathrm{rk}(F) < r$ the inequality

$$\frac{P_F(n)}{\mathrm{rk}(F)} \le \frac{P_E(n)}{\mathrm{rk}(E)}$$

holds for all $n \gg 0$. Likewise E is called Gieseker stable if we have the strict inequality

$$\frac{P_F(n)}{\mathrm{rk}(F)} < \frac{P_E(n)}{\mathrm{rk}(E)} \tag{4.3}$$

for all $n \gg 0$.

The appropriate version of (2.2) in this case is a system of differential equations for $k \in \mathbb{Z}_{\geq 0}$ defined for the curvature R_E of a holomorphic vector bundle E called the *almost Hermitian-Einstein equations*:

$$[e^{\frac{i}{2\pi}R_E + k\omega \,\mathrm{Id}_E}\,\mathrm{Td}(X)]^{2n} = \frac{P_E(k)}{\mathrm{rk}(E)}\frac{\omega^n}{n!}\,\mathrm{Id}_E \tag{4.4}$$

where $[\ ]^{2n}$ represents the projection onto the top degree of cohomology. The main result of Leung in [13] and [14] is a proof that Giesker stability is equivalent to the existence of a solution to the almost Hermitian Einstein equation (4.4) for $k \gg 0$.

Theorem 4.5 ([14]). *A vector bundle E is Gieseker semistable if and only if it admits a hermitian metric whose compatible connection has curvature which satisfies (4.4) for all $k \gg 0$.*

5. Necessary background on t-structures and tilting

Before arriving at our final notion of stability, it is necessary to discuss a technique called tilting, and the related notions of t-structures and torsion pairs. The key reference in this area is [11].

A *torsion pair* in an abelian category lets us break any object in half in a unique way. Explicitly, if $\mathcal{A}$ is an abelian category then

Definition 5.1. A *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ such that

- $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$
- For any $X \in \mathcal{A}$ there exists a short exact sequence
$$0 \to T_X \to X \to F_X \to 0$$
where $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$.

The analogous and intimately related definition, when we replace $\mathcal{A}$ by a triangulated category $\mathcal{D}$ is the following:

Definition 5.2. A bounded *t-structure* on $\mathcal{D}$ is a pair $(C^{\leq 0}, C^{\geq 0})$ of full subcategories such that if $C^{\leq n} = C^{\leq 0}[-n]$ and $C^{\geq n} = C^{\geq 0}[-n]$ then

- $\cdots \subset C^{\leq n-1} \subset C^{\leq n} \subset C^{\leq n+1} \subset \cdots$

- $\cdots \subset C^{\geq n+1} \subset C^{\geq n} \subset C^{\geq n-1} \subset \cdots$

- $\mathrm{Hom}(C^{\leq 0}, C^{\geq 1}) = 0$
- For any $X \in \mathcal{D}$ there exists an exact triangle

$$X_{\leq 0} \to X \to X_{\geq 1} \to X_{\leq 0}[1]$$

  where $X_{\leq 0} \in C^{\leq 0}$ and $X_{\geq 1} \in C^{\geq 1}$
- (boundedness) $\mathcal{D} = \bigcup_{m,n\in\mathbb{Z}}(C^{\leq n} \cap C^{\geq m})$.

If we remove the last condition then we are left with just the definition of a t-structure. The boundedness condition can be thought of as a generalization of the fact that every element in the bounded derived category has cohomology in only a finite number of degrees. To make this analogy with cohomology precise, we define the *heart* of a t-structure to be the full subcategory $C^{\leq 0} \cap C^{\geq 0}$.

It turns out that we can recover a bounded t-structure from its heart. If $\mathcal{H}$ is the heart of a bounded t-structure then if $C^{\leq 0}_{\mathcal{H}}$ is the category generated by extensions from $\cup_{n\geq 0}\mathcal{H}[n]$ and $C^{\geq 0}$ is the category generated by extensions from $\cup_{n\leq 0}\mathcal{H}[n]$, we can define the bounded t-structure $(C^{\leq 0}_{\mathcal{H}}, C^{\geq 0}_{\mathcal{H}})$ whose heart is $\mathcal{H}$. This construction is only really useful if we have independent conditions by which we can check whether or not an abelian subcategory $\mathcal{H} \subset \mathcal{D}$ is a heart of a bounded t-structure. This is provided by Proposition (5.3), which some authors (such as [15]) simply take as the definition of the phrase "heart of a bounded t-structure". Better yet, we only need a priori an additive structure on the category we intend to prove is a heart.

Proposition 5.3. *A full additive subcategory $\mathcal{H} \subset \mathcal{D}$ is the heart of a bounded t-structure if and only if*

- $\mathrm{Hom}(A, B[i]) = 0$ *for* $A, B \in \mathcal{H}$ *and* $i < 0$.
- *To any $E \in \mathcal{D}$ we can associate a collection of triangles*

$$\begin{tikzcd}
0 = E_{k-1} \arrow[rr] & & E_k \arrow[dl] \arrow[rr] & & \cdots \arrow[r] & E_{l-1} \arrow[rr] & & E_l = E \arrow[dl] \\
& A_k \arrow[ul, dashed] & & A_{k+1} \arrow[ul, dashed] & \cdots & & A_l \arrow[ul, dashed] &
\end{tikzcd}$$

  such that $A_i[i] \in \mathcal{H}$.

The filtration given is precisely what can be thought of as a generalization of providing cohomology sheaves of an element of the derived category, and if $\mathcal{H} = Coh(X)$ we have the bounded t-structure called the standard t-structure and the objects A_i are really cohomology sheaves. In general we will denote $A_i[i]$ by $H^i_{\mathcal{H}}(E)$.

Proof. Let $\mathcal{H}$ satisfy the conditions of the claim. We will show that $(C^{\leq 0}_{\mathcal{H}}, C^{\geq 0}_{\mathcal{H}})$ as defined above is a bounded t-structure. Let E be any element of $\mathcal{D}$. The first two conditions of Definition 5.2 are trivially satisfied.

To prove the third, let $B \in C^{\leq 0}_{\mathcal{H}}$ and $B' \in C^{\geq 1}_{\mathcal{H}}$ and we need to show that $\mathrm{Hom}(B, B') = 0$. Write out the filtrations on B and B' which we get from the assumptions:

$$0 = B_0 \to \cdots \to B_n = B$$
$$0 = B'_0 \to \cdots \to B'_n = B$$

By assumption on B and B', we have $H^i_{\mathcal{H}}(B) = 0$ for $i > 0$ and $H^i_{\mathcal{H}}(B') = 0$ for $i \leq 0$. Associated to the exact triangle

$$B'_{i-1} \to B'_i \to H^i(B')[-i] \to B'_{i-1}[1]$$

there is a long exact sequence

$$\mathrm{Hom}(A, B'_{i-1}) \to \mathrm{Hom}(A, B'_i) \to \mathrm{Hom}(A, H^i(B')[-i]) \to \mathrm{Hom}(A, B'_{i-1}[1]) \to \cdots$$

and supposing $\mathrm{Hom}(A, H^i(B')[-i])$ vanishes for any $i > 0$ then we can prove by induction on i that $\mathrm{Hom}(A, B') = 0$ using the exactness of the given sequence. By the assumptions of the claim this holds for $A \in \mathcal{H}$, and also $A \in \mathcal{H}[i]$ for $i \geq 0$ for in this case $\mathrm{Hom}(A, B') = \mathrm{Hom}(A[-i], B'[i]) = 0$. Now using this fact we can also show that it holds for $A = B$. This time we will take the exact triangles

$$B_{i-1} \to B_i \to H^i(B)[-i] \to B_{i-1}[1]$$

giving the long exact sequences

$$\mathrm{Hom}(B_{i-1}, B') \leftarrow \mathrm{Hom}(B_i, B') \leftarrow \mathrm{Hom}(H^i(B)[-i], B') \leftarrow \mathrm{Hom}(B_{i-1}, B')$$

and since $B_0 = 0$ and $\mathrm{Hom}(H^i(B)[-i], B') = 0$ for all i, it follows by induction that for each i we have $\mathrm{Hom}(B_i, B') = 0$ and in particular that $\mathrm{Hom}(B, B') = 0$.

Now we prove the fourth axiom of a t-structure. Writing out the filtration on any element E given by the second condition which we assume, and stopping at the 0th degree gives $E_{\leq 0}$ which is generated by extensions by $H^i_{\mathcal{H}}[i]$ for $i \geq 0$ so E_0 lies in $C^{\leq 0}_{\mathcal{H}}$. If we complete this to a triangle

$$E_{\leq 0} \to E \to E_{\geq 1} \to E_{\leq 0}[1]$$

then $E_{\geq 1}$ is generated up to extension by the other objects in the filtration, which are all $H^i_{\mathcal{H}}[i]$ for $i < 0$ and so $E_{\geq 1}$ resides in $C^{\geq 1}_{\mathcal{H}}$. Since boundedness is trivial by the finiteness of the filtration, we have proven that our heart $\mathcal{H}$ gives a t-structure.

Going the other direction, if $(C^{\leq 0}, C^{\geq 0})$ is a bounded t-structure with heart $\mathcal{H}$ then the first required property is trivial. To prove the second, we cite the fact that the inclusions $\iota : C^{\leq 0} \to \mathcal{D}$ and $\iota' : C^{\geq 0} \to \mathcal{D}$ admit a right adjoint $\tau_{\leq 0}$ and a left adjoint $\tau_{\geq 0}$ respectively [3]. If we define

$$H^i_{\mathcal{H}} := [-i] \circ \tau_{\geq 0} \circ \tau_{\leq 0} \circ [i]$$

then we can construct the filtration as follows: given $E \in \mathcal{D}$ let i be the smallest integer such that $E[i] \in C^{\leq 0}$ then $\tau_{\geq 0} \circ \tau_{\leq 0}(E[i]) = \tau_{\geq 0}(E[i])$ and we get a map $E[i] \to \tau_{\geq 0}(E[i])$ from the fact that $\tau_{\geq 0}$ is left adjoint to the inclusion so the map f comes from an isomorphism

$$\begin{gathered} \mathrm{Hom}(E[i], \iota'(\tau_{\geq 0}(E[i]))) \simeq \mathrm{Hom}(\tau_{\geq 0}(E[i]), \tau_{\geq 0}(E[i])) \\ f \longleftrightarrow \mathrm{Id}\,. \end{gathered}$$

Completing this to an exact triangle and shifting back gives

$$E' \to E \to H^i_{\mathcal{H}}(E)[-i] \to E'[1]$$

which is the rightmost triangle in the filtration. Then $E'[i+1] \in C^{\leq 0}$ and we repeat the argument with E'. The boundedness assumption shows that this process will eventually terminate. □

We can also show using Proposition (5.3) that given a torsion pair $(\mathcal{T}, \mathcal{F})$ on the heart of a bounded t-structure $\mathcal{H}$ we can produce a new heart, called the *tilt* of $\mathcal{H}$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. The construction is encoded in the following claim:

Proposition 5.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of $\mathcal{H}$ the heart of a bounded t-structure. Then*

$$\mathcal{H}_\natural = \{E \in \mathcal{D} | H^{-1}_{\mathcal{H}}(E) \in \mathcal{F}, H^0_{\mathcal{H}}(E) \in \mathcal{T}, \text{ other } H^i_{\mathcal{H}}(E) = 0\}$$

is the heart of a bounded t-structure.

Proof. For ease of notation, we will denote the $\mathcal{H}$ cohomology simply by H^i without a subscript. Now we will show $\mathrm{Hom}(X, Y[i]) = 0$ for $i < 0$ for $X, Y \in \mathcal{H}_\natural$. Produce the filtrations

$$\begin{tikzcd}
X_{-2} \arrow[rr] & & X_{-1} \arrow[dl] \arrow[rr] & & X_0 = X \arrow[dl] \\
& H^{-1}(X)[1] \in \mathcal{F}[1] \arrow[ul, dashed] & & H^0(X) \in \mathcal{T} \arrow[ul, dashed] &
\end{tikzcd}$$

and likewise for $Y[i]$:

$$\begin{tikzcd}
Y_{-2}[i] \arrow[rr] & & Y_{-1}[i] \arrow[dl] \arrow[rr] & & Y_0[i] = Y[i] \arrow[dl] \\
& H^{-1}(Y)[i+1] \in \mathcal{F}[i+1] \arrow[ul, dashed] & & H^0(Y)[i] \in \mathcal{T}[i] \arrow[ul, dashed] &
\end{tikzcd}$$

We know that for any pair of choices of $A \in \{H^{-1}(X)[1], H^0(X), X_{-2}\}$ and $B \in \{H^{-1}(Y)[i+1], H^0(Y)[i], Y_{-2}[i]\}$ we have $\mathrm{Hom}(A, B) = 0$ by either the assumption that $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ or by the assumption that the cohomology sheaves lie in the heart of a t-structure and the first claim of Proposition 5.3. Then the fact that $\mathrm{Hom}(X, Y[i]) = 0$ follows from the argument using induction on several long exact sequences given when proving the third step of that proposition.

To give the required filtration into $H^i_{\mathcal{H}_\natural}$ cohomology objects, first take any $E \in \mathcal{D}$ and write, for each i, the short exact sequence

$$0 \to T_i \to H^i(E) \to F_i \to 0$$

with $T_i \in \mathcal{T}$ and $F_i \in \mathcal{F}$ which we can do because $(\mathcal{T}, \mathcal{F})$ is a torsion pair. To produce the required filtration, the argument is almost identical to the argument in Proposition 6.2 which produces a filtration in the derived category based on filtrations on every element in the heart. Starting with the $\mathcal{H}$ cohomology filtration $E_0 \to E_1 \to \cdots \to E$ on $E \in \mathcal{D}$, the argument there is modified in the following way: first we replace the filtration on the heart with this one: $T_i \subset H^i(E)$. This produces a filtration of E with twice as many elements

$$0 = \tilde{E}_0 \to E_0 \to \tilde{E}_1 \to E_1 \to \cdots \to \tilde{E}_n \to E_n = E$$

and then we make the filtration more coarse by forgetting the objects in the original filtration, giving:

$$0 = \tilde{E}_0 \to \tilde{E}_1 \to \cdots \to \tilde{E}_n \to E = E$$

it is not hard to see that the cohomology objects in this case will be elements of $\mathcal{H}_\natural$. $\square$

6. Bridgeland Stability Conditions

Douglas [9, 10] defined a notion of stability for D-branes, called Π-stability which was intended to provide the proper analogue of a Hermitian-Einstein bundle in the context of string theory. Tom Bridgeland [5] gave a notion of stability putting these ideas on firm mathematical footing which allows one, for example, to construct moduli spaces of objects in a derived category of coherent sheaves. The basic situation is this: consider the natural cohomology filtration $0 = E^0 \to E^1 \to \cdots \to E^n = E$ on a bounded complex whose successive objects fit into exact triangles

$$\begin{tikzcd} E^{j-1} \arrow[rr] & & E^j \arrow[dl] \\ & H^j(E)[j] \arrow[ul, dashed] & \end{tikzcd}$$

giving the

cohomology of the complex. Then this filtration can be refined by using a Harder-Narasimhan filtration on the cohomology objects H^i to produce a finer filtration on whole complexes E which will break any object $E \in D(X)$ into the semistable factors of its cohomology sheaves, which don't depend on the particular representation of E as a complex. The interesting feature of Bridgeland stability is that we will actually produce filtrations which don't split the complexes into their cohomology sheaves, but semistable objects will actually be complexes, or more accurately objects in the derived category, in a non-trivial way. The semistable objects *will* however be shifts of objects of some abelian category $\mathcal{A} \subset D(X)$, which is a heart of a t-structure on $D(X)$. Thus in this picture $D(X)$ can be thought of as mediating between different abelian categories $\mathcal{A}$ with objects in one being complexes of objects in another and vice versa.

We now give the axioms which are required to define such stable objects.

Definition 6.1 ([5])**.** A (numerical) stability condition $(Z, \mathcal{P})$ on a triangulated category D is a linear map from the Grothendieck group $Z : K(D) \to \mathbb{C}$ called the central charge, together with a collection of full additive subcategories $\mathcal{P}(\phi) \subset D$ for all $\phi \in \mathbb{R}$ (the value ϕ is called a phase, think of it as $\tan^{-1}$ of a slope) which satisfy the following properties:

(1) (Numerical condition) The central charge Z factors through numerical equivalence.
(2) if $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E)/|Z(E)| = e^{i\phi}$.
(3) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(4) If $\phi_1 > \phi_2$ and $E_1 \in \mathcal{P}(\phi_1), E_2 \in \mathcal{P}(\phi_2)$ then $\mathrm{Hom}(E_1, E_2) = 0$.
(5) (Harder-Narasimhan filtration) For any $0 \neq E \in D$ there exists a sequence of exact triangles fitting into the diagram

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & \cdots \arrow[r] \arrow[dl] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & \cdots & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

where $A_j \in P(\phi_j)$ for a decreasing sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$.

There is also a technical notion of being locally finite, which we will assume for all of our stability conditions, which in particular implies that semistable objects have a finite filtration whose factors are stable. For details see [5].

Objects in the subcategories $P(\phi)$ are called semistable, and simple objects in $P(\phi)$ are called stable. Because of the fact that the central charge factors through the K theory of the derived category and property (3) once we know the value of the central charge on coherent sheaves we can recover its value on any object $E \in D(X)$.

Thus we can recover the whole stability condition from the morphism $Z : K \to \mathbb{C}$ where K is the Grothendieck group of $Coh(X)$ and a description of which coherent sheaves are semistable. Generalizing by replacing $Coh(X)$ by the heart of any bounded t-structure on $D(X)$ and we have that

Proposition 6.2 ([5], Proposition 5.3)**.** *A stability condition on $D(X)$ can equivalently be given by the following data:*

(1) A is the heart of a bounded t-structure on $D(X)$
(2) $Z : K(A) \to \mathbb{C}$ is a stability function, that is a group morphism such that
- $0 \neq E \in K(A) \implies Z(E)/|Z(E)| = e^{i\phi_E}$ *for* $\phi_E \in (0, 1]$ *and*
- *every $E \in K(A)$ has a Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that successive quotients E_i/E_{i-1} have decreasing phases and are semistable.

In the context of this proposition, F being semistable means that if $F' \subset F$ is a non-zero subobject then $\phi_{F'} \leq \phi_F$.

Proof. Here and throughout, we denote by $P(I)$ for an interval I the extension closed subcategory generated by the $P(\phi)$ for $\phi \in I$. Then given a stability condition (Z, P) the category $P(0, 1]$ is the heart of a bounded t-structure on $\mathcal{D}(X)$. We will show that is satisfies the conditions of Proposition 5.3. To show that $\mathrm{Hom}(X, Y[i]) = 0$ for $X, Y \in P(0, 1]$ and $i < 0$, we note that X is generated by extensions of semistable objects of phase $\phi \in (0, 1]$ and Y is generated by semistable objects of phase $\phi \in (i, i + 1]$ and by using induction on the length of extension required to produce X and Y and the fact that $\mathrm{Hom}(X', Y') = 0$ for $X' \in P(\phi)$ for $\phi \in (0, 1]$ and $Y' \in P(\phi')$ for $\phi' \in (i, i + 1]$ since $i < 0$.

Now to produce the filtration required by Proposition 5.3, we simply take the Harder-Narasimhan filtration of any element and forget some of the elements to give a coarser filtration, all of whose factors are extensions of semistable objects whose phases have the same integral parts.

Going the other direction, given a heart $\mathcal{A}$ and a stability function $Z : \mathcal{A} \to \mathbb{C}$ with the Harder-Narasimhan property we form the stability condition $(\tilde{Z}, P)$ where $\tilde{Z} : K(X) \to \mathbb{C}$ is given by

$$\tilde{Z}(E) = \sum_{i \in \mathbb{Z}} (-1)^i Z(H^i_A(E))$$

which is the alternating sum of the charge on its cohomology sheaves with respect to the t-structure associated to the heart, and we produce a slicing where $P(\phi)$ consists of semistable objects of $\mathcal{A}$ of phase ϕ when $\phi \in (0, 1]$ and if $\phi \in (i, i+1]$ is just defined to be $P(\phi - i)[i]$. We produce the Harder-Narasimhan filtration on any object in the derived category by first taking the filtration of Proposition 5.3 and refining by taking the Harder-Narasimhan filtration on the cohomology objects which we have by assumption: Let

$$E_{i-1} \to E_i \to H^i_{\mathcal{A}}(E) \to E_{i-1}[1]$$

be a triangle in the cohomology filtration and let

$$0 = H_0 \subset H_1 \subset \cdots \subset H_k = H^i_{\mathcal{A}}(E)$$

be the Harder-Narasimhan filtration on the cohomology object with composition factors denoted by $A_k = H_k/H_{k-1}$. Taking the composition

$$E_i \to H^i_{\mathcal{A}}(E) \to H^i_{\mathcal{A}}(E)/H_{k-1} = A_k$$

and we complete to an exact triangle

$$\begin{tikzcd} E_{i,k-1} \arrow[rr] & & E_i \arrow[dl] \\ & A_k \arrow[ul, dashed] & \end{tikzcd}$$

. To continue the filtration, we note that the composition (in $\mathcal{A}$)

$$\tau_{\geq i}\tau_{\leq i}E_{i,k-1} \to \tau_{\geq i}\tau_{\leq i}E_{i,k} \xrightarrow{f} H_k \to A_k$$

is 0. Thus the image of f lies in H_{k-1}, so we can compose with the quotient $H_{k-1} \to A_{k-1}$ to give

$$\tau_{\geq i}\tau_{\leq i}E_{i,k-1} \to H_{k-1} \to A_{k-1}$$

but since $\tau_{\geq i}$ is left adjoint to the inclusion of $\mathcal{A}[-i] \subset C^{\geq i} \hookrightarrow D(X)$ this gives a unique map

$$E_{i,k-1} = \tau_{\leq i}E_{i,k-1} \to A_{k-1}$$

continuing in this fashion we produce the required Harder-Narasimhan filtration, which terminates because eventually $A_0 = 0$. $\square$

7. Stability conditions on an elliptic curve

Bridgeland gave a complete description of the space of stability conditions on an elliptic curve. The first example, which we will call the standard stability condition on an elliptic curve, is given by the central charge $Z(E) = \deg(E) + i\,\mathrm{rk}(E)$. For a vector bundle, the value $\tan(\phi)$ really is a slope and stability is just Mumford stability. All remaining examples are obtained from this one by acting with some element of the universal cover of $\mathrm{GL}_2^+(\mathbb{R})$, the orientation preserving linear automorphisms of $\mathbb{R}^2 \simeq \mathbb{C}$ which acts on the target of the central charge.

What do these look like in light of Proposition 6.2? Let (Z_0, P_0) be a stability condition on an elliptic curve X. Shift enough times so that this stability condition satisfies $Z_0(E) = g^{-1} \circ Z(E) = g^{-1}(\deg(E) + i\,\mathrm{rk}(E))$ for $g \in \mathrm{GL}_2^+(\mathbb{R})$ and $P_0(0,1] \subset P(0,2)$. Then $P_0(0,1] = P(\theta, \theta+1]$ where $g^{-1}(1) = re^{\pi i \theta}$ for $\theta \in [0,1)$ and for $r \in \mathbb{R}_{>0}$. Then $A_\theta := P_0(0,1] = P(\theta,\theta+1]$ is the heart of a bounded t-structure and $Z_0 : A_\theta \to \mathbb{C}$ is a stability function.

Proposition 7.1. *The category A_θ is the tilt of the standard t-structure on $D(X)$ associated with the torsion pair $(\mathcal{T}, \mathcal{F})$ where $\mathcal{F}$ consists of torsion-free sheaves E of slope $\mu(E) \leq \theta$ and $\mathcal{T}$ consists of sheaves whose torsion free part have slope $> \theta$. That is*

$$A_\theta = \{E \in D(X) | H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}, \ \text{other } H^i = 0\}$$

Proof. Let $E \in A_\theta$. Take its Harder-Narasimhan filtration with respect to the standard stability condition on X. Let E_1 be the last object in the filtration such that E_1 has all semistable factors of phase $\phi > 1$. Then $E_1 \in P(1, 1+\theta] = (P(0,\theta])[1]$, which are shifts of elements of $\mathcal{F}$. Then we have an exact triangle

$$\begin{tikzcd} E_1 \arrow[rr] & & E \arrow[dl] \\ & E_2 \arrow[ul, dashed] & \end{tikzcd}$$

where the semistable factors of E_2 have phase ≤ 1, so $E_2 \in P(\theta, 1] = \mathcal{T}$ and thus E is in this tilt. In the other direction, if E is in the specified tilt, then its Harder-Narasimhan filtration with respect to the standard stability condition has semistable factors with phases in $(\theta, \theta+1]$ and since it is formed by extensions of its semistable factors, it lies in A_θ. $\square$

The key point is that this category has been proven by Polishchuk and Schwarz to be equivalent to the category of holomorphic vector bundles on a complex non-commutative two-torus when θ is irrational.

8. Complex Geometry of Non-commutative Tori

One standard way of introducing non-commutative geometry seems to be to first describe classical physics as existing on cotangent bundles T^*M with position coordinates x_i and momentum coordinates $p_i = dx_i$, which of course commute with each other as they are just smooth functions. Then one recalls the famous commutation relation from quantum physics

$$[x, p] = i\hbar$$

and imagines how nice it would be if one were able to do geometry where these x and p were coordinate functions.

There are several approaches to defining the notion of a non-commutative space. Over and over again we learn that we should think of a space as the same thing as the collection of functions on it, and a reasonable way to proceed is to start with some (mildly) non-commutative algebra of functions and see how much of geometry we can salvage. This procedure works quite well over mildly non-commutative versions of tori, called non-commutative tori, and the groundbreaking work in this area described them extensively [7]. This section will introduce the complex geometry of non-commutative tori following papers of Schwarz and coauthors [17, 18]. .

Every smooth function f on a torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ can be represented by a Fourier series

$$f(x) = \sum_{n \in Z^d} f_{\vec{n}} U_{\vec{n}}(x)$$

where if $\vec{n} = (n_1, \ldots, n_d)$ then $U_{\vec{n}}(x) = e^{is\sum n_k x_k} = e^{i\vec{n}\cdot x}$ and the $f_{\vec{n}}$ are Fourier coefficients. We will now extend this to the non-commutative case by replacing these $U_{\vec{n}}$ with non-commutative versions. Generalizing the previous definition of $U_{\vec{n}}$ we will now define an algebra of functions $C^\infty(T^d_\theta, \mathbb{C}) = C^\infty(T^d_\theta)$ on a non-commutative torus, (ignoring the obvious subterfuge that this is not actually $C^\infty(X)$ for any space X) as consisting of series

$$f = \sum_{\vec{n} \in \mathbb{Z}^d} f_{\vec{n}} U_{\vec{n}}$$

where the coefficients $f_{\vec{n}} \in \mathbb{C}$ vanish faster than any power of the $|\vec{n}|$. To give the multiplicative structure, let $e_1, \ldots, e_d$ be the standard basis for $\mathbb{Z}^d$, the $U_{\vec{n}}$ will be multiplicatively generated by the U_{e_i} and their conjugates subject to the relations

$$U_{e_i} U_{e_j} = e^{2\pi i \theta_{ij}} U_{e_j} U_{e_i}$$

where θ is a $d \times d$ anti-symmetric matrix. More concisely we can write

$$U_{\vec{n}} U_{\vec{m}} = e^{2\pi i \vec{n}\theta\vec{m}} U_{\vec{n}+\vec{m}}$$

where $U_{\vec{n}} = e^{-i\pi\vec{n}\theta\vec{n}} U_1^{n_1} \cdots U_d^{n_d}$ where we write $U_i = U_{e_i}$.

The next step is to describe vector bundles over non-commutative tori. It is well known that a the space of sections of a vector bundle over a smooth space X is a projective $C^\infty(X)$ module. Generalizing this, we define a (finite dimensional) vector bundle over T_θ to be a finite dimensional projective module over $C^\infty(T^d_\theta)$. The first examples are of course the free modules, which are the trivial vector bundles. Over a two-torus the form θ can be specified by a single real number (abusively also denoted θ) and we have the following classification of modules:

Theorem 8.1 ([17]). *Every finite dimensional right projective module over $C^\infty(T_\theta)$ is isomorphic to some $E_{n,m}(\theta)$ where $n, m \in \mathbb{Z}$, $n+\theta m \neq 0$ and $E_{n,0}(\theta) = A_\theta^{|n|}$ while for $m \neq 0$ we have $E_{n,m}(\theta)$ is the Schwartz space*

$$\mathcal{S}(\mathbb{R} \times \mathbb{Z}/m\mathbb{Z}) = \{(f_1, \ldots, f_m) | \sup_{a,b,x,i} |x^a \frac{d^b}{dx^b} f_i(x)| < \infty\}$$

with action of the functions given by

$$fU_1(x, a) = f(x - \frac{n + m\theta}{m}, a - 1)$$
$$fU_2(x, a) = e^{2\pi i (x - \frac{an}{m})} f(x, a).$$

If n and m are relatively prime then $E_{n,m}(\theta)$ is called *basic*. Further, it can be shown that in this case $E_{dn,dm}(\theta) = E_{n,m}(\theta)^{\oplus d}$.

The next step is to define complex versions of these spaces and complex vector bundles. Recall that a complex structure on a space X gives a decomposition of the complexified tangent space $TX \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ into holomorphic and antiholomorphic parts which are complex conjugate to each other. On a torus modeled on a vector space we only have to give a decomposition at the origin because we have a canonical isomorphism between the tangent space at any point and that at the origin. Returning again to any $d = 2n$ dimensional non-commutative torus, the space $L \simeq \mathbb{R}^d$ generated by $\delta_1, \ldots, \delta_d$ acts by derivations on T^d_θ by defining $\delta_i U_j = 2\pi i \delta_{ij} U_j$ and extending using linearity and the Leibniz rule. Of course in the commutative case we recover ordinary differentiation of Fourier series. Then a complex structure on T^d_θ is a decomposition $L \otimes \mathbb{C} = L^{1,0} \oplus L^{0,1}$ into complex conjugate subspaces. Write $\bar{\delta}_1, \ldots, \bar{\delta}_n$ for a basis of $L^{0,1}$.

Then we are interested in holomorphic vector bundles over these spaces. Let $\mathcal{A}^{p,q}(X)$ be the space of holomorphic (p,q)-forms. Recall that a holomorphic vector bundle can equivalently be given as a complex vector bundle E with connection $\nabla : \Gamma(E) \to \Gamma(E) \otimes \mathcal{A}^1(X)$ such that if we write $\nabla = \nabla' + \nabla''$ where $\nabla' : \Gamma(E) \to \Gamma(E) \otimes \mathcal{A}^{1,0}(X)$ and $\nabla'' : \Gamma(E) \to \Gamma(E) \otimes \mathcal{A}^{0,1}(X)$ then $\nabla'' \circ \nabla'' = 0$ where we extend ∇ and its decomposition to (p,q)-forms by linearity and Leibniz as usual. In this case $\nabla'' = \bar{\partial}$. Proceeding in the direction this analogy proposes, if E is a vector bundle over T^d_θ with complex structure given by the notation above, then a holomorphic structure on E is given by a set of operators $\bar{\nabla}_1, \ldots, \bar{\nabla}_d : E \to E$ such that

$$\bar{\nabla}_i(e \cdot f) = \bar{\nabla}_i(e) \cdot f + e \cdot \bar{\delta}_i(f)$$

for any function f on T^d_θ.

Let us again restrict to a real two dimensional non-commutative torus T_θ. Then a complex structure is given by a complex number τ with $\mathrm{im}(\tau) \neq 0$ and the corresponding $\bar{\delta}_\tau$ is defined by

$$\bar{\delta}_\tau(U_1^{n_1} U_2^{n_2}) = 2\pi i(n_1\tau + n_2)(U_1^{n_1} U_2^{n_2})$$

and following Schwarz we will denote the torus with this complex structure $T_{\theta,\tau}$. A holomorphic structure on a vector bundle E is just a map $\bar{\nabla} : E \to E$ such that

$$\bar{\nabla}(e \cdot f) = \bar{\nabla}(e) \cdot f + e \cdot \bar{\delta}_\tau(f)$$

and then we define a map of holomorphic vector bundles to be holomorphic if the action of T_θ commutes with the $\bar{\nabla}$. Now we will define what is called a *standard holomorphic structure* on one of the $E = E_{n,m}(\theta)$ of Theorem 8.1 for m and n relatively prime. Let $\mu(E) = m/(n + m\theta)$ and define

$$\bar{\nabla}_z(f)(x,a) = \frac{\partial f}{\partial x}(x,a) + 2\pi i(\tau\mu(E)x + z)f(x,a) \tag{8.2}$$

for any $z \in \mathbb{C}$. Holomorphic bundles (E, ∇_z) will be called *standard holomorphic bundles*.

9. Equivalence of the categories

Culminating in [16], there is a proof that every holomorphic vector bundle on $T_{\theta,\tau}$ is generated up to extension by standard ones and that further, the category $C(T_{\theta,\tau})$ of holomorphic vector bundles on $T_{\theta,\tau}$ is equivalent to the previously defined tilted category A_θ formed by tilting $Coh(X)$ for X and elliptic curve $\mathbb{C}/\mathbb{Z}{+}\tau\mathbb{Z}$ where bundles with standard holomorphic structures are sent to simple bundles. Using Proposition 7.1 it follows that

Corollary 9.1. *The category $C(T_{\theta,\tau})$ for θ irrational is equivalent to $P(0,1]$ for a stability condition (Z,P) on the elliptic curve X_τ, and under this equivalence standard bundles are sent to stable bundles.*

It is worthwhile to give some indication of where this equivalence comes from, to see the relationship between this equivalence and the Fourier-Mukai transform. Briefly recalling the latter, given an abelian variety X and its dual $\hat{X}$, the points of $\hat{X}$ correspond to topologically trivial line bundles on X and there is a universal line bundle P over $X \times \hat{X}$ called the Poincare line bundle. Let π_1, π_2 be the left and right projections from $X \times \hat{X}$. Then the Poincare line bundle is such that if $l \in \hat{X}$ corresponds to the line bundle L, then $L \simeq \pi_{1*}(\pi_2^*(k(l)) \otimes P)$. Replacing these functors with their derived functors (without changing the notation) we get a functor, studied by Mukai,

$$\Phi_P : D(\hat{X}) \to D(X)$$
$$\Phi_P(E) = \pi_{1*}(\pi_2^*(E) \otimes P)$$

which is called the Fourier-Mukai transform with kernel P. The key result is that Φ_P is actually a triangulated equivalence between the two derived categories.

The equivalence between $C(T_{\theta,\tau})$ and A_θ is constructed in a similar fashion. Let us define an element

$$S((E,\bar{\nabla})) \in D(X_\tau)$$

for any holomorphic vector bundle $(E,\bar{\nabla})$. We will construct a length two complex

$$\cdots 0 \to S_E \xrightarrow{d_{\bar{\nabla}}} S_E \to 0 \cdots \tag{9.2}$$

where S_E is constructed by starting with a vector bundle $\tilde{S}_E$ over $\mathbb{C}$ which is just a trivial bundle with fiber E and a map

$$\tilde{d}_{\bar{\nabla}} : \tilde{S}_E \to \tilde{S}_E$$

such that on the fiber over z we have

$$\tilde{d}_{\bar{\nabla},z}(f) = \bar{\nabla}(f) + 2\pi i z f. \tag{9.3}$$

The goal is now to show how to descend by acting by translations by 1 and τ in $\mathbb{C}$ to produce a vector bundle over the elliptic curve $X_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. Translations by 1 correspond to

$$f(z) \mapsto f(z+1)U_2$$

and by τ correspond to

$$f(z) \mapsto e^{-2\pi i\theta z} f(z+\tau)U_1$$

it follows from the fact that U_1 and U_2 are unitary operators on E that S_E is a vector bundle over the elliptic curve and to prove that we actually produce a complex from the differential $d_{\bar{\nabla}}$ defined by (9.3), we compute that under translation by 1

$$\begin{aligned} d_{\bar{\nabla}} f(z) &\mapsto (d_{\bar{\nabla}} f)(z+1)U_2 \\ &= \bar{\nabla}(f(z+1))U_2 + 2\pi i(z+1)f(z+1)U_2 \\ &= \bar{\nabla}(f(z+1))U_2 + f(z+1)\bar{\delta}_\tau U_2 + 2\pi i z f(z+1)U_2 \\ &= \bar{\nabla}(f(z+1)U_2) + 2\pi i z f(z+1)U_2 \\ &= d_{\bar{\nabla}}(f(z+1)U_2) \end{aligned}$$

and under translations by τ the parallel computation gives

$$\begin{aligned} d_{\bar{\nabla}} f(z) &\mapsto e^{-2\pi i z\theta}(d_{\bar{\nabla}} f)(z+\tau)U_1 \\ &= d_{\bar{\nabla}}(e^{-2\pi i z\theta} f(z+\tau)U_1) \end{aligned}$$

and so the complex (9.2) is actually a complex over the elliptic curve X_τ.

10. What about higher dimensional tori?

The results in this paper say that for an elliptic curve we can think of rotating the standard stability condition by an irrational angle θ consists of rotating through non-commutative versions of the elliptic curve where we introduce a non-commutativity factor $e^{2\pi i\theta}$. And if we continue rotating (ignoring the rather important fact that we can't actually rotate past rational values of θ) all the way until we hit $\theta = 1$ then we return to our original elliptic curve except we have just shifted the derived category by the translation $[1] : \mathcal{D}(X) \to \mathcal{D}(X)$. Future work will attempt to carry this description over to higher dimensional abelian varieties.

It is a hard problem to produce stability conditions on higher-dimensional varieties. Focusing still on only abelian varieties, in the case of surfaces there is Bridgeland's construction:

Theorem 10.1 ([6]). *Let X be an abelian surface, $\omega \in \mathrm{Amp}(X) \otimes \mathbb{R}$ an ample class and $\beta \in NS(X) \otimes \mathbb{R}$ Then*

$$Z(E) = \int_X e^{-\beta - i\omega} \operatorname{ch}(E)$$

is a central charge on a heart of a bounded t-structure $\mathcal{A}_{\beta,\omega}$ which satisfies the requirements of Proposition (6.2).

It should be noted that the heart in question is formed by tilting with respect to a torsion pair constructed by breaking off at a certain slope. There are also various extensions of this result to three dimensional abelian varieties in [1, 2] and many other places.

There are also extensions of the results of Polishchuk and Schwarz which provide non-commutative analogues of the Fourier-Mukai transform for higher dimensional abelian varieties, rather than just on elliptic curves. This was predicted by the aforementioned authors and this program seems to have been carried out in [4].

In general we get an equivalence between a category of modules on a complex non-commutative torus associated to the 2-form θ and a category of B-twisted modules over a commutative abelian variety where B is a gerbe associated with θ. It would be interesting to consider whether similar constructions would provide relations between Bridgeland stability on abelian varieties of dimension two and higher and with holomorphic vector bundles on non-commutative tori.

References

[1] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. *Inventiones mathematicae*, 206(3):869–933, 2016.
[2] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds i: Bogomolov-Gieseker type inequalities. *arXiv preprint arXiv:1103.5010*, 2011.
[3] Alexander A Beilinson. Faisceaux pervers. *Astérisque*, 100, 1982.
[4] Jonathan Block. Duality and equivalence of module categories in noncommutative geometry ii: Mukai duality for holomorphic noncommutative tori. *arXiv preprint math/0604296*, 2006.
[5] Tom Bridgeland. Stability conditions on triangulated categories. *Annals of Mathematics*, 166(2):317–345, 2007.
[6] Tom Bridgeland et al. Stability conditions on $K3$ surfaces. *Duke Mathematical Journal*, 141(2):241–291, 2008.
[7] Alain Connes. C-algebres et géométrie différentielle. *CR Acad. Sci. Paris Sér. AB*, 290(13):A599–A604, 1980.
[8] Simon K Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proceedings of the London Mathematical Society*, 3(1):1–26, 1985.
[9] Michael R Douglas. Dirichlet branes, homological mirror symmetry, and stability. *arXiv preprint math/0207021*, 2002.
[10] Michael R Douglas, Bartomeu Fiol, and Christian Römelsberger. Stability and BPS branes. *Journal of High Energy Physics*, 2005(09):006, 2005.
[11] Dieter Happel, Idun Reiten, and Sverre O Smalø. *Tilting in abelian categories and quasitilted algebras*, volume 575. American Mathematical Soc., 1996.
[12] Shoshichi Kobayashi. *Differential geometry of complex vector bundles*. Princeton University Press, 2014.

[13] Nai-Chung Conan Leung. *Differential geometric and symplectic interpretations of stability in the sense of Gieseker*. PhD thesis, Massachusetts Institute of Technology, 1993.

[14] Naichung Conan Leung et al. Einstein type metrics and stability on vector bundles. *J. Differential Geom*, 45(3):514–546, 1997.

[15] Emanuele Macrì and Benjamin Schmidt. Lectures on Bridgeland stability. In *Moduli of Curves*, pages 139–211. Springer, 2017.

[16] Alexander Polishchuk. Classification of holomorphic vector bundles on noncommutative two-tori. *Doc. Math*, 9:163–181, 2004.

[17] Alexander Polishchuk and Albert Schwarz. Categories of holomorphic vector bundles on noncommutative two-tori. *Communications in mathematical physics*, 236(1):135–159, 2003.

[18] Albert Schwarz. Theta functions on noncommutative tori. *Letters in Mathematical Physics*, 58(1):81–90, 2001.

[19] Karen Uhlenbeck and Shing-Tung Yau. On the existence of Hermitian-Yang-Mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics*, 39(S1), 1986.