COMBINATORIAL STRUCTURES IN THE COORDINATE RINGS OF SCHUBERT VARIETIES

David C. Lax

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Approved by:

Robert A. Proctor

Prakash Belkale

Ivan Cherednik

Shrawn Kumar

Richárd Rimányi

ABSTRACT

David C. Lax: Combinatorial Structures in the Coordinate Rings of Schubert Varieties
(Under the direction of Robert A. Proctor)

Given an increasing sequence of dimensions, a flag in a vector space is an increasing sequence of subspaces with those dimensions. The set of all such flags (the flag manifold) can be projectively coordinatized using products of minors of a matrix. These products are indexed by tableaux on a Young diagram. A basis of “standard monomials” for the vector space generated by such projective coordinates over the entire flag manifold has long been known. A Schubert variety is a subset of flags specified by a permutation. Lakshmibai, Musili, and Seshadri gave a standard monomial basis for the smaller vector space generated by the projective coordinates restricted to a Schubert variety. Reiner and Shimozono made this theory more explicit by giving a straightening algorithm for the products of the minors in terms of the right key of a Young tableau. This dissertation uses the recently introduced notion of scanning tableaux to give more-direct proofs of the spanning and the linear independence of the standard monomials. This basis is a weight basis for the dual of a Demazure module for a Borel subgroup of the general linear group.

The most famous of the above flag manifolds are the Grassmann manifolds of flags that consist of a single subspace. The Plücker relations which define the Grassmann manifolds as projective varieties are well known. Grassmann manifolds are examples of minuscule flag manifolds. We study the generalized Plücker relations for minuscule flag manifolds independent of Lie type. To do this we combinatorially model the Plücker coordinates based on Wildberger’s construction of minuscule Lie algebra representations; it uses the colored partially ordered sets known as minuscule posets. We obtain, uniformly across Lie type, descriptions of the Plücker relations of “extreme weight”. We show that these are “supported” by “double-tailed diamond” sublattices of minuscule lattices. From this, we obtain a complete set of Plücker relations for the exceptional minuscule flag manifolds. These Plücker relations are straightening laws for their coordinate rings.

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CHAPTER 1

Introduction

Flag manifolds and their Schubert subvarieties are fundamental geometric objects that have algebraic realizations. In these realizations, one can study the flag manifolds through their coordinate rings. Although we are motivated by geometry, no geometric knowledge is required of the reader. This dissertation contains two parts, each of which concerns the coordinate rings of certain kinds of flag manifolds or their Schubert subvarieties. Each part addresses a different question about the coordinate rings for its special kind of manifolds. We briefly describe these two parts here: each part contains its own more detailed introduction (Sections 2.1 and 6.1 respectively).

To specify a flag manifold, one chooses a Dynkin diagram and a subset of its nodes. To specify an algebraic realization of this flag manifold, one then additionally chooses a dominant integral weight whose "support" in the Dynkin diagram is the chosen subset of nodes.

Part I of this dissertation works with flag manifolds of Lie type A. Here we consider every algebraic realization of the manifold, which arise from all appropriate dominant integral weights. In this classical case, the flag manifolds truly consist of flags in a vector space. Their coordinate rings can be studied using determinants of matrices. No knowledge of Lie theory is needed for Part I. The coordinate rings of these flag manifolds have been well understood since the 1950s, but the more complicated coordinate rings of their Schubert subvarieties were not described until the 1980s. For a fixed flag manifold, the set of its Schubert subvarieties forms a commonly studied filtration of the flag manifold. We seek a "standard" basis for the coordinate ring of each of these Schubert varieties. Here the "Plücker" relations among the coordinate generators of the ring are well known. We use these well-known Plücker relations to obtain our basis. A certain spanning set of the ring is commonly indexed by Young tableaux. Previous descriptions of a basis have used the notion of "right keys" to specify a subset of this spanning set that forms a basis. We show how to use the recently introduced notion of "scanning tableau" to do the same. Our description of a basis is combinatorially simpler than the older right key version. Moreover, we show more explicitly how to express each of the vectors in the spanning set in terms of this basis.

Part II of this dissertation works with flag manifolds uniformly across Lie type that are specified by the singleton subsets of "minuscule" nodes in the Dynkin diagram. These flag manifolds are called minuscule flag manifolds. For each of these, we consider its algebraic realization specified by the corresponding minuscule weight. All but three Lie types have at least one minuscule weight. The techniques used in this case require the background knowledge of simple complex Lie algebras. We seek a description of the coordinate rings of the minuscule flag manifolds that is based solely on the combinatorics of minuscule weights, independent of the particular Lie type. Here our understanding of the coordinate rings is essentially inverted from our understanding in Part I: A standard basis of the coordinate ring for any of these minuscule flag manifolds is known independent of its Lie type. However, there is no description of the Plücker relations that is uniform across Lie types. For some types, the Plücker relations have only been presented as recently as 2013, and for the "exceptional" minuscule flag manifolds the Plücker relations have apparently never been fully presented. We uniformly describe a subset of the Plücker relations in all minuscule flag manifold coordinate rings: These "extreme" Plücker relations are expressed as alternating sums over a certain lattice structure. In particular we present *all* of the Plücker relations in the exceptional type cases, apparently for the first time. The coordinate rings for minuscule flag varieties are known to have the structure of algebras with straightening laws. The Plücker relations which we present are explicit straightening laws for that structure.

CHAPTER 2

Coordinates of Schubert varieties[1]

2.1 Introduction to Part I

The main results of this paper are accessible to anyone who knows basic linear algebra: the Laplace expansion of a determinant is the most advanced linear algebra technique used. Otherwise, the most sophisticated fact needed is that the application of a multivariate polynomial may be moved inside a limit. Readers may replace our field $\mathbb{C}$ with any field of characteristic zero, such as $\mathbb{R}$.

Let $n \geq 2$ and $1 \leq k \leq n-1$. Fix $0 < q_1 < q_2 < \cdots < q_k < n$ and let Q denote the set $\{q_1, \ldots, q_k\}$. A Q-flag of $\mathbb{C}^n$ is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n$ such that $\dim(V_j) = q_j$ for $1 \leq j \leq k$. The set $\mathcal{F}\ell_Q$ of Q-flags has long been studied by geometers. It is known as a *flag manifold (for GL_n)*. Given a fixed sequence of integers $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_m$ with $\zeta_i \in Q$ for $1 \leq i \leq m$, one can form projective coordinates for $\mathcal{F}\ell_Q$ as follows: First, any flag can be represented with a sequence of n column vectors of length n. The juxtaposition of these vectors forms an $n \times n$ matrix f. For each $1 \leq i \leq m$, form a left-initial $\zeta_i \times \zeta_i$ minor of f by selecting ζ_i of its n rows. We refer to a product of such minors as a "monomial" for the given ζ_j's. Let N be the number of such possible monomials. One can inefficiently coordinatize $\mathcal{F}\ell_Q$ in $\mathbb{P}(\mathbb{C}^N)$ by evaluating all of these monomials over the flag manifold. The sequence $\zeta_1, \ldots, \zeta_m$ can be viewed as the lengths of the columns of a Young diagram λ. Hodge and Pedoe [1] used a basis theorem of Young [2] to index an efficient subset of these coordinates with the semistandard Young tableaux on the diagram λ. This subset is a basis of "standard" monomials for the vector space spanned by all monomials over the flag manifold. One can group flags into subsets known as *Schubert varieties* using a form of

[1] Chapters 2-5 originally appeared in the journal *Linear Algebra and its Applications*. The original citation is: D. C. Lax, "Accessible Proof of Standard Monomial Basis for Coordinatization of Schubert Sets of Flags," *Linear Algebra Appl.*, vol. 494, pp. 105-137, 2016. DOI: 10.1016/j.laa.2016.01.003

Gaussian elimination on their matrix representatives; these can be indexed by n-permutations. For a given Schubert variety, the coordinatization by the set of monomials indexed by semistandard tableaux is inefficient. Utilizing recent developments in tableau combinatorics, this paper gives a new derivation of a basis of standard monomials for the vector space generated by all monomials restricted to a Schubert variety.

The most famous flag manifolds are the sets of d-dimensional subspaces of $\mathbb{C}^n$. These are the cases $k := 1$ and $q_1 := d$ above and are known as the Grassmannians. Here the basis result for Schubert varieties may be readily deduced once it is known for the entire Grassmannian. The next-most studied flag manifold is the "complete" flag manifold, which is the case $k := n-1$ above.

It was not until the late 1970s that Lakshmibai, Musili, and Seshadri first gave [3] a standard monomial basis for any Schubert variety of a general flag manifold (for GL_n). Their solution used sophisticated geometric methods and was expressed in the language of the representation theory of semisimple Lie groups. In 1990, Lascoux and Schützenberger defined [4] the "right key" of a semistandard tableau. In 1997, Reiner and Shimozono used the notion of right key to give [5] a new derivation of the standard monomial basis for any Schubert variety of the complete flag manifold. They provided a "straightening algorithm" for products of minors that expressed the monomial specified by a given tableau as a linear combination in the standard monomial basis. In 2013, Willis defined [6] the "scanning tableau" of a semistandard tableau and showed that it is the right key of Lascoux and Schützenberger. The scanning tableau appears to be the simplest description of the right key.

We show how scanning tableaux can be used to improve the proofs of [5] for the spanning and the linear independence of the standard monomials. All aspects of our presentation consider all Schubert varieties of all flag manifolds for GL_n, i.e. for any $1 \leq k \leq n-1$. The statements of our basis theorem, Theorem 2.18, and both its spanning and linear independence parts differ from the analogous statements in [5]: We do not limit ourselves to the $k = n-1$ complete flag case. Here we use the scanning tableau to determine whether the monomial of a given tableau is a member of our standard basis for a given Schubert variety. In that article, membership is determined by using a "jeu de taquin" procedure to compute the right key of a tableau. The use of scanning tableaux allows for a direct and widely accessible proof of this theorem which is entirely self-contained. As a consequence of our basis theorem, we obtain a weighted tableaux summation expression, Corollary 5.1, that is associated to the vector space at hand. It is the "Demazure polynomial" of [7], or the "key polynomial" of [8] (which is given in terms of right keys). The derivation of this character expression is also self-contained: In particular, the original notion of right key is not needed.

Our spanning proof uses scanning tableaux to give a straightening algorithm in the spirit of [5]. The determinantal identity from [9] used there is also used here; more details are given for its application to the projective coordinates of a Schubert variety. Combinatorialists' interest in straightening algorithms goes back at least to [10, 11]. Apart from motivation, the spanning proof does not need any mention of Q-flags or Schubert varieties. All of the necessary definitions for the spanning theorem, Theorem 3.12, make sense for matrices with entries from any commutative ring R. The theorem statement itself makes sense over R when "spans" is replaced by "generates as an R-module." The proof presented in this paper is valid at that level of generality.

Our linear independence proof follows the general inductive strategy used in [3] and [5]. However, the simpler combinatorics of scanning tableaux allow those proofs to be simplified. One simpler aspect is that now only single Schubert varieties need be considered in the induction, rather than the unions of Schubert varieties that arose in the earlier papers. The statement of the linear independence theorem, Theorem 4.15, makes sense over any field. The proof presented here is valid for any field of characteristic zero; we make this assumption to obtain a self-contained development. The related proof in [5] does not need characteristic zero since it refers to a standard fact concerning the closure of a "Bruhat cell." There it is assumed the base field is algebraically closed, but given [12], they actually do not need that assumption for this fact. Hence the basis results in [5] and here hold over any field. See the appendix for details.

We need a number of well-known facts about Schubert varieties for our linear independence proof. There are references for these facts at varying levels of sophistication for the Grassmannians [13, 14] or the complete flag manifold [15]. However, we have not found a comprehensive source at any level of sophistication. Nor have we found a combination of sources that are accessible to readers without advanced educations in pure mathematics. So we have included elementary proofs of these standard facts for all flag manifolds for GL_n: Sections 2.2, 2.3, 4.1, and 4.2 of this paper can serve as an accessible introduction to the subject. The appendix provides an interface with the modern literature on flag manifolds and Schubert varieties. Using this appendix, the reader can transition from this paper to the reductive Lie group and representation theory contexts of references such as [13, 14]. There we describe how the standard monomial basis provides a basis of global sections for a certain line bundle on a homogeneous space of GL_n. This is a weight basis for the dual of a Demazure module for a Borel subgroup of GL_n. For coordinatizing Schubert varieties, it is sufficient to consider Young diagrams with columns of length less than n. Such diagrams would also suffice if one were interested only in realizing representations of SL_n. But we allow our Young diagrams to have columns of length n so that we can realize all of the irreducible polynomial representations of GL_n in the appendix.

Combinatorial tools are introduced in Section 2.2. Section 2.3 presents the definitions of flag varieties, Schubert varieties, and their projective coordinates. Our main theorem, Theorem 2.18, is motivated and stated there. Sections 3.1 and 3.2 prove the spanning parts of Theorems 2.16 and 2.18. Sections 4.1 and 4.2 present the facts needed to projectively coordinatize Schubert varieties. Section 4.3 proves the linear independence parts of Theorems 2.16 and 2.18. Section 5.1 presents the Demazure polynomial summation. Section 5.2 is the appendix of contemporary terminology.

2.2 Combinatorial tools

The needed combinatorial tools are "Q-chains", which we use to index Schubert varieties, and "tabloids", which we use to index some projective coordinates for flag manifolds.

Fix $n \geq 2$ and a nonempty subset $Q \subseteq \{1, 2, \ldots, n-1\}$ throughout the paper. Set $k := |Q|$ and index the elements of Q in increasing order: $1 \leq q_1 < q_2 < \cdots < q_k < n =: q_{k+1}$. Define $[n] := \{1, 2, \ldots, n\}$. A *$Q$-chain* is a sequence of subsets $P_1 \subset P_2 \subset \cdots \subset P_k \subseteq [n]$ such that $|P_j| = q_j$ for $1 \leq j \leq k$.

An *n-partition* is an n-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 := \lambda_{n+1}$. Fix an n-partition λ. The *shape* of λ, also denoted λ, is an array of n rows of boxes that has λ_r boxes in row r. The column lengths of the shape λ are denoted $n \geq \zeta_1 \geq \cdots \geq \zeta_{\lambda_1}$. Denote the set of distinct column lengths of λ that are less than n by $Q(\lambda)$. Refer to a location in λ with column index $1 \leq c \leq \lambda_1$ and row index $1 \leq r \leq \zeta_c$ by (r, c). Sets of locations in λ are called *regions*. A *tabloid* T of shape λ is a filling of the shape λ with values from $[n]$ such that the values strictly increase down each column. The value of T at location (r, c) is denoted $T(r, c)$. Partially order the tabloids of shape λ by defining $T \preceq U$ if $T(r, c) \leq U(r, c)$ for all locations $(r, c) \in \lambda$. We use the term *column tabloid* to refer to a tabloid of shape 1^d for some *length* $d \leq n$. Given a subset $P \subseteq [n]$, define $Y(P)$ to be the column tabloid of length $|P|$ filled with the values of P in increasing order. There is a unique column tabloid of length n, namely $Y([n])$. A *(semistandard Young) tableau* is a tabloid whose values weakly increase across each row. In Theorem 2.16 we use tableaux to index the standard monomial basis for a flag manifold.

Given a Q-chain $\pi = (P_1, \ldots, P_k)$, its *key* $Y(\pi)$ is the tabloid whose shape has one column each of the lengths $q_k, q_{k-1}, \ldots, q_1$ and which is obtained by juxtaposing the columns $Y(P_k), Y(P_{k-1}), \ldots, Y(P_1)$ as in Example 2.1 below. It can be seen that $Y(\pi)$ is a tableau. The *Bruhat order* on Q-chains is the following partial order: For two Q-chains ρ and π, define $\rho \preceq \pi$ if $Y(\rho) \preceq Y(\pi)$. The *Q-carrels* for an n-tuple are the following $k+1$ sets of positions: the first q_1 positions, the next $q_2 - q_1$ positions, and so on through the last $n - q_k$ positions. To each Q-chain π, we associate the permutation $\overline{\pi}$ of $[n]$: In n-tuple form, the Q-carrels of $\overline{\pi}$ respectively display the elements of the $k+1$ sets $P_1, P_2 \setminus P_1, \ldots, P_k \setminus P_{k-1}, [n] \setminus P_k$, with the elements of each set listed in increasing order. A *Q-permutation* is a permutation of $[n]$ in n-tuple form such that the values within each Q-carrel increase from left to right. It is easy to see that the creation of $\overline{\pi}$ describes a bijection from the set of Q-chains to the set of Q-permutations.

For $1 \leq i < j \leq n$, define the *reflection* σ_{ij} to be the following operator on Q-chains: Let $\pi = (P_1, \ldots, P_k)$ be a Q-chain. For $1 \leq \ell \leq k$, form the following sets: If $i \in P_\ell$ and $j \notin P_\ell$, set $P'_\ell := (P_\ell \setminus \{i\}) \cup \{j\}$. If $j \in P_\ell$ and $i \notin P_\ell$, set $P'_\ell := (P_\ell \setminus \{j\}) \cup \{i\}$. Otherwise, set $P'_\ell := P_\ell$. It can be seen that $P'_1 \subset \cdots \subset P'_k$; this is the Q-chain $\sigma_{ij}\pi$. If there exists $1 \leq \ell \leq k$ such that $j \in P_\ell$ and $i \notin P_\ell$, then $Y(\sigma_{ij}\pi)$ is produced from $Y(\pi)$ by decreasing some values from j to i (and sorting the resulting columns), so $\sigma_{ij}\pi \prec \pi$.

Example 2.1. *Set $n := 7$ and $Q := \{1, 2, 4\}$. The chain of sets $\pi := \{5\} \subset \{3, 5\} \subset \{1, 3, 4, 5\} \subset [7]$ is a Q-chain. Its Q-permutation is $\overline{\pi} = (5; 3; 1, 4; 2, 6, 7)$, where the semicolons separate the Q-carrels of $\overline{\pi}$. The result of the reflection $\sigma_{1,5}$ on π is the Q-chain $\sigma_{1,5}\pi = \{1\} \subset \{1, 3\} \subset \{1, 3, 4, 5\} \subset [7]$. The keys of π and $\sigma_{1,3}\pi$ are depicted below. One can see that $\sigma_{1,5}\pi \prec \pi$.*

$$Y(\pi) = \begin{ytableau} 1 & 3 & 5 \\ 3 & 5 \\ 4 \\ 5 \end{ytableau} \qquad\qquad Y(\sigma_{1,5}\pi) = \begin{ytableau} 1 & 1 & 1 \\ 3 & 3 \\ 4 \\ 5 \end{ytableau}$$

The following lemma says that we can find a reflection to step down in the Bruhat order between two Q-chains:

Lemma 2.2. *Let ρ, π be Q-chains. If $\rho \prec \pi$, then there exists $1 \leq i < j \leq n$ such that $\rho \preceq \sigma_{ij}\pi \prec \pi$.*

Proof. Write $\rho = (R_1, \ldots, R_k)$ and $\pi = (P_1, \ldots, P_k)$. Find the rightmost column where the keys $Y(\rho)$ and $Y(\pi)$ differ: these columns are $Y(R_h)$ and $Y(P_h)$ respectively for some $1 \leq h \leq k$. Find the minimal $i \in R_h \setminus P_h$ and the minimal $j \in P_h \setminus R_h$. Since $Y(R_h) \prec Y(P_h)$, we have $i < j$. Form the Q-chain $\sigma_{ij}\pi = (P'_1, \ldots, P'_k)$. By the above remark $\sigma_{ij}\pi \prec \pi$.

We verify that $\rho \preceq \sigma_{ij}\pi$: For the values of $1 \leq \ell \leq k$ such that $P'_\ell = P_\ell$, we have $Y(R_\ell) \preceq Y(P_\ell) = Y(P'_\ell)$. For the other values of ℓ, we have $P'_\ell = (P_\ell \setminus \{j\}) \cup \{i\}$. Let $1 \leq p \leq q_\ell$ denote the row index of the value j in $Y(P_\ell)$. In the rows below row p the value in $Y(P'_\ell)$ is the same as the value in $Y(P_\ell)$ since these are the values of P'_ℓ and P_ℓ which are greater than j. So here the value in $Y(R_\ell)$ is at most the value in $Y(P'_\ell)$. For rows at and above row p, the value in $Y(R_\ell)$ is at most the value in $Y(P'_\ell)$ since R_ℓ contains all of the p values of P'_ℓ which are less than j. $\square$

Fix an n-partition λ with $Q(\lambda) \subseteq Q$. For $1 \leq \ell \leq k$, the number of columns of length q_ℓ in λ is $\lambda_{q_\ell} - \lambda_{q_{\ell+1}}$. The number of columns of length n in λ is λ_n. Given a Q-chain $\pi = (P_1, \ldots, P_k)$, its *λ-key* is the tableau $Y_\lambda(\pi)$ of shape λ obtained by juxtaposing λ_n copies of the column $Y([n])$, $\lambda_{q_k} - \lambda_{q_{k+1}}$ copies of the column $Y(P_k)$, $\lambda_{q_{k-1}} - \lambda_{q_k}$ copies of the column $Y(P_{k-1})$, $\ldots$ and $\lambda_{q_1} - \lambda_{q_2}$ copies of the column $Y(P_1)$.

Lemma 2.3. *Let ρ, π be Q-chains. If $\rho \preceq \pi$, then $Y_\lambda(\rho) \preceq Y_\lambda(\pi)$. When $Q(\lambda) = Q$ the converse holds: if $Y_\lambda(\rho) \preceq Y_\lambda(\pi)$, then $\rho \preceq \pi$.*

Proof. Every column of length n is $Y([n])$, and every column of $Y_\lambda(\pi)$ of length less than n appears in $Y(\pi)$. When $Q(\lambda) = Q$, every column of $Y(\pi)$ also appears in $Y_\lambda(\pi)$. $\square$

We now describe the scanning algorithm of [6]. Fix a sequence $(b_1, b_2, b_3, \ldots)$. Define its *earliest weakly increasing subsequence* (EWIS) to be the subsequence $(b_{i_1}, b_{i_2}, b_{i_3}, \ldots)$, where $i_1 = 1$ and for $j > 1$ the index i_j is the smallest index such that $b_{i_j} \geq b_{i_{j-1}}$. The EWIS of the sequence $(6, 6, 4, 3, 5)$ which arises in Example 2.4 below is $(6, 6)$. For any tableau T of shape λ, construct its *scanning tableau* $S(T)$ as follows: Begin with an empty shape λ. Form the sequence of the bottom-most values of the columns of T from left to right. Find the EWIS of this sequence. When a value is added to the EWIS, mark its location in T. The sequence of locations just marked is called a *scanning path.* Fill the lowest available location of the leftmost available column of $S(T)$ with the last member of the EWIS. Iterate this process as if the marked locations are no longer part of T. Using the row condition on the filling of T, it can be seen that at each stage the unmarked locations form the shape of some n-partition. This implies that every location in T is marked once the leftmost column of $S(T)$ has been filled. To find the values of the next column of $S(T)$:

1. Ignore the leftmost column of T and λ.

2. Remove the marks from the remaining locations.

3. Repeat the above process.

Continue until the shape has been completely filled with values: this is the scanning tableau $S(T)$ of T. For a location $(r, c) \in \lambda$, let $P(T; r, c)$ denote the scanning path found to fill location (r, c) of $S(T)$.

Example 2.4. *Set* $n := 6$*,* $Q := \{1, 2, 4, 5\}$*, and* $\lambda := (5, 3, 2, 2, 1, 0)$*. Below are shown a tableau* T*, its scanning tableau* $S(T)$*, and a figure using different symbols to depict the scanning paths found while filling the leftmost column of* $S(T)$*.*

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad S(T) = \begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 5 & 5 \\ \hline 3 & 3 & 5 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \diamondsuit & \heartsuit \\ \hline \diamondsuit & \diamondsuit & \spadesuit \\ \cline{1-3} \spadesuit & \spadesuit \\ \cline{1-2} \heartsuit & \clubsuit \\ \cline{1-2} \clubsuit \\ \cline{1-1} \end{array}$$

We need four lemmas concerning the scanning tableau $S(T)$ of a tableau T of shape λ. Only the first is needed to prove Theorem 3.12, the main spanning theorem. The other three along with Lemmas 2.2 and 2.3 are used in Sections 4.2 and 4.3.

Lemma 2.5. *Let* $1 \leq c \leq \lambda_1$ *and* $1 \leq r \leq \zeta_c - 1$*. For any location* (p, b) *in the scanning path* $P(T; r, c)$*, there exists a location* (u, v) *in the previous scanning path* $P(T; r + 1, c)$ *such that* $v \leq b$ *and* $T(u, v) > T(p, b)$*.*

Proof. Since the scanning algorithm is defined recursively for column bottoms, we may reduce to the case that $(r + 1, c)$ is a column bottom of T.

First, suppose (p, b) is a column bottom of T. The location (p, b) is not in $P(T; r+1, c)$, but it does belong to the sequence of column bottoms of T which is scanned to form $P(T; r+1, c)$. Hence its value $T(p, b)$ was not in that previous earliest weakly increasing subsequence. Therefore there is a column bottom (u, v) of T in the scanning path $P(T; r+1, c)$ strictly to the left of (p, b) such that $T(u, v) > T(p, b)$.

Now suppose (p, b) is not a column bottom of T. Since (p, b) is scanned in the formation of $P(T; r, c)$, the location $(p+1, b)$ was marked as part of the previous scanning path $P(T; r+1, c)$. By the column strict condition on tabloids, its value satisfies $T(p+1, b) > T(p, b)$. Take $u := p+1$ and $v := b$. $\square$

Lemma 2.6. *Every value in the rightmost column of T appears in every column of $S(T)$. In particular, the rightmost column of $S(T)$ is the rightmost column of T.*

Proof. Fix a column index $1 \leq c \leq \lambda_1$. As was noted above, every location in T to the right of column c is marked in the construction of column c of $S(T)$. So every location in the rightmost column of T belongs to a scanning path $P(T; r, c)$ for some $1 \leq r \leq \zeta_c$. These locations must be the end of their respective scanning paths. $\square$

Let λ' denote the partition obtained from λ by omitting the rightmost column of its shape. Given a tableau T of shape λ, let T' denote the tabloid of shape λ' obtained by omitting the rightmost column of T.

Lemma 2.7. *Deleting the rightmost column both before and after forming the scanning tableau, we find that $S(T') \preceq [S(T)]'$.*

Proof. Let $(r, c) \in \lambda'$. In the two applications of the scanning algorithm, the same locations are marked and removed from within the region $\lambda' \subset \lambda$ of T as are from T'. So the scanning path $P(T; r, c)$ is the path $P(T'; r, c)$ with at most one location appended from the rightmost column of T. Since the values within a scanning path weakly increase, the value at the end of $P(T'; r, c)$ is less than or equal to the value at the end of $P(T; r, c)$. The value at location (r, c) in $S(T')$ is the value at the end of $P(T'; r, c)$, and the value at (r, c) in $S(T)$ is the value at the end of $P(T; r, c)$. $\square$

Now we fix a Q-chain π. Form its λ-key $Y_\lambda(\pi)$. In this π-specific environment, the notion of tableau is more complicated:

Definition 2.8. A tableau T of shape λ is *π-Demazure* if its scanning tableau satisfies $S(T) \preceq Y_\lambda(\pi)$.

In Theorem 2.18 we use π-Demazure tableaux to index the standard monomial basis for the Schubert variety indexed by π.

Lemma 2.9. *If a tableau T of shape λ is π-Demazure, then the tableau T' of shape λ' is π-Demazure.*

Proof. By the previous lemma, we have $S(T') \preceq [S(T)]' \preceq [Y_\lambda(\pi)]' = Y_{\lambda'}(\pi)$. □

2.3 Flags of subspaces and tabloid monomials

We now introduce the main objects of the paper: flags of subspaces, Bruhat cells, Schubert varieties, and tabloid monomials. For Sections 3.1 and 3.2, only the definitions concerning tabloid monomials are needed. Along the way we mention five facts about these structures for motivation which are formally stated and proved in Sections 4.1 and 4.2. Our main result, Theorem 2.18, is stated at the end of this section.

Definition 2.10. A *Q-flag* of $\mathbb{C}^n$ is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n$ such that $\dim(V_j) = q_j$ for $1 \leq j \leq k$.

We denote the set of Q-flags in $\mathbb{C}^n$ by $\mathcal{F}\ell_Q$. An ordered basis $(v_1, v_2, \ldots, v_n)$ of column vectors for $\mathbb{C}^n$ is presented in this paper as the $n \times n$ invertible matrix $[v_1, v_2, \ldots, v_n]$ whose columns from left to right are $v_1, \ldots, v_n$. Define a map Φ_Q from the ordered bases for $\mathbb{C}^n$ to $\mathcal{F}\ell_Q$ by sending an ordered basis $f := [v_1, \ldots, v_n]$ to the Q-flag $\Phi_Q(f)$ of subspaces V_j =span($\{v_i | i \leq q_j\}$) for $1 \leq j \leq k$. Any Q-flag can be represented in this way by many ordered bases. Special Q-flags can be made using the axis basis vectors $e_1, \ldots, e_n$ for $\mathbb{C}^n$: For each Q-chain $\pi = (P_1, P_2, \cdots, P_k)$, construct the *$Q$-chain flag* $\varphi(\pi)$ of subspaces V_j :=span($\{e_i | i \in P_j\}$) for $1 \leq j \leq k$. Given a Q-chain π, form the Q-permutation $\overline{\pi}$ as in Section 2.2. Define the $n \times n$ matrix s_π to be the permutation matrix whose $(\overline{\pi}_j, j)$ entry is 1 for $1 \leq j \leq n$. It is clear that $\varphi(\pi) = \Phi_Q(s_\pi)$, when s_π is viewed as an ordered basis.

Let B denote the subgroup of upper triangular matrices within GL_n, the group of invertible matrices.

Definition 2.11. Let π be a Q-chain. The *Bruhat cell* $C(\pi)$ is the set $\{\Phi_Q(bs_\pi) | b \in B\}$ of Q-flags which can be produced from the ordered basis s_π for the Q-chain flag $\varphi(\pi)$ with the action of the upper triangular matrices.

We will see (Fact 4.8) that every Q-flag belongs to a unique Bruhat cell. The following disjoint unions of Bruhat cells are important subsets of $\mathcal{F}\ell_Q$:

Definition 2.12. Let π be a Q-chain. We define the *Schubert variety* $X(\pi)$ to be the union of cells $\bigsqcup_{\rho \preceq \pi} C(\rho)$.

Our goal is to develop a coordinatization of $\mathcal{F}\ell_Q$. Recall that projective space $\mathbb{P}(\mathbb{C}^n)$ is the set of lines through the origin in $\mathbb{C}^n$; hence it is the set $\mathcal{F}\ell_{\{1\}}$ of $\{1\}$-flags. The set $\mathbb{P}(\mathbb{C}^n)$ does not have global coordinates in the usual (affine) sense. But it can be coordinatized by *projective coordinates*: A point $L \in \mathbb{P}(\mathbb{C}^n)$ is indexed by an equivalence class $[(p_1, p_2, \dots, p_n)]$ of n-tuples, where $(p_1, p_2, \dots, p_n) \in \mathbb{C}^n$ is a nonzero point on the line L and two n-tuples $(p_1, p_2, \dots, p_n)$ and $(p'_1, p'_2, \dots, p'_n)$ are equivalent if there is a nonzero $\alpha \in \mathbb{C}$ such that $(p_1, p_2, \dots, p_n) = (\alpha p'_1, \alpha p'_2, \dots, \alpha p'_n)$.

From now on, fix an n-partition λ such that $Q(\lambda) \subseteq Q$. Now we begin to form projective coordinates for $\mathcal{F}\ell_Q$ from tabloids of shape λ. Let $\mathbb{C}[x_{ij}]$ denote the ring of polynomials in the n^2 coordinates of a sequence of n vectors from $\mathbb{C}^n$. Fix $1 \leq p \leq n$. Let f be an $n \times n$ matrix. For any $1 \leq q \leq n$, define the *q-initial submatrix of f with rows* $r_1, \dots, r_p$ to be the $p \times q$ matrix whose i^{th} row consists of the first q entries of the r_i^{th} row of f. When $p = q$, in $\mathbb{C}[x_{ij}]$ we form for f its *q-initial minor with rows* $r_1, \dots, r_q$: this is the determinant of its q-initial submatrix with rows $r_1, \dots, r_q$.

Definition 2.13. Let T be a tabloid of shape λ. For each column index $1 \leq c \leq \lambda_1$, form in $\mathbb{C}[x_{ij}]$ the ζ_c-initial minor with indices $T(1, c), \dots, T(\zeta_c, c)$. The *monomial* of T, denoted by the corresponding Greek letter τ, is the product of these minors. Let π be a Q-chain. In particular, the monomial of the λ-key $Y_\lambda(\pi)$ is denoted $\psi_\lambda(\pi)$.

Example 2.14. *Set $n := 4$ and $Q := \{1, 2, 3\}$ and $\lambda := (5, 2, 1, 0)$. Form the tableau*

$$T := \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}.$$

As we begin a recurring example, set $x_{j1} =: x_j$, $x_{j2} =: y_j$, and $x_{j3} =: z_j$. Then the monomial τ of T

is the following product:

$$\tau = det\left(\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{bmatrix}\right) det\left(\begin{bmatrix} x_1 & y_1 \\ x_3 & y_3 \end{bmatrix}\right) det([x_1])det([x_2])det([x_4]).$$

Let F be a Q-flag. We will see (Lemma 4.9) that the sequence of the valuations of all tabloid monomials of shape λ on the ordered bases for F is projectively well defined: Varying the choice of basis f such that $\Phi_Q(f) = F$ will scale all these values equally. We will also see (Fact 4.14) that when $Q(\lambda) = Q$, this sequence of monomials give a faithful projective coordinatization of the set $\mathcal{F}\ell_Q$ of Q-flags.

Definition 2.15. Let Γ_λ denote the vector subspace of polynomials in $\mathbb{C}[x_{ij}]$ that are linear combinations of the tabloid monomials of shape λ.

While it is useful to consider the set of all tabloid monomials, the following long-known result shows that that set is much larger than is needed to span Γ_λ:

Theorem 2.16. *Let λ be an n-partition. The monomials of the semistandard tableaux of shape λ form a basis of the vector space Γ_λ.*

Such monomials are called *tableau monomials*. The spanning and linear independence parts of this basis theorem are reproved here as Theorem 3.8 and Corollary 4.16. This theorem implies that when $Q(\lambda) = Q$, the sequence of tableau monomials gives an efficient coordinatization of $\mathcal{F}\ell_Q$.

Now we return to having a Q-chain π fixed, as at the end of Section 2.2. Again form its λ-key $Y_\lambda(\pi)$. Define the subspace $Z_\lambda(\pi) \subseteq \Gamma_\lambda$ to be the span of the monomials of tabloids T such that $T \not\leq Y_\lambda(\pi)$.

Definition 2.17. Let π be a Q-chain. The *Demazure quotient* for π is the vector space $\Gamma_\lambda(\pi) := \Gamma_\lambda / Z_\lambda(\pi)$.

We will see (Lemma 4.11) that all tabloid monomials in $Z_\lambda(\pi)$ are zero on the Schubert variety $X(\pi)$. If moreover $Q(\lambda) = Q$, then $X(\pi)$ is the zero set in $\mathcal{F}\ell_Q$ of $Z_\lambda(\pi)$ (Fact 4.13). Hence we consider $\Gamma_\lambda(\pi)$ to be the span of the restrictions of the tabloid monomials to $X(\pi)$. We simply write "monomial" to refer to the residue of a monomial in $\Gamma_\lambda(\pi)$. Since the set of tableau monomials is now much larger than is needed to span $\Gamma_\lambda(\pi)$, we need an analog of Theorem 2.16 for the space $\Gamma_\lambda(\pi)$. Our main result is a new proof of the following theorem that is based on the scanning tableaux $S(T)$:

Theorem 2.18. *Fix a nonempty $Q \subseteq [n-1]$. Let λ be an n-partition such that $Q(\lambda) \subseteq Q$ and let π be a Q-chain. The monomials of the π-Demazure tableaux of shape λ form a basis of the vector space $\Gamma_\lambda(\pi)$.*

Such monomials are called π-*Demazure monomials*. The spanning and linear independence parts of this basis theorem are Theorem 3.12 and Theorem 4.15. This theorem implies that when $Q(\lambda) = Q$, the sequence of π-Demazure monomials gives an efficient coordinatization of the Schubert variety $X(\pi)$ of $\mathcal{F}\ell_Q$.

CHAPTER 3

Spanning Theorems

3.1 Tableau monomials span Γ_λ

Before we prove the spanning part of our main result, Theorem 2.18, in the next section, we must first prove the spanning of Γ_λ by tableau monomials in Theorem 2.16. We begin by presenting a translation of a classical determinantal identity into the language of tabloid monomials. This is a "master" identity that we use in two ways to prove the two spanning results by establishing relations amongst certain monomials. The idea of both proofs is the same: Using a total order on the set of tabloids, we provide straightening algorithms for applying the master identity. Each use of the identity progresses in the same direction under this order. The control afforded by the total order implies the termination of the algorithm. This is a common strategy; it was also used in [5].

Fix an n-partition λ; the sets Q and $Q(\lambda)$ play no role in this section. Fix a tabloid T of shape λ and a region $\mu \subseteq \lambda$. The region μ selects which locations are "active" in the master identity. The multiset of values of T within μ is denoted $T(\mu)$. For $1 \leq j \leq \lambda_1$, let T_j denote column j of T and let μ_j denote the intersection of μ with column j of λ. Let $\bar{\mu}$ denote the region of λ complementary to μ.

Definition 3.1. A *μ-shuffle of T* is a permutation of the values of T that can be obtained by the composition of two permutations as follows: First permute the values within the region μ such that the values within a given column are distinct. Then sort the values within each column into ascending order to obtain a tabloid.

Given a μ-shuffle σ of T, the resulting tabloid is denoted T_σ and its monomial is denoted τ_σ. Let $\epsilon(\sigma)$ denote the sign of σ as a permutation. For a tabloid T with repeated values, it is possible that for μ-shuffles $\sigma_1 \neq \sigma_2$ of T we have $T_{\sigma_1} = T_{\sigma_2}$.

We prepare to construct a square compound matrix $M_\mu(T)$ based on T and μ. Let g be the $n \times n$ matrix (x_{ij}) of n^2 indeterminants. First we split each of the initial square submatrices whose minors in g form the monomial τ of T into two rectangular parts. The next two definitions are illustrated at j = 1 in the example below. For each $1 \le j \le \lambda_1$ form the $\zeta_j \times |\mu_j|$ "active" matrix A_j by transposing the ζ_j-initial submatrix of g whose rows are specified by the values of $T_j(\mu)$. Also form the $\zeta_j \times |\bar{\mu}_j|$ "inactive" matrix N_j by transposing the ζ_j-initial submatrix of g similarly specified by the values of $T_j(\bar{\mu})$. The total number of columns in A_j and N_j is $|\mu_j| + |\bar{\mu}_j| = \zeta_j$. Let $A_j \sqcup N_j$ denote the $\zeta_j \times \zeta_j$ concatenation of the matrices A_j and N_j. Except for the order of its columns, the matrix $A_j \sqcup N_j$ is the transpose of the ζ_j-initial submatrix specified by the column T_j. So its determinant is the monomial τ_j of T_j, up to a sign. These $\zeta_j \times \zeta_j$ matrices form the main diagonal blocks of the compound matrix $M_\mu(T)$.

Now in addition let $1 \le i \le \lambda_1$. Form the rectangular matrix $A_j^{<i>}$ by transposing the ζ_i-initial submatrix of g whose rows are specified by the values of $T_j(\mu)$. Then let $A_j^{<i>} \sqcup 0$ denote the $\zeta_i \times \zeta_j$ concatenation of the matrix $A_j^{<i>}$ with a $\zeta_i \times |\bar{\mu}_j|$ zero matrix. These $\zeta_i \times \zeta_j$ matrices form the off-diagonal blocks of the compound matrix $M_\mu(T)$.

Define the matrix $M_\mu(T)$ to be the $(\zeta_1 + \cdots + \zeta_{\lambda_1})$-square compound matrix whose j^{th} diagonal block is $A_j^{<j>} \sqcup N_j$ and whose non-diagonal block in the (i,j) block position is $A_j^{<i>} \sqcup 0$:

$$M_\mu(T) := \begin{bmatrix} A_1^{<1>} \sqcup N_1 & A_2^{<1>} \sqcup 0 & \cdots & A_{\lambda_1}^{<1>} \sqcup 0 \\ A_1^{<2>} \sqcup 0 & A_2^{<2>} \sqcup N_2 & \cdots & A_{\lambda_1}^{<2>} \sqcup 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ A_1^{<\lambda_1>} \sqcup 0 & A_2^{<\lambda_1>} \sqcup 0 & \cdots & A_{\lambda_1}^{<\lambda_1>} \sqcup N_{\lambda_1} \end{bmatrix}$$

Example 3.2. *Set $n := 4$ and $\lambda := (5,2,1,0)$. Use the tableau T and the notation of Example 2.14, and let $\mu \subset \lambda$ be the region indicated by dots in the second figure:*

$$T := \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline & & & \bullet & \\ \hline & \bullet \\ \cline{1-2} \bullet \\ \cline{1-1} \end{array}$$

In the first column of T the value 4 lies in the region μ, while the values 1 and 2 do not. Hence the first "active" matrix is $A_1 = \begin{bmatrix} x_4 \\ y_4 \\ z_4 \end{bmatrix}$, and the first "inactive" matrix is $N_1 = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$. The compound *matrix $M_\mu(T)$ is:*

$$\left[\begin{array}{ccc|cc|c|c|c}
x_4 & x_1 & x_2 & x_3 & 0 & 0 & x_2 & 0 \\
y_4 & y_1 & y_2 & y_3 & 0 & 0 & y_2 & 0 \\
z_4 & z_1 & z_2 & z_3 & 0 & 0 & z_2 & 0 \\
\hline
x_4 & 0 & 0 & x_3 & x_1 & 0 & x_2 & 0 \\
y_4 & 0 & 0 & y_3 & y_1 & 0 & y_2 & 0 \\
\hline
x_4 & 0 & 0 & x_3 & 0 & x_1 & x_2 & 0 \\
\hline
x_4 & 0 & 0 & x_3 & 0 & 0 & x_2 & 0 \\
\hline
x_4 & 0 & 0 & x_3 & 0 & 0 & x_2 & x_4
\end{array}\right]$$

The following lemma is our master identity. It says that $|M_\mu(T)|$ is a polynomial in Γ_λ. It is the translation hinted at by Reiner and Shimozono of the left side of the determinantal identity (III.11) in [9] that they use as the left side of equation (5.3) in [5] . Note that the product of the determinants of the diagonal blocks of $M_\mu(T)$ is the monomial τ, up to a sign. This sign is the sign ambiguity in the statement of the lemma. This ambiguity vanishes in our applications.

Lemma 3.3. *Let T be a tabloid of shape λ and let $\mu \subseteq \lambda$. The determinant $|M_\mu(T)|$ is, up to sign, the signed sum of monomials $\sum \epsilon(\sigma)\tau_\sigma$, where the sum runs over all μ-shuffles σ of T.*

Proof. We calculate $|M_\mu(T)|$ by an iterated Laplace expansion process. We then show that the nonzero terms in this expansion correspond to the μ-shuffles of T. Begin to calculate $|M_\mu(T)|$ by Laplace expansion on the first ζ_1 rows, which form the first row of blocks. This expresses the determinant as the sum of the products of $\zeta_1 \times \zeta_1$ "primary" minors and $(\zeta_2 + \cdots + \zeta_{\lambda_1})$-square "complementary" minors.

Most of the products in this sum vanish because one of the two minors has a zero column or because the primary minor has repeated columns. Fix a summand that does not vanish for either of these reasons. (We allow the complementary minor to have repeated columns; the vanishing of such a minor appears as cancellation in the summation in the lemma statement.) Since the complementary minor at hand cannot have a zero column, its primary minor must include all of the columns of N_1. The primary minor's other columns come from various $A_j^{<1>}$ blocks. Recall that the columns of N_1 are the initial segments of the rows of g indexed by the values of $T_1(\bar{\mu})$ and that the columns of $A_j^{<1>}$ are the initial segments of the rows of g indexed by the values of $T_j(\mu)$. Because the primary minor does not have repeated columns, the values of $T_1(\bar{\mu})$ and those of

$T(\mu)$ that correspond to the columns of the $A_j^{<1>}$ blocks that contribute to the primary minor are distinct. Therefore we may define a column tabloid U_1 of length ζ_1 that is filled with all the values of $T_1(\bar{\mu})$ together with these values from $T(\mu)$. Except for the order of its columns, the primary minor is the determinant of the ζ_1-initial submatrix specified by the column U_1. So this primary minor is the monomial υ_1 of U_1, up to a sign.

Now consider the complementary minor of our fixed summand. Begin the next iteration by computing this determinant by Laplace expansion on its first ζ_2 rows, which form its first row of blocks. Fix a summand as above, if possible; otherwise the minor on hand is zero. Analogously define a column tabloid U_2 of length ζ_2 filled with all the values of $T_2(\bar{\mu})$ and the values of $T(\mu)$ corresponding to this new primary minor. Again this primary minor is the monomial υ_2 of U_2, up to a sign. Note that the values of $T(\mu)$ used here come from different locations within μ than those in the first iteration.

Continue iterating this Laplace expansion process. If our fixed term is nonzero, then it is the product $\upsilon_1 \cdots \upsilon_{\lambda_1}$, up to a sign. This is the monomial υ of the tabloid U of shape λ that is formed by the juxtaposition of the column tabloids $U_1, \ldots, U_{\lambda_1}$. The value from each location of T was used exactly once to construct U. Hence, the tabloid U was formed by a permutation σ of the values of T. Express σ as a composition of the following two permutations: First permute the values within μ so that the values used from $T(\mu)$ for U_i appear in the i^{th} column. As noted earlier, the values within each column are distinct. Then sort each column to obtain the tabloid U. Hence the permutation σ is actually a μ-shuffle of T. Therefore each nonzero term of this iterated Laplace expansion is the monomial τ_σ for a μ-shuffle σ of T, up to a sign. The sign is the product of the signs from the Laplace expansion process and the signs from presenting each $\zeta_i \times \zeta_i$ minor as the monomial of a column tabloid. If the sign of the diagonal term $|A_1^{<1>} \sqcup N_1| \cdots |A_{\lambda_1}^{<\lambda_1>} \sqcup N_{\lambda_1}|$ of this expansion agrees with the sign of the monomial τ of the identity μ-shuffle of T, then for each μ-shuffle σ of T the sign of the τ_σ term is $\epsilon(\sigma)$. Otherwise the diagonal term is $-\tau$; then the sign of each τ_σ term is $-\epsilon(\sigma)$.

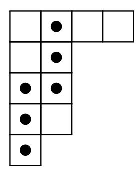
It is clear that the μ-shuffles of T associated to any two nonzero terms are distinct. It is also true that every μ-shuffle of T is associated to one of these terms: Fix a μ-shuffle σ of T. At step i in the iterated Laplace expansion, choose the primary minor to consist of the block N_i together with the columns from the $A_j^{<i>}$ blocks that correspond to the values of $T(\mu)$ that σ moves to column i

of T. $\square$

We will choose regions μ based on the tabloid T such that we can show $|M_\mu(T)| = 0$. Here the sign ambiguity vanishes, and Lemma 3.3 produces relations among the tabloid monomials. The choice of μ below yields a well-known relation. The presentation of this rederivation prepares the reader for the proof of the new Proposition 3.10.

Proposition 3.4. *Let T be a tabloid of shape λ. Let $1 \leq c \leq \lambda_1 - 1$ and $1 \leq r \leq \zeta_{c+1}$. Let $\mu \subseteq \lambda$ be the region $\{(i,c)|r \leq i \leq \zeta_c\} \cup \{(j, c+1)|1 \leq j \leq r\}$. Then $\sum \epsilon(\sigma)\tau_\sigma = 0$, where the sum runs over the μ-shuffles σ of T.*

Example 3.5. *For $n = 5$,the n-partition $\lambda = (4,2,2,2,1)$, and $(r,c) = (3,1)$, the region μ of Proposition 3.4 is the dotted region in the following figure:*



Proof. Construct the matrix $M_\mu(T)$ as above. By Lemma 3.3, it is sufficient to show that $|M_\mu(T)| = 0$. Since μ only has two columns, indexed c and $c+1$, the blocks $A_j^{<i>}$ within $M_\mu(T)$ are empty for $j \neq c$ or $c+1$. Hence the determinant $|M_\mu(T)|$ simplifies to $\tau_1 \ldots \tau_{c-1} det(*) \tau_{c+2} \ldots \tau_{\lambda_1}$, where $*$ is the $(\zeta_c + \zeta_{c+1})$-square matrix $\begin{bmatrix} A_c^{<c>} \sqcup N_c & A_{c+1}^{<c>} \sqcup 0 \\ A_c^{<c+1>} \sqcup 0 & A_{c+1}^{<c+1>} \sqcup N_{c+1} \end{bmatrix}$.

Note that $\zeta_c \geq \zeta_{c+1}$. Subtract the first ζ_{c+1} rows of the matrix $*$ from its last ζ_{c+1} rows to get the matrix $\begin{bmatrix} A_c^{<c>} \sqcup N_c & A_{c+1}^{<c>} \sqcup 0 \\ 0 \sqcup -N_c^{<c+1>} & 0 \sqcup N_{c+1} \end{bmatrix}$, where $N_c^{<c+1>}$ is the submatrix of N_c formed by its first ζ_{c+1} rows. The determinant is unchanged. The $|\mu| = \zeta_c + 1$ columns $\begin{bmatrix} A_c^{<c>} & A_{c+1}^{<c>} \\ 0 & 0 \end{bmatrix}$ have only ζ_c nonzero rows. $\square$

Let U and T be column tabloids of the same length. If the string of values of U read from top to bottom precedes the string of values of T in lexicographic order, then we define $U \leq T$. Let U and T be two tabloids of shape λ. If the string of columns of U read left to right precedes the string of columns of T in lexicographic order, then we define $U \leq T$. This is a total order; it extends the partial order $\preceq$ of Section 2.2.

Our goal is to re-express the monomial of any tabloid T which is not a tableau in terms of monomials of tabloids $U < T$. The following easy lemma is the first step.

Lemma 3.6. *Let T be a tabloid of shape λ. Let U be the tabloid obtained by sorting the columns of T of a given length in ascending order according to the total order $\leq$. Then $U \leq T$, and these tabloids have the same monomial.*

The relations given by Proposition 3.4 are sufficient for the following:

Proposition 3.7. *Let T be a tabloid of shape λ which is not a tableau. Then there exist coefficients $a_U = \pm 1$ such that $\tau = \sum a_U \upsilon$, where the sum is over some tabloids U such that $U < T$.*

Proof. If T does not already have its columns sorted by the total order, apply Lemma 3.6 to get $\tau = \upsilon$ where υ is the monomial of a tabloid $U < T$.

Now suppose T has sorted columns. Since T is not a tableau, there exists a location $(r, c) \in \lambda$ such that $T(r, c) > T(r, c+1)$. With this c and r, take μ as in Proposition 3.4. Then the relation $\sum \epsilon(\sigma)\tau_\sigma = 0$ holds, where the sum runs over all μ-shuffles of T. We can solve for $\tau = \tau_{id}$ to obtain $\tau = -\sum \epsilon(\sigma)\tau_\sigma$, where the sum now runs over all non-identity μ-shuffles of T.

Consider a non-identity μ-shuffle σ of T. We show $T_\sigma < T$. By the column filling property of tabloids and our choice of μ, every value in $T_c(\mu)$ is larger than every value in $T_{c+1}(\mu)$. Since $\sigma \neq id$, it replaces some value in $T_c(\mu)$ with a value from $T_{c+1}(\mu)$ and then sorts column c. Then in the highest location of column c in which T and T_σ differ, the smaller value is in T_σ. Therefore column c of T_σ precedes T_c in the total order, while all columns to its left are unchanged. Thus $T_\sigma < T$.

Since μ does not have repeated values, each tabloid T_σ is distinct. Therefore when we sum over tabloids instead of μ-shuffles of T, the coefficients of the monomials remain ± 1. $\square$

We are ready to prove the spanning part of Theorem 2.16. Elements of this proof reappear in the spanning proof for Theorem 2.18.

Theorem 3.8. *Let λ be an n-partition. The monomials of the semistandard tableaux of shape λ span the vector space Γ_λ.*

Proof. The space Γ_λ was defined to be the span of all tabloid monomials. Given a tabloid U, we show that its monomial υ is in the span of the tableau monomials. Suppose that U is not already a tableau.

Apply Proposition 3.7 to express υ as a linear combination of monomials of tabloids preceding U. If any of these tabloids is not a tableau, apply Proposition 3.7 to the largest among them according to the total order $\leq$ and iterate this step. After each iteration, the largest tabloid which is not a tableau that appears precedes that of the previous iteration. Since there are finitely many tabloids of shape λ, this process must terminate. When the process terminates, we have an expression for υ as a linear combination of tableau monomials. $\square$

3.2 Demazure monomials span the Demazure quotient

This section is a continuation of Section 3.1. Returning to the context at the end of Section 2.3, again fix an n-partition λ with $Q(\lambda) \subseteq Q$ and a Q-chain π. By Theorem 3.8, the space $\Gamma_\lambda(\pi) := \Gamma_\lambda / Z_\lambda(\pi)$ is spanned by the residues of tableau monomials. Here we re-express the (residue) monomial in $\Gamma_\lambda(\pi)$ of any tableau which is not π-Demazure by choosing an appropriate region μ for an application of Lemma 3.3. We write a bar over a polynomial of Γ_λ to indicate its residue in $\Gamma_\lambda(\pi)$. Proposition 3.10, Proposition 3.11, and Theorem 3.12 are respectively analogous to Proposition 3.4, Proposition 3.7, and Theorem 3.8.

If a tableau T is not π-Demazure, then there exists a location $(r, c) \in \lambda$ such that $S(T)[r, c] > Y_\lambda(\pi)[r, c]$. Our region μ will consist of locations that are associated to each of the locations $(r, c), (r+1, c), \ldots, (\zeta_c, c)$ in column c: These $\zeta_c - r + 1$ other locations will be indexed by $r, r+1, \ldots, \zeta_c$. The last value of the scanning path $P(T; r, c)$ is $S(T)[r, c]$. Define (p_r, b_r) to be the location of the first value in the path $P(T; r, c)$ which is larger than $Y_\lambda(\pi)[r, c]$. If (r, c) is a column bottom, then take μ to be the region $\{(p_r, b_r)\}$. Otherwise do the following: By Lemma 2.5, there exists at least one location (u, v) in $P(T; r+1, c)$ such that $v \leq b_r$ and $T(u, v) > T(p_r, b_r)$. Define (p_{r+1}, b_{r+1}) to be the first such location in $P(T; r+1, c)$. Continue in this fashion until $(p_{\zeta_c}, b_{\zeta_c})$ in $P(T; \zeta_c, c)$ has been defined: we have found locations $(p_r, b_r), (p_{r+1}, b_{r+1}), \ldots, (p_{\zeta_c}, b_{\zeta_c})$ with $b_r \geq b_{r+1} \geq \cdots \geq b_{\zeta_c} \geq c$ and $Y_\lambda(\pi)[r, c] < T(p_r, b_r) < T(p_{r+1}, b_{r+1}) < \cdots < T(p_{\zeta_c}, b_{\zeta_c})$. Take μ to be the region $\{(p_r, b_r), \ldots, (p_{\zeta_c}, b_{\zeta_c})\}$.

Example 3.9. *Set $n := 4$, $Q := \{1, 2, 3\}$, $\lambda := (5, 2, 1, 0)$, and re-use the tableau T from Example 3.2. Let π be the Q-chain $\{4\} \subset \{3, 4\} \subset \{1, 3, 4\}$. The scanning tableau $S(T)$ is not dominated by the* *key $Y_\lambda(\pi)$ in location $(1,1)$.*

$$T := \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&4\\\hline 2&3\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad S(T) = \begin{array}{|c|c|c|c|c|}\hline 2&2&4&4&4\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array} \qquad Y_\lambda(\pi) = \begin{array}{|c|c|c|c|c|}\hline 1&3&4&4&4\\\hline 3&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

For $(r,c)=(1,1)$, the region μ constructed above is the same as the region μ from Example 3.2.

Proposition 3.10. *Let π be a Q-chain. Let T be a tableau of shape λ which is not π-Demazure. Let $\mu \subseteq \lambda$ be the region just defined. Then $\sum \epsilon(\sigma)\overline{\tau}_\sigma = 0$ in $\Gamma_\lambda(\pi)$, where the sum runs over the μ-shuffles σ of T.*

Proof. Construct the matrix $M_\mu(T)$ as in Section 3.1. By Lemma 3.3, it is sufficient show $\overline{|M_\mu(T)|} = 0$ in $\Gamma_\lambda(\pi)$. We refer to the definitions above pertaining to the region μ. Since the leftmost column of μ is column $b_{\zeta_c} \geq c$, the determinant $|M_\mu(T)|$ simplifies to $\tau_1 \ldots \tau_{c-1} det(*)$ where $*$ is the lower right $(\zeta_c + \cdots + \zeta_{\lambda_1})$-square submatrix:

$$\begin{bmatrix} A_c^{<c>} \sqcup N_c & A_{c+1}^{<c>} \sqcup 0 & \cdots & A_{\lambda_1}^{<c>} \sqcup 0 \\ A_c^{<c+1>} \sqcup 0 & A_{c+1}^{<c+1>} \sqcup N_{c+1} & \cdots & A_{\lambda_1}^{<c+1>} \sqcup 0 \\ \vdots & \vdots & & \vdots \\ A_c^{<\lambda_1>} \sqcup 0 & A_{c+1}^{<\lambda_1>} \sqcup 0 & \cdots & A_{\lambda_1}^{<\lambda_1>} \sqcup N_{\lambda_1} \end{bmatrix}.$$

Because $S(T)$ fails to be dominated by $Y_\lambda(\pi)$ in column c, the index c is emphasized over the index b_{ζ_c}.

For $1 \leq j \leq i \leq \lambda_1$, let $N_j^{<i>}$ denote the submatrix of N_j formed by its first ζ_i rows. For each $c+1 \leq i \leq \lambda_1$, the first ζ_i rows of $*$ are contained in its first row of blocks. Subtract these rows from its $(i+1-c)^{th}$ row of blocks to get the matrix

$$*' := \begin{bmatrix} A_c^{<c>} \sqcup N_c & A_{c+1}^{<c>} \sqcup 0 & \cdots & A_{\lambda_1}^{<c>} \sqcup 0 \\ 0 \sqcup -N_c^{<c+1>} & 0 \sqcup N_{c+1} & \cdots & 0 \sqcup 0 \\ \vdots & \vdots & & \vdots \\ 0 \sqcup -N_c^{<\lambda_1>} & 0 \sqcup 0 & \cdots & 0 \sqcup N_{\lambda_1} \end{bmatrix}.$$

We calculate $det(*') = det(*)$ by Laplace expansion on the first ζ_c rows, which form its first row of blocks.

Most terms in the expansion vanish. Fix a term such that neither the primary nor complementary minor has a zero column and the primary minor has no repeated columns, if possible; otherwise $det(*') = 0$ and we are done. Since the complementary minor cannot have a zero column, the $\zeta_c \times \zeta_c$ primary minor must use all of the columns of the blocks $A_c^{<c>}, \ldots, A_{\lambda_1}^{<c>}$: These blocks $A_c^{<c>}, \ldots, A_{\lambda_1}^{<c>}$ have a total of only $|\mu| = \zeta_c - r + 1$ columns. Since the primary minor does not have repeated columns, the values of $T_c(\bar{\mu})$ that correspond to the $r-1$ columns of N_c that contribute to the primary minor are distinct from the values of $T(\mu)$. Therefore we may define a column tabloid U_c of size ζ_c containing all the values of $T(\mu)$ and these values from $T_c(\bar{\mu})$. Except for the order of its columns, the primary minor is the determinant of the ζ_c-initial submatrix of g specified by the column taboid U_c. So this primary minor is the monomial υ_c of U_c, up to a sign. By the choice of μ, the $\zeta_c - r + 1$ values of $T(\mu)$ are all larger than $Y_\lambda(\pi)[r, c]$. Hence at most $r - 1$ values in U_c are less than or equal to $Y_\lambda(\pi)[r, c]$. In particular $U_c(r) > Y_\lambda(\pi)[r, c]$.

Now consider the complementary minor of our fixed summand. We compute this minor by an iterated Laplace expansion analogous to the one in the proof of Lemma 3.3. Begin the iteration by computing this determinant by Laplace expansion on its first ζ_{c+1} rows, which form its first row of blocks. Fix a summand as above, if possible; otherwise the minor on hand is zero. Since the current complementary minor cannot have a zero column, this primary minor must include all of the columns of N_{c+1}. The primary minor's other columns come from the $-N_c^{<c+1>}$ block. Define a column tabloid U_{c+1} of length ζ_{c+1} filled with all the values of $T_{c+1}(\bar{\mu})$ and the values of $T_c(\bar{\mu})$ that correspond to the columns of $-N_c^{<c+1>}$ that contribute to the primary minor. Again the primary minor is the monomial υ_{c+1} of U_{c+1}, up to a sign.

Iterate this process. If our fixed term is nonzero, then we end up with column tabloids $U_c, U_{c+1}, \ldots, U_{\lambda_1}$ of respective lengths $\zeta_c, \zeta_{c+1}, \ldots, \zeta_{\lambda_1}$. We find that our fixed nonzero term is the product $\tau_1 \cdots \tau_{c-1} \upsilon_c \cdots \upsilon_{\lambda_1}$, up to a sign. This is the monomial υ of the tabloid U that is formed by the juxtaposition of the column tabloids $T_1, \ldots, T_{c-1}, U_c, \ldots, U_{\lambda_1}$. But $U \not\preceq Y_\lambda(\pi)$, by the observation about $U(r, c) = U_c(r)$ above. Hence its monomial υ belongs to the subspace $Z_\lambda(\pi)$ of Γ_λ. Therefore $\overline{\upsilon} = 0$ in $\Gamma_\lambda(\pi)$. So all of the terms in the iterated Laplace expansion that are not zero in Γ_λ are in $Z_\lambda(\pi)$. Hence $\overline{|M_\mu(T)|} = 0$. ☐

We now show that this result can be used to re-express the monomial of a tableau T which is not π-Demazure in terms of monomials of tabloids $U < T$.

Proposition 3.11. *Let π be a Q-chain. Let T be a tableau of shape λ which is not π-Demazure. If $\overline{\tau} \neq 0$ in $\Gamma_\lambda(\pi)$, then there exist coefficients $a_U = \pm 1$ such that $\overline{\tau} = \sum a_U \overline{v}$, where the sum is over some tabloids U such that $U < T$.*

Proof. Since T is not π-Demazure, there exists a region μ as for Proposition 3.10. Then the relation $\sum \epsilon(\sigma)\overline{\tau}_\sigma = 0$ holds in $\Gamma_\lambda(\pi)$, where the sum runs over all μ-shuffles of T.

If the identity permutation is the only μ-shuffle of T, then this equation states that $\overline{\tau} = 0$ in $\Gamma_\lambda(\pi)$. Otherwise, solve for $\overline{\tau} = \overline{\tau}_{id}$. Consider a non-identity μ-shuffle σ of T. We show $T_\sigma < T$. Let b be the index of the leftmost column of λ such that σ replaces some of the values of $T_b(\mu)$. The replacement values must arrive from later columns of μ. By our choice of μ, each value in $T_b(\mu)$ is strictly larger than all of the values in the later columns of μ. Then in the highest location of column b in which T and T_σ differ, the smaller value is in T_σ. Therefore column b of T_σ precedes T_b, while all columns to its left are unchanged. Thus $T_\sigma < T$. Since μ does not have repeated values, each tabloid T_σ is distinct. Therefore when we sum over tabloids instead of μ-shuffles of T, the coefficients of the monomials remain ± 1. $\square$

Now we are ready to prove the spanning part of Theorem 2.18.

Theorem 3.12. *Fix a nonempty $Q \subseteq [n-1]$. Let λ be an n-partition with $Q(\lambda) \subseteq Q$ and let π be a Q-chain. The monomials of the π-Demazure tableaux of shape λ span the vector space $\Gamma_\lambda(\pi)$.*

Proof. By Theorem 3.8, the space $\Gamma_\lambda(\pi)$ is spanned by the tableau monomials. Given a tableau U, we show its monomial $\overline{v}$ is in the span of the π-Demazure monomials. Suppose that U is not already π-Demazure.

Apply Proposition 3.11 to express $\overline{v}$ as a linear combination of monomials for tabloids preceding U. If any of these tabloids is not a tableau, apply Proposition 3.7 to the largest among them according to the total order $\leq$ and iterate this step. As in Theorem 3.8, this process terminates. We have now expressed $\overline{v}$ as a linear combination of monomials of tableaux preceding U. If any of the tableaux is not π-Demazure, apply Proposition 3.11 to the largest among them and then repeatedly apply Proposition 3.7 to the resulting tabloids. After each iteration of Propositions 3.11 and 3.7, the largest tabloid which is not a π-Demazure tableau precedes that appearing in the previous iteration.

Since there are finitely many tabloids of shape λ, this process must terminate. When the process terminates, we have an expression for $\overline{v}$ as a linear combination of π-Demazure monomials. $\square$

CHAPTER 4

A Linear Independence Theorem

4.1 Preferred bases and Bruhat cells

The linear independence of tableau monomials for Theorem 2.16 is shown directly in [15] by organizing the leading terms of tableau monomials with respect to an order on the indeterminants x_{ij} for $\mathbb{C}[x_{ij}]$. No similarly direct proof for the linear independence part of Theorem 2.18 is known. Instead we assume that our field has characteristic zero and, following [3, 5], we evaluate a linear combination of monomials at some ordered basis to verify that it is nonzero. We analyze these evaluations based on the membership of the corresponding Q-flag in a Bruhat cell or a Schubert variety. So here and in the next section we return to the context of Section 2.3 and present the standard facts concerning tabloid monomials, Bruhat cells, and Schubert varieties. A statement in these sections is displayed as a "Lemma" if it is needed for Section 4.3 and as a "Fact" if it is included only for motivation. The n-partition λ plays no role in this section.

Recall that the *Q-carrels* for an n-tuple are the following $k+1$ sets of positions: the first q_1 positions, the next $q_2 - q_1$ positions, and so on through the last $n - q_k$ positions. Given an ordered basis $f = [v_1, v_2, \ldots, v_n]$ of $\mathbb{C}^n$ in matrix form, the formation of the Q-flag $\Phi_Q(f) = (V_1, \ldots, V_k)$ can be viewed using these Q-carrels: The vectors from the first Q-carrel of f span V_1, and for $2 \leq j \leq k$ the vectors from the j^{th} Q-carrel of f extend V_{j-1} to the space V_j. The *pivot* of a nonzero column vector is its last nonzero coordinate.

Definition 4.1. An ordered basis f is *Q-preferred* if:

1. Within a Q-carrel, the pivots descend from left to right.

2. Each vector $v \in f$ has a value of 1 in its pivot coordinate.

3. All of the coordinate values to the right of a pivot are 0.

Example 4.2. *Let $n = 6$ and let $Q = \{1, 4\}$. The ordered basis f below is Q-preferred, while the ordered basis h is not. The vertical bars separate the Q-carrels of the basis.*

$$f := \left[\begin{array}{c|ccc|cc} 1 & 4 & 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \qquad h := \left[\begin{array}{c|ccc|cc} 2 & 5 & -3 & -1 & 0 & 4 \\ -4 & 1 & -1 & 0 & 2 & 1 \\ -6 & 1 & 1 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

In the following lemma, the Q-permutation corresponding to f is $\rho = (4; 2, 3, 6; 1, 5)$. The two Q-flags $\Phi_Q(f)$ and $\Phi_Q(h)$ are equal, as seen in the proof of Lemma 4.5 below.

The pivots of a Q-preferred basis give information concerning its Q-flag:

Lemma 4.3. *Let $f = [v_1, \ldots, v_n]$ be a Q-preferred basis with pivot coordinates $\rho_1, \ldots, \rho_n$. For each $1 \leq j \leq k$, define $R_j := \{\rho_1, \ldots, \rho_{q_j}\}$. Then $\rho = (R_1, \ldots, R_k)$ is a Q-chain. The list $(\rho_1, \ldots, \rho_n)$ is the Q-permutation $\overline{\rho}$. The set R_j is the set of possible pivot coordinates for vectors in $V_j :=$span$(v_1, \ldots, v_{q_j})$.*

Proof. By the third property of Q-preferred bases, the $\rho_1, \ldots, \rho_n$ are distinct. The first conclusion follows immediately. The condition on the values within the Q-carrels of a Q-permutation follows for $(\rho_1, \ldots, \rho_n)$ from the first property of Q-preferred bases; it is clearly $\overline{\rho}$. No nonzero linear combination of vectors with distinct pivot coordinates produces a vector with a new pivot coordinate. □

A Q-preferred basis f is a distinctive representative for $\Phi_Q(f)$ in the following way:

Lemma 4.4. *Let $f = [v_1, \ldots, v_n]$ be a Q-preferred basis with pivot coordinates $\rho_1, \ldots, \rho_n$. Fix $1 \leq m \leq n$, and let $1 \leq j \leq k+1$ be minimal such that $m \leq q_j$. Then v_m is the unique vector in $V_j :=$span$(v_1, \ldots, v_{q_j})$ that has a value of 1 at its pivot coordinate ρ_m and a value of 0 at coordinates $\rho_1, \rho_2, \ldots, \rho_{m-1}$.*

Proof. Let w be any such vector. Set $u := w - v_m$. Since the pivots within a Q-carrel of f descend from left to right, we have that $\rho_m < \rho_{m+1} < \cdots < \rho_{q_j}$. Since w and v_m both have pivot coordinate

ρ_m, the vector u has a value of 0 at the coordinates $\rho_{m+1}, \ldots, \rho_{q_j}$. Since w and v_m both have a value of 1 at coordinate ρ_m, the vector u has a value of 0 at coordinate ρ_m. Then u is a vector in V_j with a value of 0 at the coordinates $\rho_1, \rho_2, \ldots, \rho_{q_j}$. By the preceding lemma, this vector cannot have any other pivot coordinate. Therefore $u = 0$, and so $w = v_m$. $\square$

Each Q-flag has a unique Q-preferred representative:

Lemma 4.5. *The restriction of the map Φ_Q to the set of Q-preferred bases is a bijection to the set $\mathcal{F}\ell_Q$ of Q-flags. Hence Lemma 4.3 associates to each Q-flag a unique Q-chain.*

Proof. We begin by showing that the restriction of the map Φ_Q is injective. Suppose there are at least two Q-preferred bases. Then $Q \neq \{n\}$, since here the identity matrix depicts the only Q-preferred basis. Let $f_1 = [v_1, v_2, \ldots, v_n]$ and $f_2 = [w_1, w_2, \ldots, w_n]$ be distinct Q-preferred bases. Suppose that for some $1 \leq j \leq k$, the sets of pivot coordinates of the first q_j vectors of f_1 and of f_2 are different. Let V_j :=span$(v_1, \ldots, v_{q_j})$ and W_j :=span$(w_1, \ldots, w_{q_j})$. Let $1 \leq m \leq q_j$ be such that v_m has a pivot coordinate different from those of $w_1, \ldots, w_{q_j}$. By Lemma 4.3, no vector in W_j has the same pivot coordinate as v_m. Hence $v_m \notin W_j$, and so $V_j \neq W_j$. Otherwise the pivot coordinates $\rho_1, \ldots, \rho_n$ for f_1 and f_2 are the same. Let $1 \leq m \leq n$ be such that $v_m \neq w_m$ and let $1 \leq j \leq k+1$ be minimal such that $m \leq q_j$. The vectors v_m and w_m both have a value of 1 in their pivot coordinate ρ_m and a value of 0 in the coordinates $\rho_1, \ldots, \rho_{m-1}$. By the preceding lemma applied to w_m, we see that $v_m \notin W_j$. So again $V_j \neq W_j$. Hence $\Phi_Q(f_1) \neq \Phi_Q(f_2)$.

We now provide the inverse map: Fix a Q-flag F and choose *any* ordered basis h such that $\Phi_Q(h) = F$. The following elementary column operations on h preserve its Q-flag:

1. Swap two columns within the same Q-carrel of h.
2. Multiply a column of h by a nonzero scalar.
3. Add a multiple of a column of h to a column to its right.

Using these column operations, a Gaussian elimination algorithm can be performed on h so that its output f is a Q-preferred basis with $\Phi_Q(f) = F$. In Example 4.2 above, six of these operations can be used to convert the ordered basis h into the Q-preferred basis f. We verify that the output is independent of the choice of the basis used to represent F: Suppose two

Q-preferred bases f_1 and f_2 can be produced from two representatives h_1 and h_2 for F. Then $\Phi_Q(f_1) = \Phi_Q(h_1) = \Phi_Q(h_2) = \Phi_Q(f_2)$. Since the restriction of Φ_Q to the set of Q-preferred bases is injective, we have $f_1 = f_2$. So this process is a well-defined pre-inverse of the restriction of Φ_Q. If the input to the process is Q-preferred, no action is taken. Hence this function is also a post-inverse of the restriction of Φ_Q to the set of Q-preferred bases. $\square$

Now we present some facts about Bruhat cells. In Section 2.3 we associated to each Q-chain π the $n \times n$ permutation matrix s_π. It is easy to see that s_π is Q-preferred when viewed as an ordered basis.

Fact 4.6. *The set of Q-flags can be expressed as the union of Bruhat cells over all Q-chains: $\mathcal{F}\ell_Q = \bigcup_\rho C(\rho)$.*

Proof. Fix any Q-flag F; find its Q-preferred basis f as in the proof of Lemma 4.5. Let $\rho = (R_1, \ldots, R_k)$ be the Q-chain for f from Lemma 4.3: The Q-chain ρ records the pivots of f. It also records the pivots of the permutation matrix s_ρ. Since the pivots of both descend within each Q-carrel, the pivots of f are the locations of the 1s in s_ρ. It can be seen that $b := fs_\rho^{-1}$ is the matrix obtained by sorting the columns of f so that all of its pivots descend from left to right. So the matrix b is upper triangular. Then $F = \Phi_Q(f) = \Phi_Q(bs_\rho) \in C(\rho)$. $\square$

Fact 4.7. *Let π, ρ be Q-chains. If $\pi \neq \rho$, then the intersection of Bruhat cells $C(\pi) \cap C(\rho)$ is empty. So for each Q-flag F, the Q-chain π such that $F \in C(\pi)$ is unique.*

Proof. The action of B preserves the pivots of an ordered basis. So the Q-preferred basis for a Q-flag $F \in C(\pi) \cap C(\rho)$ would have the pivots listed in both $\overline{\pi}$ and $\overline{\rho}$. But $\pi \neq \rho$. $\square$

Together, these two facts show:

Fact 4.8. *The Bruhat cells $C(\pi)$ for all Q-chains π partition $\mathcal{F}\ell_Q$.*

4.2 Tabloid monomials, Bruhat cells, and Schubert varieties

Here we present some facts concerning tabloid monomials, Bruhat cells, and Schubert varieties. As in Section 3.2, fix an n-partition λ such that $Q(\lambda) \subseteq Q$.

Lemma 4.9. *Let g and h be ordered bases with $\Phi_Q(g) = \Phi_Q(h)$. There exists one $\alpha \neq 0$ such that for all tabloid monomials $\tau \in \Gamma_\lambda$, the equation $\tau(g) = \alpha\tau(h)$ holds.*

Proof. Recall that column c of any tabloid of shape λ specifies a ζ_c-initial minor of h. Since $Q(\lambda) \subseteq Q$, the sequence of elementary column operations in the proof of Lemma 4.5 that produces the Q-preferred basis f from h is also a sequence of elementary column operations when restricted to any ζ_c-initial submatrix of h. Hence any ζ_c-initial minor of h is a nonzero multiple, say $\kappa_c(h)$, of the same ζ_c-initial minor of f. Then $\tau(h) = \left(\prod_{c=1}^{\lambda_1} \kappa_c(h)\right) \tau(f)$. The valuation $\tau(g)$ also differs by some uniform nonzero scalar multiple from $\tau(f)$ for all tabloid monomials τ. □

Fix a Q-flag F. We can evaluate the sequence of all tabloid monomials of shape λ at any ordered basis representative for F. By the above lemma, the projective equivalence class of this sequence of valuations does not depend on the choice of representative. In this way we define a map $\Omega_\lambda : \mathcal{F}\ell_Q \to \mathbb{P}(\mathbb{C}^N)$, where N is the number of tabloids of shape λ.

Now fix a Q-chain $\pi = (P_1, \ldots, P_k)$. The next four results consider whether the tabloid monomials vanish or not at an ordered basis h when $\Phi_Q(h)$ is in the Bruhat cell $C(\pi)$ or the Schubert variety $X(\pi)$.

Lemma 4.10. *At any ordered basis h with $\Phi_Q(h) \in C(\pi)$, the monomial $\psi_\lambda(\pi)$ of the λ-key $Y_\lambda(\pi)$ does not vanish.*

Proof. By Lemma 4.5 there is a Q-preferred basis f such that $\Phi_Q(f) = \Phi_Q(h)$. There is a $b \in B$ such that $f = bs_\pi$. When the columns of f are sorted so that their pivots are in descending order, we produce an upper triangular matrix with 1s on the diagonal. Since the minor specified by any column of length n is the determinant of f, we see that such a minor is ± 1. For any $1 \leq j \leq k$, the minor of f specified by a column of $Y_\lambda(\pi)$ of length q_j is the determinant of the q_j-initial submatrix of h with rows given by P_j. The pivot coordinates in the first q_j columns of $f = bs_\pi$ are also given by P_j. Hence when the columns of this q_j-initial submatrix of f are sorted so that these pivots are in descending order, we produce an upper triangular matrix with 1s on the diagonal. Multiplying these minors, we see that the value of $\psi_\lambda(\pi)$ is ± 1 at f. By Lemma 4.9, the value is also nonzero at h. □

Lemma 4.11. *At any ordered basis h with $\Phi_Q(h) \in X(\pi)$, any tabloid monomial $\tau \in Z_\lambda(\pi)$ vanishes.*

Proof. Let f be the Q-preferred basis such that $\Phi_Q(f) = \Phi_Q(h)$. Then $\Phi_Q(f) \in C(\rho)$ for some $\rho \preceq \pi$. By Lemma 2.3, we have $Y_\lambda(\rho) \preceq Y_\lambda(\pi)$. There is a $b \in B$ such that $f = bs_\rho$. Since $\tau \in Z_\lambda(\pi)$, it is the monomial of a tabloid T such that $T \not\preceq Y_\lambda(\pi)$. Find a location $(r, c) \in \lambda$ such that $T(r, c) > Y_\lambda(\pi)[(r, c)]$. The r highest pivots of the first ζ_c columns of $f = bs_\rho$ are the coordinates $Y_\lambda(\rho)[(1, c)],\ Y_\lambda(\rho)[(2, c)],\ \ldots,\ Y_\lambda(\rho)[(r, c)]$. These coordinates are at or above the coordinate $Y_\lambda(\pi)[(r, c)]$ since $Y_\lambda(\rho)[(r, c)] \leq Y_\lambda(\pi)[(r, c)]$. On the other hand, the minor in τ specified by column c of T is the determinant of a ζ_c-initial submatrix m of f whose final $\zeta_c - r + 1$ rows are the rows $T(r, c), T(r + 1, c), \ldots, T(\zeta_c, c)$ of f. Since $T(r, c) > Y_\lambda(\pi)[(r, c)]$, the r columns of $f = bs_\rho$ with the pivots listed above have zeros in the last $\zeta_c - r + 1$ rows used for τ. This leaves at most $\zeta_c - r$ columns of m which can be nonzero in their last $\zeta_c - r + 1$ rows. Hence $\det(m) = 0$, and so $\tau(f) = 0$. By Lemma 4.9, we also have $\tau(h) = 0$. $\square$

Fact 4.12. *Suppose $Q(\lambda) = Q$. At any ordered basis h with $\Phi_Q(h) \notin X(\pi)$, there exists a tabloid monomial $\tau \in Z_\lambda(\pi)$ that does not vanish.*

Proof. By Fact 4.8, there is a unique Q-chain ρ such that $\Phi_Q(h) \in C(\rho)$. Since $\Phi_Q(h) \notin X(\pi)$, we have $\rho \not\preceq \pi$. Since $Q(\lambda) = Q$, Lemma 2.3 concludes $Y_\lambda(\rho) \not\preceq Y_\lambda(\pi)$. Hence $\psi_\lambda(\rho) \in Z_\lambda(\pi)$. By Lemma 4.10, the value of $\psi_\lambda(\rho)$ at h is nonzero. $\square$

The preceding three statements actually depended only on the Q-flag of an ordered basis, since the vanishing and nonvanishing of tabloid monomials is preserved under the scaling found in Lemma 4.9. So these statements are useful when the tabloid monomials are used as projective coordinates for $\mathcal{F}\ell_Q$. The last two statements show:

Fact 4.13. *If $Q(\lambda) = Q$, then the Schubert variety $X(\pi)$ is the zero set in $\mathcal{F}\ell_Q$ of $Z_\lambda(\pi)$.*

Using the above facts, we can finally show:

Fact 4.14. *If $Q(\lambda) = Q$, then the sequence of all tabloid monomials of shape λ distinguishes Q-flags. That is, the map Ω_λ is injective and faithfully parameterizes $\mathcal{F}\ell_Q$.*

Proof. Let F, G be Q-flags. Find the Q-preferred bases f, g of F, G. Let π, ρ be the Q-chains such that $F \in C(\pi)$ and $G \in C(\rho)$. Suppose $\pi \not\preceq \rho$. Since $Q(\lambda) = Q$, Lemma 2.3 concludes $Y_\lambda(\pi) \not\preceq Y_\lambda(\rho)$. So the monomial $\psi_\lambda(\pi)$ is in $Z_\lambda(\rho)$. By Lemmas 4.10 and 4.11, the monomial $\psi_\lambda(\pi)$ is nonzero at f and zero at g. If $\pi \prec \rho$, apply the argument above to $\rho \not\preceq \pi$.

Otherwise $\pi = \rho$, and so f and g have the same list of pivots $\overline{\pi} = (\pi_1, \dots, \pi_n)$. From the proof of Lemma 4.10, the monomial $\psi_\lambda(\pi)$ is either 1 at both f and g or -1 at both. Write $\pi = (P_1, \dots, P_k)$. If $P_j = \{1, 2, \dots, q_j\}$ for all $1 \leq j \leq k$, then $\overline{\pi}$ is the identity permutation. Here the identity matrix depicts the only Q-preferred basis, and so there is only one Q-flag. So suppose there is some $1 \leq j \leq k$ such that $P_j \neq \{1, 2, \dots, q_j\}$. Then in a Q-preferred basis, there exists a matrix entry unconstrained by the three Q-preferred properties. Now suppose $F \neq G$. Then $f \neq g$. Find an entry (r, c) where $f \neq g$. It must lie above the pivot in column c, so $r < \pi_c$. Let $1 \leq \ell \leq k$ be minimal such that $\pi_c \in P_\ell$. One of f or g is nonzero at entry (r, c). By the third property of Q-preferred bases, the column with pivot coordinate r lies in a later Q-carrel than does column c. Hence $r \notin P_\ell$. Let T be the tabloid obtained from $Y_\lambda(\pi)$ by replacing one of its columns $Y(P_\ell)$ with the column $Y\left(P_\ell \setminus \{\pi_c\} \cup \{r\}\right) \prec Y(P_\ell)$. Following a proof similar to that of Lemma 4.10, it can be seen that the evaluation of its monomial τ at f and g gives (up to sign) their (r, c) entries. Therefore the two valuations of the pair of monomials $(\psi_\lambda(\pi), \tau)$ at f and at g are not multiples of each other. $\square$

4.3 Linear independence of the Demazure monomials

The n-partition λ such that $Q(\lambda) \subseteq Q$ remains fixed. The objective of this section is to prove:

Theorem 4.15. *Fix a nonempty $Q \subseteq [n-1]$. Let λ be an n-partition such that $Q(\lambda) \subseteq Q$ and let π be a Q-chain. The monomials of the π-Demazure tableaux of shape λ are linearly independent in the vector space $\Gamma_\lambda(\pi)$.*

A particular application of Theorem 4.15 gives the linear independence of the tableau monomials for Theorem 2.16:

Corollary 4.16. *Let λ be an n-partition. The monomials of the semistandard tableaux of shape λ form a basis of the vector space Γ_λ.*

Proof. Take $Q := Q(\lambda)$. Let π_0 be the maximal Q-chain of subsets $P_j := \{n - q_j + 1, n - q_j + 2, \dots, n\}$ for $1 \leq j \leq k$. It can be seen that every tabloid of shape λ is dominated by the λ-key $Y_\lambda(\pi_0)$ in the partial order $\preceq$. So every tableau is π_0-Demazure. Here, the subspace $Z_\lambda(\pi_0)$ of Γ_λ is $\{0\}$. So we have $\Gamma_\lambda(\pi_0) = \Gamma_\lambda / Z_\lambda(\pi_0) = \Gamma_\lambda$. $\square$

Fix a Q-chain π from now on. Let ξ be any polynomial in Γ_λ. Suppose we can find an ordered basis f such that its Q-flag $\Phi_Q(f)$ lies in the Schubert variety $X(\pi)$ and $\xi(f) \neq 0$. By Lemma 4.11, the latter property implies that $\xi \notin Z_\lambda(\pi)$. Then the residue $\overline{\xi}$ in $\Gamma_\lambda(\pi) = \Gamma_\lambda/Z_\lambda(\pi)$ is nonzero. Since $C(\pi) \subseteq X(\pi)$, this observation implies that the theorem follows from:

Proposition 4.17. *Let π be a Q-chain. Let $T_1, \ldots, T_\ell$ be π-Demazure tableaux of shape λ. For any nonzero coefficients $a_1, \ldots, a_\ell$, there is some ordered basis f with $\Phi_Q(f)$ in the Bruhat cell $C(\pi)$ such that $\sum_{i=1}^{\ell} a_i \tau_i(f) \neq 0$.*

We will prove this proposition using induction on the number of columns of λ.

Before proving this proposition, we now elaborate on the “efficiency” claim from Section 2.3. Let N denote the number of tabloids of shape λ. Consider the map from the set GL_n to $\mathbb{C}^N$ given by the evaluation of the sequence of all tabloid monomials of shape λ. The coordinatization $\Omega_\lambda : \mathcal{F}\ell_Q \to \mathbb{P}(\mathbb{C}^N)$ of Section 4.2 was given by observing that the set of matrix representatives for a given flag maps to a unique projective equivalence class in $\mathbb{C}^N$. This coordinatization is inefficient: By the spanning Theorem 3.8, the coordinatization of $\mathcal{F}\ell_Q$ in $\mathbb{C}^N$ up to scalar multiples is contained in a subspace V that is parameterized by the coordinates corresponding to the tableau monomials. By Proposition 4.17 applied to $\pi := \pi_0$ as in the proof of Corollary 4.16, this subspace V is the minimal subspace that contains the image of $\mathcal{F}\ell_Q$. So we can actually coordinatize $\mathcal{F}\ell_Q$ with $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{C}^N)$. Let M denote the number of tableau monomials of shape λ. Then $V \cong \mathbb{C}^M$ and one may more efficiently coordinatize $\mathcal{F}\ell_Q$ in $\mathbb{P}(\mathbb{C}^M)$ by evaluating only the sequence of tableau monomials.

But this new coordinatization is inefficient for a proper Schubert variety $X(\pi)$. By Theorem 3.12, the coordinatization of $X(\pi)$ in $\mathbb{C}^M$ up to scalar multiples is contained in a subspace $V(\pi)$ that is parameterized by the coordinates corresponding to the π-Demazure monomials. By Proposition 4.17, this subspace $V(\pi)$ is the minimal subspace that contains the image of $X(\pi)$. So we can actually coordinatize $X(\pi)$ with $\mathbb{P}(V(\pi)) \subset \mathbb{P}(\mathbb{C}^M)$. Let $M(\pi)$ denote the number of π-Demazure monomials of shape λ. Then $V(\pi) \cong \mathbb{C}^{M(\pi)}$, and one may more efficiently coordinatize $X(\pi)$ in $\mathbb{P}(\mathbb{C}^{M(\pi)})$ by evaluating only the sequence of π-Demazure monomials.

Now we assume our field has characteristic zero. The corollary to the following proposition is used as the last step in the proof of Proposition 4.17. Recall the reflection operator σ_{ij} defined in Section 2.2 which acts on Q-chains by swapping the elements i and j. Here the limit in the set of ordered bases of $\mathbb{C}^n$ is found with respect to the usual metric on the n^2 entries of ordered bases when they are viewed as $n \times n$ complex matrices.

Proposition 4.18. *Let π be a Q-chain. Let $1 \leq i < j \leq n$ and use the reflection σ_{ij} to define $\rho := \sigma_{ij}\pi$. Let F be a Q-flag in the Bruhat cell $C(\rho)$. If $\rho \prec \pi$, then there is a path $\beta(t)$ in the set of ordered bases of $\mathbb{C}^n$ with $\Phi_Q(\beta(t)) \in C(\pi)$ for $0 < t < \frac{1}{2}$ such that $F = \Phi_Q\left(\lim_{t\to 0}\beta(t)\right)$.*

Proof. Let s_π be the $n \times n$ permutation matrix associated to π as in Section 2.3. Construct a path $\gamma(t)$ in the space of $n \times n$ matrices by altering s_π as follows: Let c_i and c_j be the column indices such that entries (i, c_i) and (j, c_j) of s_π are 1. Since $\rho \neq \pi$, columns c_i and c_j are in different Q-carrels of s_π. And since $\rho \prec \pi$, we have $c_j < c_i$. The submatrix at rows (i, j) and columns (c_j, c_i) of s_π is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Change these entries to $\begin{bmatrix} 1-t & t \\ t & 1-t \end{bmatrix}$.

For $0 < t < \frac{1}{2}$, compute the Q-preferred basis of $\gamma(t)$ by subtracting $\frac{1-t}{t}$ times column c_j from column c_i and re-scaling. Then we see that $\Phi_Q(\gamma(t))$ is still in $C(\pi)$. However, the limit $\lim_{t\to 0}\gamma(t)$ is the permutation matrix formed from s_π by switching columns c_i and c_j. Up to a reordering of columns within the affected Q-carrels, this is the permutation matrix s_ρ for the Q-chain ρ. So the Q-flag for this limit is $\Phi_Q(s_\rho)$. Let $b \in B$ be such that $F = \Phi_Q(bs_\rho)$. Here F is also $\Phi_Q\left(b\lim_{t\to 0}\gamma(t)\right)$. Define $\beta(t) := b\gamma(t)$. Then we have $\Phi_Q(\beta(t)) \in C(\pi)$ for $0 < t < \frac{1}{2}$. Note that $\lim_{t\to 0} b\gamma(t) = b\lim_{t\to 0}\gamma(t)$, since the entries in this product by b are linear combinations of the original matrix entries. Finally we have $\Phi_Q\left(\lim_{t\to 0}\beta(t)\right) = \Phi_Q\left(b\lim_{t\to 0}\gamma(t)\right) = F$. $\square$

The following corollary relates the vanishing of a polynomial in Γ_λ on the Bruhat cell $C(\pi)$ to its vanishing on the Schubert variety $X(\pi)$. Its proof uses the fact that the application of a polynomial from $\mathbb{C}[x_{ij}]$ commutes with forming a limit in the $n \times n$ complex matrices.

Corollary 4.19. *Let π be a Q-chain. Let f be an ordered basis with $\Phi_Q(f) \in X(\pi)$ and fix a polynomial $\xi \in \Gamma_\lambda$. If $\xi(h) = 0$ for every ordered basis h with $\Phi_Q(h) \in C(\pi)$, then $\xi(f) = 0$.*

Proof. Since $\Phi_Q(f) \in X(\pi)$, we have $\Phi_Q(f) \in C(\rho)$ for some Q-chain $\rho \preceq \pi$. The conclusion is trivial if $\rho = \pi$, so suppose that $\rho \prec \pi$. By Lemma 2.2, there exist $1 \leq i < j \leq n$ such that $\pi_1 := \sigma_{ij}\pi$ satisfies $\rho \preceq \pi_1 \prec \pi$. Since there are finitely many Q-chains, we can iterate Lemma 2.2 until we have a sequence of reflected Q-chains $\rho = \pi_m \prec \pi_{m-1} \prec \cdots \prec \pi_1 \prec \pi =: \pi_0$ for some $m > 0$. Let ℓ run from 0 to $m-1$ and iterate the following: Let $h_{\ell+1}$ be any ordered basis with $\Phi_Q(h_{\ell+1}) \in C(\pi_{\ell+1})$. Denote this Q-flag by F. Here the proposition constructed a path $\beta(t)$ with $\Phi_Q(\beta(t)) \in C(\pi_\ell)$

for $0 < t < \frac{1}{2}$ such that $F = \Phi_Q\left(\lim_{t\to 0}\beta(t)\right)$. By induction: For every ordered basis h_ℓ with $\Phi_Q(h_\ell) \in C(\pi_\ell)$, we had $\xi(h_\ell) = 0$. Since $\xi \in \mathbb{C}[x_{ij}]$, we have $\xi\left(\lim_{t\to 0}\beta(t)\right) = \lim_{t\to 0}\xi(\beta(t)) = \lim_{t\to 0} 0 = 0$. By Lemma 4.9 applied to the ordered bases $\lim_{t\to 0}\beta(t)$ and $h_{\ell+1}$ for F, we have $\xi(h_{\ell+1}) = 0$. When beginning the $\ell = m-1$ iteration, take $h_{\ell+1} := f$. $\square$

Now we are prepared to prove Proposition 4.17:

Proof of Proposition 4.17. The base case for our induction on the number of columns of λ is when every column of λ has length n, perhaps vacuously. Here the only tableau consists of the columns $Y([n])$. It is π-Demazure. Its monomial is a nonnegative power of the determinant, which is nonzero. So in this case the proposition holds.

Suppose λ has at least one column of length less than n. As for Lemma 2.7, let λ' denote the partition obtained from λ by deleting the rightmost column of its shape. Note that $Q(\lambda') \subseteq Q$. Suppose by induction that for any Q-chain ρ and linear combination ξ of ρ-Demazure monomials of shape λ', there is some ordered basis f with $\Phi_Q(f) \in C(\rho)$ such that $\xi(f) \neq 0$.

Write $\pi = (P_1, P_2, \ldots, P_k)$. Let $1 \leq h \leq k$ be minimal such that $q_h \in Q(\lambda)$. Examine the rightmost columns of the tableaux $T_1, \ldots, T_\ell$ and identify a minimal column among these with respect to the order $\preceq$. Suppose m of the tableaux share this minimal column, which has length q_h. Reindex the tableaux so that $T_1, \ldots, T_m$ have this rightmost column. We now form a Q-chain $(R_1, \ldots, R_k) =: \rho$ from this minimal column and π in such a way that ρ is small enough to have $\rho \preceq \pi$ and large enough to have $Y_\lambda(\rho) \succeq S(T_i)$ for $1 \leq i \leq m$. For $1 \leq j \leq h$, take R_j to be the set of the q_j smallest tableau values in this minimal column. So this minimal column is $Y(R_h)$. By Lemma 2.6 this column is also the rightmost column of $S(T_i)$ for $1 \leq i \leq m$. Since $S(T_i) \preceq Y_\lambda(\pi)$, we have $Y(R_h) \preceq Y(P_h)$. This implies that $Y(R_j) \preceq Y(P_j)$ for any $1 \leq j \leq h$.

For $h+1 \leq j \leq k$, form R_j by evolving P_j using R_h as follows: List the elements $r_1 < \cdots < r_{q_h}$ of R_h in increasing order. As t runs from 1 to q_h, successively replace the smallest element of P_j that is larger than or equal to r_t with the element r_t. Such an element exists since $P_j \supset P_h$ and $Y(P_h) \succeq Y(R_h)$. Visualize this replacement using the column $Y(P_j)$: by our replacement rule, replacing this value in $Y(P_j)$ by r_t in the same position preserves the property that the filling increases down the column. Define R_j to be the set resulting from the q_h iteration. Then the column $Y(R_j)$ is produced from $Y(P_j)$ by decreasing some of its values to values from R_h without reordering. So we have $Y(R_j) \preceq Y(P_j)$. For $h+1 \leq j \leq q_k$, we can see that $R_{j-1} \subset R_j$ as follows: Let $r \in R_{j-1}$. If $r \in R_h$, then $r \in R_j$. On the other hand if $r \notin R_h$, then $r \in P_{j-1} \subset P_j$. Both R_{j-1} and R_j were formed by replacing elements of P_{j-1} and P_j respectively with elements of the same set R_h. The element r was not replaced when R_{j-1} was formed from P_{j-1}, so it also was not replaced when R_j was formed from P_j. Then $\rho := (R_1, R_2, \ldots, R_k)$ is a Q-chain, and $\rho \preceq \pi$.

Fix $m+1 \leq i \leq \ell$. The rightmost column of $Y_\lambda(\rho)$ is $Y(R_h)$, which was minimal among the rightmost columns of $T_1, \ldots, T_\ell$. Since T_i does not share this minimal rightmost column, we can see that $T_i \not\preceq Y_\lambda(\rho)$. So by definition we have $\tau_{m+1}, \ldots, \tau_\ell \in Z_\lambda(\rho)$. Then by Lemma 4.11, at any ordered basis f with $\Phi_Q(f) \in X(\rho)$ we have $\sum_{i=1}^{\ell} a_i\tau_i(f) = \sum_{i=1}^{m} a_i\tau_i(f)$.

We want to show that each of $T_1, \ldots, T_m$ is ρ-Demazure. Fix $1 \leq i \leq m$. We know that $S(T_i) \preceq Y_\lambda(\pi)$. Fix a location $(b,c) \in \lambda$. From the construction of ρ, the value $Y_\lambda(\rho)[b,c]$ is $Y_\lambda(\pi)[b,c]$ or else a value from R_h. Suppose $Y_\lambda(\rho)[b,c] = Y_\lambda(\pi)[b,c]$. Since $S(T_i) \preceq Y_\lambda(\pi)$, we have $S(T_i)[b,c] \leq Y_\lambda(\rho)[b,c]$. Now suppose $Y_\lambda(\rho)[b,c]$ is some value $r \in R_h$. By Lemma 2.6, the value r appears in column c of $S(T_i)$. Let $1 \leq d \leq \zeta_c$ denote the row index such that $r = S(T_i)[d,c]$. Since $S(T_i) \preceq Y_\lambda(\pi)$, we have $Y_\lambda(\pi)[d,c] \geq r$. But from the construction of ρ, the value $Y_\lambda(\pi)[b,c]$ is the smallest value in its column larger than or equal to r. Since the filling $Y_\lambda(\pi)$ increases down each column, we have $b \leq d$. Hence $S(T_i)[b,c] \leq S(T_i)[d,c] = r = Y_\lambda(\rho)[b,c]$. Therefore $S(T_i) \preceq Y_\lambda(\rho)$ in both cases. So the tableaux $T_1, \ldots, T_m$ are ρ-Demazure. Then by Lemma 2.9 the corresponding tableaux $T_1', \ldots, T_m'$ of shape λ' are ρ-Demazure. By the inductive hypothesis, there is an ordered basis f with $\Phi_Q(f) \in C(\rho)$ such that $\sum_{i=1}^{m} a_i\tau_i'(f) \neq 0$.

The rightmost column of $Y_\lambda(\rho)$ is $Y(R_h)$. Hence the minor specified by $Y(R_h)$ is a factor of the monomial $\psi_\lambda(\rho)$. By Lemma 4.10, the value of $\psi_\lambda(\rho)$ at f is nonzero. Hence the minor specified by $Y(R_h)$ has some value $\alpha \neq 0$ at f. Therefore the valuation $\sum_{i=1}^{\ell} a_i\tau_i(f) = \sum_{i=1}^{m} a_i\tau_i(f) = \alpha \sum_{i=1}^{m} a_i\tau_i'(f)$. Thus we have $\sum_{i=1}^{\ell} a_i\tau_i(f) \neq 0$, where $\Phi_Q(f) \in C(\rho)$.

For the sake of contradiction, suppose that for every ordered basis h with $\Phi_Q(h) \in C(\pi)$ we have $\sum_{i=1}^{\ell} a_i\tau_i(h) = 0$. By design we have $\rho \preceq \pi$, and so $\Phi_Q(f) \in X(\pi)$. Then by Corollary 4.19 we also have $\sum_{i=1}^{\ell} a_i\tau_i(f) = 0$, a contradiction. $\square$

CHAPTER 5

Character Formulas

5.1 Summation formula for Demazure polynomials

Let H be the abelian subgroup of B consisting of its diagonal matrices $diag(y_1^{-1}, \ldots, y_n^{-1})$. The group H acts on ordered bases from the left. This induces an action on our polynomial subspace Γ_λ of $\mathbb{C}[x_{ij}]$: For a monomial $\tau \in \Gamma_\lambda$, an element $h \in H$, and an ordered basis f of $\mathbb{C}^n$ in matrix form, one has $h.\tau(f) := \tau(h^{-1}f)$. Since τ is a product of minors and the multiplication here by h^{-1} scales the rows of f, we see that $\mathbb{C}\tau$ is H-invariant. Given a tableau T, let c_i be the number of values in T equal to i. Then the character of H acting on $\mathbb{C}\tau$ is $y^T := \prod_{i=1}^n y_i^{c_i}$.

Now fix a Q-chain π. The subspace $Z_\lambda(\pi)$ is H-invariant. The character of the induced representation on $\Gamma_\lambda(\pi) := \Gamma_\lambda / Z_\lambda(\pi)$ follows from Theorem 2.18:

Corollary 5.1. *The character of H on $\Gamma_\lambda(\pi)$ is $\sum y^T$, where the sum runs over all π-Demazure tableaux of shape λ.*

This polynomial is the *Demazure polynomial* of [7]. This terminology will be justified in the appendix, where we note that $\Gamma_\lambda(\pi)^*$ is a Demazure module for B. Given that the scanning tableau of a tableau T is the right key of T, this polynomial is also the "key polynomial" of Lascoux and Schützenberger [8, Theorem 1].

5.2 Contemporary terminology

Here we provide a dictionary for relating the objects of this paper to the contemporary algebraic geometry literature. We also identify the character from Section 5.1 using the representation theory of GL_n. Continue to use the definitions from Section 5.1. Here we require an algebraically closed field of characteristic zero; we use $\mathbb{C}$.

Our subgroup H of diagonal matrices in GL_n is called the *torus*, and our subgroup B of upper triangular matrices in GL_n is called the *Borel* subgroup. Fix a nonempty $Q \subseteq \{1, 2, \ldots, n-1\}$ and set $k := |Q|$. Let E be the Q-flag of subspaces $V_j = \text{span}(\{e_i | i \le q_j\})$ for $1 \le j \le k$. The action of GL_n on ordered bases of $\mathbb{C}^n$ induces an action on the set $\mathcal{F}\ell_Q$ of Q-flags. Let P be the "parabolic" subgroup of GL_n that stabilizes E. Note that B stabilizes E, so $B \subseteq P$. Let $(v_1, \ldots, v_n)$ be an ordered basis and let $f := [v_1, \ldots, v_n] \in GL_n$ be the corresponding invertible matrix. Let F be any Q-flag. If $\Phi_Q(f) = F$, then $f.E = F$. So Lemma 4.5 implies that the action of GL_n on $\mathcal{F}\ell_Q$ is transitive. From the definition of P, we see that $\mathcal{F}\ell_Q$ is isomorphic to the coset space GL_n/P as a GL_n-set. The three operations from the proof of Lemma 4.5 generate the right action of P on GL_n when GL_n is considered as the set of all ordered bases for $\mathbb{C}^n$. That lemma found a preferred representative in GL_n for each coset in GL_n/P. So the map Φ_Q can be used to describe an isomorphism from GL_n/P to $\mathcal{F}\ell_Q$.

Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}$, and $\mathfrak{p}$ denote the Lie algebras of GL_n, H, B, and P respectively. The Lie algebra $\mathfrak{g}$ is reductive. Let $\phi_1, \ldots, \phi_n$ denote the basis of $\mathfrak{h}^*$ such that $\phi_i(h)$ is the entry of h in position (i,i) for any $h \in \mathfrak{h}$. Equip $\mathfrak{h}^*$ with the inner product for which $\phi_1, \ldots, \phi_n$ is an orthonormal basis. For each $1 \leq j \leq n-1$ set $\alpha_j := \phi_j - \phi_{j+1}$ and $\omega_j := \sum_{i=1}^{j} \phi_i$. For the semisimple part of $\mathfrak{g}$, the $\alpha_1, \ldots, \alpha_{n-1}$ depict the positive simple roots and the $\omega_1, \ldots, \omega_{n-1}$ depict the fundamental weights. Set $\omega_n := \sum_{i=1}^{n} \phi_i$. This weight is orthogonal to $\alpha_1, \ldots, \alpha_{n-1}$; it corresponds to the center of $\mathfrak{g}$. Set $J := [n-1] \setminus Q$. It can be seen that $\mathfrak{p}$ is the direct sum of $\mathfrak{b}$ and the root subspaces for the negative roots in the span of $\{\alpha_j | j \in J\}$. For each weight $\mu \in \mathfrak{h}^*$, there is a corresponding character $exp(\mu)$ of the torus H. For $1 \leq i \leq n$ set $x_i := exp(\phi_i)$. Let λ be an n-partition. For $1 \leq i \leq n$, set $a_i := \lambda_i - \lambda_{i+1}$; this is the number of columns of length i in the shape of λ. Then we have $Q(\lambda) = \{1 \leq i \leq n-1 | a_i \neq 0\}$. Use λ to also denote the weight $\sum_{i=1}^{n} \lambda_i \phi_i = \sum_{i=1}^{n} a_i \omega_i$. Let V_λ denote an irreducible representation of GL_n with highest weight λ.

The contragredient representation of H on Γ_λ defined in Section 5.1 extends to a representation of GL_n: Since a monomial τ of a tableau T is a product of minors, it can be seen that $g.\tau$ is again a polynomial in Γ_λ. Here τ is a weight vector of Γ_λ of weight $\sum_{i=1}^{n} -c_i \phi_i$.

Now fix an n-partition λ such that $Q(\lambda) \subseteq Q$. Let ϵ be the minimal Q-chain of subsets $E_j := \{1, \ldots, q_j\}$ for $1 \leq j \leq k$. Notate the λ-key monomial $\psi_\lambda(\epsilon)$ with ψ. It can be seen that for all ordered bases f and any $p \in P$, we have $\psi(fp) = \theta_\lambda(p)\psi(f)$ for a certain scalar $\theta_\lambda(p)$. Since the function θ_λ on P is multiplicative, it defines a character of P that is realized in $GL(\mathbb{C}\psi)$. Define an equivalence relation $\sim$ on $GL_n \times \mathbb{C}\psi$ by setting $(g, z) \sim (gp, \theta_\lambda(p) z\psi)$ for any $g \in GL_n$ and $p \in P$ and $z \in \mathbb{C}$. Define a line bundle $\mathcal{L}_\lambda$ on GL_n/P to be $(GL_n \times \mathbb{C}\psi)/\sim$. There is a contragredient representation of GL_n on its space of global sections $\Gamma(GL_n/P, \mathcal{L}_\lambda)$: For $\xi \in \Gamma(GL_n/P, \mathcal{L}_\lambda)$, a matrix $g \in GL_n$, and coset $f \in GL_n/P$, we define $g.\xi(f) := \xi(g^{-1}f)$. The Borel-Weil theorem says [16, Section 4] that this representation is irreducible with lowest weight $-\lambda$.

For the monomial τ of any tabloid T, we more generally have $\tau(fp) = \theta_\lambda(p)\tau(f)$ for any $f \in GL_n$ and $p \in P$. This is because the right multiplication of ordered bases by p is generated by the column operations of Lemma 4.5, while the filling of T specifies the rows used to form minors for τ. So τ can be used to define a section of $\mathcal{L}_\lambda$ that sends the coset fP of GL_n/P to the equivalence class $[f, \tau(f)\psi]$ of $(GL_n \times \mathbb{C}\psi)/\sim$. Hence Γ_λ can be viewed as a submodule of the global section space $\Gamma(GL_n/P, \mathcal{L}_\lambda)$ of this bundle. Since $\Gamma(GL_n/P, \mathcal{L}_\lambda)$ is irreducible, this entire space is realized by Γ_λ. It can be seen that the section defined by ψ is a lowest weight vector of $\Gamma(GL_n/P, \mathcal{L}_\lambda)$ for the lowest weight $-\lambda$. Here Theorem 2.16 says that the (semistandard) tableau monomials describe a basis for $\Gamma(GL_n/P, \mathcal{L}_\lambda)$. Moreover, this basis is a weight basis. Since the lowest weight of $\Gamma(GL_n/P, \mathcal{L}_\lambda)$ is $-\lambda$, the highest weight of $\Gamma(GL_n/P, \mathcal{L}_\lambda)^*$ is λ. Hence $\Gamma_\lambda^* \cong V_\lambda$. Since we have allowed λ to have columns of length n, we can have positive powers of the determinant in our characters. Hence each of the irreducible polynomial representations of GL_n can be realized with some Γ_λ^* (for all Q). If suitable notation were introduced, our treatment could also handle negative powers of the determinant. Then each of the irreducible rational representations of GL_n could be realized with some Γ_λ^*.

The Weyl group W of the semisimple part of $\mathfrak{g}$ is generated by the simple reflections $s_1, \ldots, s_{n-1}$ corresponding to the simple roots. Using the depiction of the simple roots in $\mathfrak{h}^*$ above, we can depict the action of a simple reflection on the ϕ basis as follows: For $1 \leq i \leq n-1$ we have $s_i.\phi_i = \phi_{i+1}$ and $s_i.\phi_{i+1} = \phi_i$, with $s_i.\phi_j = \phi_j$ for all other j. By considering only the subscripts here, we can model the action of W with the group S_n of permutations of $[n]$. Corresponding to the simple reflection s_i, the transposition $(i, i+1)$ swaps the values i and $i+1$ in an n-sequence of values from $[n]$. Given a permutation $\pi \in S_n$, write the result of $\pi.(1, 2, \ldots, n)$ in one-row form as $(\pi_1, \pi_2, \ldots, \pi_n)$. Then π models the element $w \in W$ such that $w.\phi_i = \phi_{\pi_i}$ for $1 \leq i \leq n$.

The length of an element $w \in W$ is the smallest number of simple reflections needed to express w. Let W_J denote the subgroup of W generated by the reflections s_j for $j \in J$. Since $Q(\lambda) \subseteq Q$, it can be seen that the group W_J stabilizes λ. Each coset of W/W_J has a unique minimal length representative. Let $W^J \subseteq W$ denote the set of such representatives. It can be seen that each

Q-permutation (Section 2.2) models some $w \in W^J$ and any $w \in W^J$ is correspondingly modeled by some Q-permutation. So the map sending a Q-chain π to the Q-permutation $\overline{\pi}$ can be viewed as a bijection from the set of Q-chains to W^J. Under this bijection, our partial order $\preceq$ on Q-chains agrees [17, Theorem 2.6.3] with the Bruhat order on W restricted to W^J. The Weyl group can also be depicted in GL_n relative to H as the group of $n \times n$ permutation matrices. Here the Q-chain π is represented by the matrix s_π from Section 2.3.

Given $w \in W$, let $v_{w\lambda}$ be a weight vector in V_λ of weight $w\lambda$. Let $D_\lambda(w)$ denote the Demazure B-module $\mathbb{C}[B].v_{w\lambda}$. Since W_J stabilizes λ, the module $D_\lambda(w)$ only depends on the coset of w in W/W_J. So we can name this Demazure module $D_\lambda(\pi)$, where π is the Q-chain corresponding to the representative of this coset in W^J.

Using Q-preferred bases, it is can be seen that the flags $\varphi(\pi)$ for Q-chains π are exactly the H-invariant Q-flags. The Bruhat cells are the B-orbits of GL_n/P. Corollary 4.19 can be strengthened as follows: Given a Q-chain π, the Schubert variety $X(\pi)$ is the closure of the Bruhat cell $C(\pi)$ in the Zariski topology on GL_n/P. This is proved over any algebraically closed field in e.g. [13, Section 10.6], but since GL_n is "split" that proof works here over any field [12]. If one accepts this substitute for Corollary 4.19, then every result in this paper other than Proposition 4.18 is valid over any field.

Now fix a Q-chain π. Let $\mathcal{L}_\lambda(\pi)$ denote the restriction of $\mathcal{L}_\lambda$ to the Schubert variety $X(\pi)$. The global section space $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$ of this bundle is not a GL_n-module since $X(\pi)$ is not GL_n-invariant in GL_n/P. But $X(\pi)$ is B-invariant, and so the restriction of the GL_n representation on $\Gamma(GL_n/P, \mathcal{L}_\lambda)$ to the subgroup B induces a representation of B on $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$. It is known [18] that its dual is isomorphic to the Demazure module defined above: $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))^* \cong D_\lambda(\pi)$. The section defined by the monomial $\psi_\lambda(\epsilon)$ is again a lowest weight vector of $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$ for the lowest weight $-\lambda$. The section defined by the monomial $\psi_\lambda(\pi)$ is a highest weight vector of $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$ for the highest weight $-w\lambda$, where $w \in W$ is modeled by the Q-permutation $\overline{\pi}$. Analagously, the vector space $\Gamma_\lambda(\pi) := \Gamma_\lambda / Z_\lambda(\pi)$ is not a GL_n-module since $Z_\lambda(\pi)$ is not GL_n-invariant in Γ_λ. But one can see that the action of B on a tabloid monomial produces a combination of monomials for tabloids with larger values. Then $Z_\lambda(\pi)$ is B-invariant, and so the restriction of the GL_n representation on Γ_λ to the subgroup B induces a representation of B on $\Gamma_\lambda(\pi)$. Fact 4.13 and the isomorphism $\Gamma_\lambda \cong \Gamma(GL_n/P, \mathcal{L}_\lambda)$ above imply that these B-modules

$\Gamma_\lambda(\pi)$ and $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$ are isomorphic. Here Theorem 2.18 says that the π-Demazure monomials describe a basis for $\Gamma(X(\pi), \mathcal{L}_\lambda(\pi))$. Moreover, this basis is a weight basis. By this isomorphism, we have $\Gamma_\lambda(\pi)^* \cong D_\lambda(\pi)$. Corollary 5.1 gives the character of $D_\lambda(\pi)$ as the Demazure polynomial $\sum x^T$, where the sum runs over all π-Demazure tableaux of shape λ. This implies that the dimension of $D_\lambda(\pi)$ is the number of π-Demazure tableaux of shape λ. See the appendix of [7] for more information concerning the concrete description of the coordinatized Demazure modules of $B \subset GL_n$.

CHAPTER 6

Minuscule Flag Manifolds

6.1 Introduction to Part II

The Grassmann manifold $Gr(d, n)$ is the complex manifold which consists of the d-dimensional complex subspaces of the vector space $\mathbb{C}^n$. The algebraic study of this manifold dates back to Julius Plücker in the 1800s. It is the manifold of Q-flags from Section 2.3 where $|Q| = 1$ and $q_1 = d$. Embed the Grassmann manifold in projective space using the embedding defined there for the one-column partition $\lambda = 1^d$. The homogeneous coordinates for this embedding are called the Plücker coordinates. Plücker coordinates are usually indexed by d-element subsets of $\{1, 2, \ldots, n\}$, which we arranged into column tableaux and ordered in Chapter 2. We studied the Plücker coordinates combinatorially using this index set. In particular, the linear relations we studied among the monomials for partitions $\lambda = r^d$ for some $r \geq 2$ are algebraic relations among the Plücker coordinates. For $r = 2$ these quadratic relations are called Plücker relations, and they have a nice combinatorial formulation.

Grassmann manifolds are examples of generalized flag manifolds, which are constructed as quotients of semisimple complex Lie groups. The special "minuscule" flag manifolds share many properties with the Grassmann manifolds. The minuscule flag manifolds are the Grassmann manifolds (Lie type A), the maximal orthogonal Grassmannians (types B and D), the even quadrics (also type D), and two "exceptional" manifolds: the complex Cayley plane (type E_6) and the Freudenthal variety (type E_7). All flag manifolds can be embedded in projective space. One property that a minuscule flag manifold shares with the Grassmann manifolds is that its homogeneous coordinates under its foremost embedding are indexed by a natural partially ordered set (that is a distributive lattice). Such ordered coordinates are also called Plücker coordinates; the minuscule Plücker relations are the quadratic relations among them.

We seek a uniform combinatorial description of the quadratic Plücker relations for the minuscule flag manifolds that is independent of Lie type. We can uniformly describe a certain kind of Plücker relation for all minuscule flag manifolds. For the two exceptional cases this leads to a complete description of all of the Plücker relations, apparently for the first time. The vector space spanned by the Plücker coordinates is a Lie algebra module. Our approach uses a combinatorial model for these "minuscule" representations due to Wildberger. For the simply laced Lie algebras, he constructed these representations with the order filters of the corresponding "minuscule posets." These are colored posets whose filters encode the weights of a minuscule representation. Uncolored minuscule posets and their associated minuscule lattices were introduced by Proctor in the 1980s.

The simplest nontrivial minuscule lattices correspond to the natural representations of the even orthogonal algebras $\mathfrak{o}(2n)$ for $n \geq 3$. Here the Hasse diagram of the minuscule lattice is a "double-tailed diamond." The structure of the relations found in these model cases is also possessed by the most accessible of the Plücker relations for the other minuscule cases. Our main results can be summarized as follows:

Theorem 6.1. *The "extreme weight" minuscule Plücker relations are "standard straightening laws" on "double-tailed diamond sublattices" of the Plücker coordinates. These are all of the Plücker relations for the complex Cayley plane (type E_6). For the Freudenthal variety (type E_7) we obtain a complete set of Plücker relations by supplementing the extreme weight relations with seven "zero weight" relations.*

We can explain this statement in more detail once we have the context provided by Section 6.2.

In this chapter we review past results and the necessary representation theory of Lie algebras. Chapter 7 derives in detail the Plücker relations for a model family of minuscule flag manifolds. Chapter 8 presents Wildberger's construction of minuscule representations from minuscule posets. Chapter 9 develops our main results.

6.2 Known Results

The minuscule flag manifolds in type A_n are the Grassmann manifolds of d-planes in $\mathbb{C}^n$. In this case the Plücker coordinates are specified by the one-column tableaux. The relevant n-partitions from Section 2.3 are the rectangles $\lambda = r^d$ of column length d. Hence every column of a tabloid of this shape is drawn from the same partially ordered set of column tableaux of length d. In fact, this order is a distributive lattice. Then a (semistandard) tableau of shape λ is nothing more than a length r chain in this lattice. Therefore, a standard monomial of degree r for the Grassmann manifold is a product of the Plücker coordinates specified by the elements in an r-chain in the lattice of column tableaux. The relations given in Proposition 3.4 are Plücker relations: A close inspection reveals that each of these relations re-expresses a product of two Plücker coordinates that are indexed by an incomparable pair of column tableaux as a signed sum. In Theorem 3.8, we used such relations to express a nonstandard monomial as a linear combination of standard monomials that preceded it in a lexicographic order.

These remarkable properties led Eisenbud to introduce the concept of an "algebra with straightening law" in [19]. We recall the definition here. Suppose R is a ring, A an R-algebra, and H a finite poset contained in A whose elements generate A as an R-algebra. A *standard monomial* of A is the product of a chain in H, *i.e.* an element of the form $a_1a_2\dots a_k$ with $a_i \in H$ for each $1 \le i \le k$ and $a_1 \preceq a_2 \preceq \cdots \preceq a_k$.

Definition 6.2. The R-algebra A is an *algebra with straightening law* on H if:

1. The algebra A is a free R-module whose basis is the set of standard monomials.

2. For each incomparable pair $a, b \in H$, it is required that: If $ab = \sum_{i=1}^{s} r_i h_{i1} h_{i2} \dots h_{ik_i}$ is the unique expression for ab as a linear combination of standard monomials, then it must be the case that $h_{i1} \preceq a$ and $h_{i1} \preceq b$ for every $1 \le i \le s$. Such a relation is called the *straightening law* for the incomparable pair a and b.

In the context of property (1), the straightening laws of property (2) can be shown to algebraically generate all of the relations in A on the set of generators H: Eisenbud showed that any monomial M on H can be written as a linear combination of standard monomials through the repeated application of straightening laws which involve incomparable pairs of factors of M.

Given an R-algebra generated on a poset H, one wants to know explicitly the straightening laws for as many incomparable pairs in H as possible. For the coordinate ring of a Grassmann manifold, the relations of Proposition 3.4 for the two column "rectangular" partitions are not themselves straightening laws. However, the algorithm in Theorem 3.8 uses them to obtain straightening laws in a finite number of steps. Such a process is called a "straightening algorithm". Hence the coordinate ring of the Grassmann manifold under the one-column embedding from Section 2.3 is an algebra with straightening law on the lattice of column tableaux.

As noted above, each minuscule flag manifold has a projective embedding such that its homogeneous Plücker coordinates are indexed by a certain natural partially ordered set. We call this its *standard* embedding. Seshadri launched the "Standard Monomial Theory" program in [20]; its main result can be reformulated as follows:

Theorem 6.3 (Seshadri)**.** *The coordinate ring of a minuscule flag manifold under its standard embedding is an algebra with straightening law on its Plücker coordinates.*

Seshadri's proof does not explicitly present the straightening laws. He proved that standard monomials form a basis without indicating how to express each monomial in the standard monomial basis. The second condition above follows without explicitly presenting the straightening laws, by considering the weight properties of a Lie algebra action on the coordinate ring. The straightening laws themselves are quadratic Plücker relations. We refine our goal to now seek a uniform description of straightening laws for these coordinate rings. The minuscule Plücker relations we obtain in Chapter 9 are explicit presentations of some of these straightening laws.

The fact that any monomial in an algebra with straightening law can be written as a linear combination of standard monomials through the repeated application of its straightening laws is generally proved by induction on the "indiscreteness" of the algebra. However the coordinate ring of a flag manifold is special in that it is graded and its Plücker coordinates all have degree one. Here, given the existence of straightening laws (2) this proof could be approached by implementing an algorithm similar to the one in the proof of Theorem 3.8. This strategy does not require the *a priori* knowledge that the standard monomials span the coordinate ring. Hence the spanning part of property (1) is deduced from property (2).

The poset of standard minuscule Plücker coordinates is actually a lattice, meaning it has a meet ($\wedge$) and a join ($\vee$) operation. Lakshmibai and Gonciulea used the lattice structure in [21] to study minuscule flag manifolds. However, they were only able to give the relations that define a certain flat degeneration of the manifold. Let H be the set of Plücker coordinates for a minuscule flag manifold X under its standard embedding.

Theorem 6.4 (Gonciulea and Lakshmibai)**.** *The minuscule flag variety X degenerates flatly to the toric variety defined by the relations* $\{ab = (a \wedge b)(a \vee b)\}_{a,b \in H}$.

We will use the lattice structure in Chapter 9 to describe the Plücker relations we obtain. The "leading term" of one of our relations is indeed the product of the meet and join of the incomparable pair. Hence the flat degeneration above truncates our straightening laws after one term. We give a description of all of the terms for our Plücker relations: the standard monomials which appear are products of "sister" elements in a "double-tailed diamond" sublattice structure.

We know of two type-specific studies of minuscule Plücker relations beyond the Grassmann manifolds. In 2013, Chirivì and Maffei studied [22] the Plücker relations for the maximal orthogonal Grassmannians by using Pfaffians to model the Plücker coordinates. They were able to give a straightening algorithm for the coordinate ring using relations among these Pfaffians. This parallels the straightening algorithm presented here in Theorem 3.8 for the Grassmann manifolds. Since our type-independent results only give "extreme weight" Plücker relations, in general those straightening algorithms produce many more relations for their respective manifolds. Chirivì, Littelmann, and Maffei briefly considered the Plücker relations of the Freudenthal variety in [23]. There they gave a single relation which can in principle be used to generate all of the Plücker relations with a Lie algebra action. Our results for this variety and for the complex Cayley plane go much further: We explicitly state all of the relations for the complex Cayley plane and the seven zero weight relations for the Freudenthal variety. Our method of generating relations can be applied to produce the other (extreme) 126 relations for the Freudenthal variety.

Within the context established above, we can now explain Theorem 6.1 in more detail. Our goal is:

Problem 6.5. *Describe as many of the straightening laws for the coordinate rings of the standard embeddings of minuscule flag manifolds as possible.*

We describe the most accessible of these straightening laws. They are all of the same form as the single straightening law for a model $\mathfrak{o}(2n)$ example. This straightening law is "supported" by a double-tailed diamond lattice. In particular, the "first" term in the straightening law is the product of the meet and join of the incomparable pair. In general the set of straightening laws we describe is not the entire set of straightening laws. But in the cases where we obtain all of the straightening laws, including the complex Cayley plane and the Freudenthal variety, these laws can be used to give constructive algorithmic proofs that the standard monomials span the coordinate rings.

6.3 Minuscule representations

Fix a simple complex Lie algebra $\mathfrak{g}$ of rank n with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ denote the set of roots. Let S denote the nodes of the Dynkin diagram. In this section we index these nodes as in [24]. We normalize the Killing form on $\mathfrak{g}$ so that in the induced inner product $(\cdot\,,\cdot)$ on $\mathfrak{h}^*_{\mathbb{R}}$, every long root α satisfies $(\alpha, \alpha) = 2$. Let $\{h_\alpha\}_{\alpha \in \Phi}$ denote the coroots in $\mathfrak{h}$, and let $\langle \lambda, \alpha \rangle := \lambda(h_\alpha) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ to be the application of a weight λ to the coroot for α. Let $\{\alpha_i\}_{i \in S}$ in Φ and $\{\omega_i\}_{i \in S}$ in $\mathfrak{h}^*$ denote the simple roots and the fundamental weights. The Weyl vector ρ is the sum of fundamental weights or, equivalently, the half sum of positive roots. Let W be the Weyl group of Φ, which acts on $\mathfrak{h}^*_{\mathbb{R}}$ by reflection over the root hyperplanes. The Weyl group is generated by the simple reflections $\{s_i\}_{i \in S}$ over the simple root hyperplanes. There is a standard partial order on the set of weights: For weights $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \preceq \lambda$ if $\lambda - \mu$ is a nonnegative integral sum of the simple roots. A weight λ is said to be dominant integral if $\langle \lambda, \alpha \rangle$ is a nonnegative integer for every positive root α. Given a dominant integral weight λ, let V_λ denote the irreducible $\mathfrak{g}$-module of highest weight λ as constructed in [24, Section 20]. Every finite dimensional irreducible $\mathfrak{g}$-module is isomorphic to V_λ for some dominant integral weight λ.

Definition 6.6. A dominant integral weight $\lambda \in \mathfrak{h}^*$ is *minuscule* if every weight of the irreducible $\mathfrak{g}$-module V_λ lies in the Weyl group orbit of λ. The following is the complete list of minuscule weights by Lie type:

$$\text{Type } A_n\text{: } \omega_1, \ldots, \omega_r; \quad \text{Type } B_n\text{: } \omega_n; \quad \text{Type } C_n\text{: } \omega_1; \quad \text{Type } D_n\text{: } \omega_1, \omega_{n-1}, \omega_n;$$
$$\text{Type } E_6\text{: } \omega_1, \omega_6; \quad \text{Type } E_7\text{: } \omega_7.$$

Given a minuscule weight λ, a module isomorphic to V_λ is called a minuscule representation of $\mathfrak{g}$. It is small in the following sense: Every finite dimensional highest weight module has a non-empty weight space for each weight in the Weyl group orbit of its highest weight. In a minuscule representation, there are no other weights.

The weights of any finite-dimensional $\mathfrak{g}$-module form an interval in the integral weight lattice under the order $\preceq$. Hence they form a finite distributive lattice. We call this lattice a *minuscule* lattice and denote it L_λ. Any weight basis of a minuscule $\mathfrak{g}$-module is in bijection with the lattice of its weights. Hence, we focus our study of minuscule representations on the minuscule lattice L_λ.

CHAPTER 7

A model family of minuscule flag manifolds

7.1 Classical geometry approach

In this chapter we find the sole Plücker relation for each flag manifold in a family of model examples, the even quadrics. These minuscule flag manifolds correspond to the natural representations of the type D_r Lie algebras $\mathfrak{o}(2r)$. We will work through these fundamental examples from two viewpoints. We begin in this section with the classical geometric approach, written to be accessible to everyone. We then proceed in the next section with a method based on representation theory. The representation theoretic approach is later reused in the general case.

Let V be a $2r$-dimensional complex vector space equipped with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. All such spaces are isomorphic, so we have the freedom to take a basis of V such that the form's matrix is any nonsingular symmetric matrix we choose. Our choice of matrix will be informed by the following:

Definition 7.1. A vector $v \in V$ is *isotropic* if it satisfies $\langle v, v \rangle = 0$. A line $L \subset V$ is istropic if any vector which spans L is an isotropic vector.

The set of isotropic lines of V is a subset of the projective space $\mathbb{P}(V)$, the set of all lines of V. We want to show that this subset is a variety by finding its defining equations.

Now we choose the matrix of our bilinear form. Since we are interested in isotropic lines, we choose a basis of isotropic vectors. Now we take advantage of the foreknowledge that the minuscule lattice L_θ associated to this example is the double-tailed diamond lattice pictured in Figure 7.2. For $r = 1$, the double-tailed diamond consists of two unordered elements. There is an intuitive way to pair the elements of the lattice according to its rank symmetry. Since there is no natural order to the two middle rank elements, we denote this incomparable pair $0^\sharp$ and $0^\flat$. We name the elements in the upper tail $1, 2, \ldots, r-1$ going up from the incomparable pair and their paired elements in the lower tail $\overline{1}, \overline{2}, \ldots, \overline{r-1}$ going down. We index our basis of V with these lattice elements

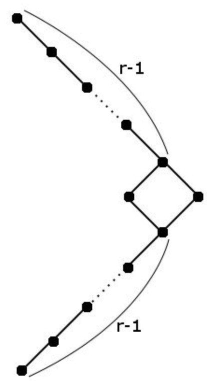


Figure 7.2: The double-tailed diamond lattice L_θ.

$v_{r-1}, \ldots, v_{\overline{r-1}}$, resisting the urge to unnaturally order the elements $0^\sharp$ and $0^\flat$. We want the only nonzero pairings of basis vectors in our bilinear form to be $\langle v_{0^\sharp}, v_{0^\flat}\rangle$ and $\langle v_i, v_{\bar{i}}\rangle$ for $1 \leq i \leq r-1$. To obtain a "clean" action of the Lie algebra $\mathfrak{o}(V)$ which stabilizes our form (as will be evident later), we alternate the values of these pairings between $+1$ and -1. The incomparable pair $0^\sharp$ and $0^\flat$ is taken as the starting point; hence we have $\langle v_{0^\sharp}, v_{0^\flat}\rangle = 1$, $\langle v_1, v_{\bar{1}}\rangle = -1$, $\langle v_2, v_{\bar{2}}\rangle = 1$, and so on. The matrix B of our bilinear form is displayed in Figure 7.3, where the middle two rows and columns corresponding to $0^\sharp$ and $0^\flat$ can be taken in either order.

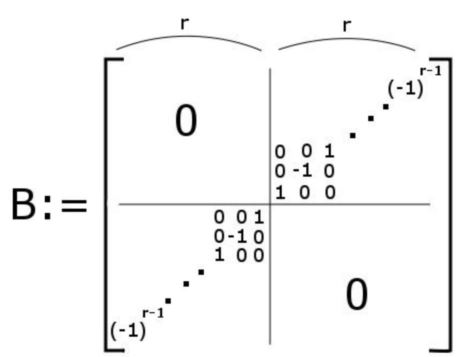


Figure 7.3: The matrix representing our bilinear form.

We use the bilinear form to create the dual basis $v^{r-1} := \langle v_{r-1}, \cdot\rangle, \ldots, v^{\overline{r-1}} := \langle v_{\overline{r-1}}, \cdot\rangle$ of the vector space V^*. Using these as coordinate functions, a vector $x \in V$ can be written as the $2r$-tuple $(v^{r-1}(x), \ldots, v^{\overline{r-1}}(x))$ of its valuations by the dual basis vectors. Let L be a line in V, which is an element of the projective space $\mathbb{P}(V)$. The line L is specified by its *projective coordinates*, which is an equivalence class of $2r$-tuples where two $2r$-tuples are equivalent if they differ by a nonzero scaling. Here if $x \in V$ spans the line L, then $L = [(v^{r-1}(x), \ldots, v^{\overline{r-1}}(x))]$. The following characterization of isotropic lines is the coordinatized expression of the quadratic form associated to our bilinear form:

Proposition 7.4. *The subset of isotropic lines in $\mathbb{P}(V)$ is the zero set of the single quadratic equation*

$$v^{0^\sharp}v^{0^\flat} - v^1v^{\overline{1}} + \cdots + (-1)^{r-1}v^{r-1}v^{\overline{r-1}} = 0.$$

Hence the set of isotropic lines of V is a projective variety. We can solve for the product $v^{0^\sharp}v^{0^\flat}$ to obtain the following relation:

$$v^{0^\sharp}v^{0^\flat} = v^1v^{\overline{1}} - \cdots + (-1)^r v^{r-1}v^{\overline{r-1}}.$$

For this example the coordinates $v^{r-1}, \ldots, v^{\overline{r-1}}$ were our Plücker coordinates, and the relation above is a Plücker relation. We have written it in the form of a straightening law on the double-tailed diamond lattice L_θ. This proposition verifies Seshadri's theorem for the even quadrics. Also notice that $0^\sharp \wedge 0^\flat = \overline{1}$ and $0^\sharp \vee 0^\flat = 1$, and so the leading term on the right hand side of the straightening law is the product of the meet and join of the incomparable pair. Lastly, note that the last sign in the above relation depends on the parity of r. This is a common phenomenon for type D_r objects.

7.2 Representation theory approach

In this section, we want to derive the relation of Proposition 7.4 in a representation theory context. This approach uses more sophisticated machinery, but it has the advantage of generalizing to other minuscule flag manifolds. We continue to refer to the terminology from Section 7.1.

Let $\mathfrak{o}(V)$ be the orthogonal Lie algebra for V, *i.e.* the algebra of matrices M such that $BM + M^TB = 0$ that is equipped with the commutator bracket. The following is a well-known fact:

Fact 7.5. *For $r \geq 3$, the Lie algebra $\mathfrak{o}(V)$ is simple with root system of type D_r. The representation of $\mathfrak{o}(V)$ over V is irreducible. Its highest weight is the fundamental weight associated to the top node of the type D Dynkin diagram in Figure 7.6.*

We denote this fundamental weight θ. From now on, assume $r \geq 3$. Note that we have labelled our Dynkin diagram with indices that are related to the indices for our basis of V. This is because

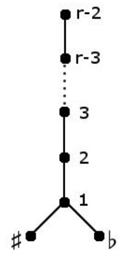


Figure 7.6: A labeling of the type D_r Dynkin diagram.

the simple roots correspond to the transitions between "adjacent" basis vectors. We defer the full description of the representation of $\mathfrak{o}(V)$ to the next section. For now we only need the following:

Fact 7.7. *With respect to the conventions of Section 7.3, the vector v_{r-1} is a highest weight vector of V.*

The ring generated by the coordinates of V is denoted $Sym(V^*)$. There is a natural representation of $\mathfrak{o}(V)$ on $Sym(V^*)$. The $\mathfrak{o}(V)$-module $Sym^2(V^*)$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{o}(V)).(v^{r-1})^2 \oplus I$ of $\mathfrak{o}(V)$-submodules for a unique submodule I. A result of Kostant relates this representation to the coordinate ring of the variety of isotropic lines:

Proposition 7.8. *The quotient of $Sym(V^*)$ by the ideal generated by the submodule $I \subset Sym^2(V^*)$ is the homogeneous coordinate ring of the variety of isotropic lines in $\mathbb{P}(V)$.*

Hence the study of the Plücker relations for the variety of isotropic lines in $\mathbb{P}(V)$ has been converted to the study of a submodule $I \subset Sym^2(V^*)$. The following lemma corresponds to the fact that Proposition 7.4 used only a single equation to define the variety of isotropic lines.

Lemma 7.9. *The submodule $I \subset Sym^2(V^*)$ has dimension 1.*

Proof. From Proposition 7.8, we have $\dim(I) = \dim[Sym^2(V^*)] - \dim[\mathfrak{U}(\mathfrak{g}).(v^{r-1})^2]$. Since $\dim(V) = 2r$, we have $\dim[Sym^2(V^*)] = \binom{2r+1}{2}$.

Since v_{r-1} spans the highest weight space of V, we have that $(v^{r-1})^2$ spans the highest weight space of $Sym^2(V^*)$ of weight 2θ. We can compute the dimension of the module $\mathfrak{U}(\mathfrak{g}).(v^{r-1})^2 \cong V_{2\theta}$ by using the Weyl dimension formula. Rather than evaulate the full formula, we compare it to the formula for $dim(V_\theta)$:

$$\dim(V_{2\theta}) = \prod_{\alpha \succ 0} \frac{\langle 2\theta + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\alpha \succ 0} \frac{\langle 2\theta + \rho, \alpha \rangle}{\langle \theta + \rho, \alpha \rangle} \frac{\langle \theta + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \dim(V_\theta) \prod_{\alpha \succ 0} \frac{\langle 2\theta + \rho, \alpha \rangle}{\langle \theta + \rho, \alpha \rangle} = 2r \prod_{\alpha \succ 0} \frac{\langle 2\theta + \rho, \alpha \rangle}{\langle \theta + \rho, \alpha \rangle}.$$

Index the simple roots of the type D_r root system as in Figure 7.6. The weight θ is the fundamental weight ω_{r-2}. When the positive roots are written in the simple root basis, we see that for roots α with no α_{r-2} component we have $\langle 2\theta + \rho, \alpha\rangle = \langle \theta + \rho, \alpha\rangle$ and the corresponding factor in the formula above is 1. The positive roots with nonzero α_{r-2} component are:

$$\begin{gathered}
\alpha_{r-2} \\
\alpha_{r-3} + \alpha_{r-2} \\
\vdots \\
\alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} \\
\alpha_\sharp + \alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} \\
\alpha_\flat + \alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} \\
\alpha_\sharp + \alpha_\flat + \alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} \\
\alpha_\sharp + \alpha_\flat + 2\alpha_1 + \alpha_2 + \cdots + \alpha_{r-2} \\
\vdots \\
\alpha_\sharp + \alpha_\flat + 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-3} + \alpha_{r-2}.
\end{gathered}$$

Denote this set Φ_{r-2}. Then one can compute $\prod_{\alpha \succ 0} \frac{\langle 2\theta+\rho,\alpha\rangle}{\langle\theta+\rho,\alpha\rangle}$ as:

$$\prod_{\alpha\in\Phi_{r-2}} \frac{\langle 2\theta + \rho, \alpha\rangle}{\langle \theta + \rho, \alpha\rangle} = \frac{3 \cdot 4 \cdots r(r+1)(r+1)(r+2)(r+3)\cdots(2r-1)}{2 \cdot 3 \cdots (r-1)r \cdot r(r+1)(r+2)\cdots(2r-2)} = \frac{(r+1)(2r-1)}{2r}.$$

Finally we have obtained $\dim(V_{2\theta}) = \frac{(r+1)(2r-1)}{2r} 2r = 2r^2 + r - 1 = (2r+1)r - 1 = \binom{2r+1}{2} - 1$. $\square$

To find a vector that spans I, we make use of the following lemma:

Lemma 7.10. *Let $\Omega \in \mathfrak{U}(\mathfrak{o}(V))$ denote the universal Casimir operator for $\mathfrak{o}(V)$. The image of the operator $\Omega - 4r$ on $Sym^2(V^*)$ is contained in the submodule I.*

Proof. The universal Casimir operator acts by a scalar on each irreducible component of $Sym^2(V^*)$. The component $\mathfrak{U}(\mathfrak{g}).(v^{r-1})^2$ is irreducible with highest weight 2θ. Hence by [24, Section 22.3], the Casimir operator acts on it by the scalar $(2\theta+\rho, 2\theta+\rho) - (\rho,\rho) = 4[(\theta,\theta)+(\theta,\rho)] = 4[1+(r-1)] = 4r$. Any vector of $Sym^2(V^*)$ can be written as $u + w$ with $u \in \mathfrak{U}(\mathfrak{g}).(v^{r-1})^2$ and $w \in I$. Then we find that $(\Omega - 4r).(u + w) = \Omega.u - 4r.u + (\Omega - 4r).w = 4r.u - 4r.u + (\Omega - 4r).w = (\Omega - 4r).w$. $\square$

The following is the main proposition of this section. Its proof requires a detailed description of the action of the Casimir operator Ω on $Sym^2(V^*)$ and is postponed to the next section.

Proposition 7.11. *The submodule $I \subset Sym^2(V^*)$ is spanned by the alternating sum:*

$$v^{0^\sharp}v^{0^\flat} - v^1v^{\overline{1}} + v^2v^{\overline{2}} - \cdots + (-1)^{r-1}v^{r-1}v^{\overline{r-1}}.$$

Hence the following relation has been re-obtained in the quotient $Sym^2(V^*)/I$:

$$v^{0^\sharp}v^{0^\flat} = v^1v^{\overline{1}} - v^2v^{\overline{2}} + \cdots + (-1)^r v^{r-1}v^{\overline{r-1}}.$$

7.3 Concrete Lie algebra actions

Here we explicitly describe the actions of a basis of $\mathfrak{o}(V)$ on our basis of V, using the labelling of the Dynkin diagram in Figure 7.6. Then we prove Proposition 7.11.

For elements i, j of the double-tailed diamond lattice L_θ which indexes our basis of V, let E_{ij} denote the linear transformation which sends v_j to v_i and all other basis vectors to zero. Recall that a matrix M belongs to $\mathfrak{o}(V)$ if $BM + M^TB = 0$, where B is the matrix of our bilinear form. It is straightforward to check that the following r diagonal matrices form a basis for the subspace of $\mathfrak{o}(V)$ consisting of diagonal matrices:

$$h_\sharp = (E_{1,1} - E_{0^\sharp,0^\sharp}) - (E_{\overline{1},\overline{1}} - E_{0^\flat,0^\flat}) \qquad h_\flat = (E_{1,1} - E_{0^\flat,0^\flat}) - (E_{\overline{1},\overline{1}} - E_{0^\sharp,0^\sharp})$$
$$h_1 = (E_{2,2} - E_{1,1}) - (E_{\overline{2},\overline{2}} - E_{\overline{1},\overline{1}})$$
$$\vdots$$
$$h_{r-2} = (E_{r-1,r-1} - E_{r-2,r-2}) - (E_{\overline{r-1},\overline{r-1}} - E_{\overline{r-2},\overline{r-2}})$$

Here we can take the subalgebra of diagonal matrices as our Cartan subalgebra, for which these vectors form a basis. We also take the subalgebra of upper triangular matrices as our Borel subalgebra. The fact that the indices $0^\sharp$ and $0^\flat$ are unordered causes no confusion here: it can be seen that for a matrix M to satisfy $BM + M^TB = 0$, both the $(0^\sharp, 0^\flat)$ and $(0^\flat, 0^\sharp)$ entries of M must be zero. With these conventions, the above basis of our Cartan subalgebra is the usual simple coroot basis as indexed by Figure 7.6. Define the coroots $\{h_\alpha\}$ for the non-simple roots α as sums of the above, according the simple root expansions of the corresponding roots. One can check that the following are simple root vectors:

$$e_\sharp := E_{1,\sharp} + E_{\flat,\overline{1}} \qquad e_\flat := E_{1,\flat} + E_{\sharp,\overline{1}}$$

$$e_1 := E_{2,1} + E_{\bar{1},\bar{2}}$$
$$\vdots$$
$$e_{r-2} := E_{r-1,r-2} + E_{\overline{r-2},\overline{r-1}}$$

Note that each of these simple root vectors is formed from a pair of transitions between basis vectors of V. These transitions follow covering relations in the lattice L_θ. The nonnegative action of these simple root vectors on our basis is our reward for choosing the alternating ± 1s in the matrix B for our bilinear form.

When expressed as a sum of simple roots, there are three types of positive roots in a type D root system: those with no $\sharp/\flat$ component, those with either a $\sharp$ or a $\flat$ component, and those with both a $\sharp$ and $\flat$ component. The first type is of the form $\alpha := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for some $1 \leq i \leq j \leq r-2$. Our preferred root vector for such a root is $e_\alpha := E_{j+1,i} + (-1)^{j-i} E_{\bar{i},\overline{j+1}}$. The second type is of the form $\alpha := \alpha_\star + \alpha_1 + \cdots + \alpha_i$ for some $0 \leq i \leq r-2$ and $\star \in \{\sharp, \flat\}$. Take $\bar{\star}$ to be the opposite of $\star$ in $\{\sharp, \flat\}$. Our preferred root vector for such a root is $e_\alpha := E_{i+1,0^\star} + (-1)^i E_{0^{\bar{\star}},\overline{i+1}}$. Finally, the last type of root is of the form $\alpha := \alpha_\sharp + \alpha_\flat + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for some $0 \leq i < j \leq r-2$. Our preferred root vector for such a root with $i = 0$ is $e_\alpha := E_{j+1,\bar{i}} + (-1)^{j+1} E_{i,\overline{j+1}}$. Our preferred vector for such a root with $i > 0$ is $e_\alpha := E_{j+1,\bar{i}} + (-1)^{j-i} E_{i,\overline{j+1}}$. These choices correspond to a specific realization of a construction of Wildberger that appears in Section 8.2. For each root α we use the matrix $f_\alpha := e_\alpha^T$ as our negative root vector. The triple $\{e_\alpha, f_\alpha, h_\alpha\}$ is a standard $\mathfrak{sl}_2$ triple.

Fact 7.7 is now evident. To prove Proposition 7.11, we must recall the mechanics of a dual representation. For an element $g \in \mathfrak{o}(V)$ and functional $\phi \in V^*$, the action of g on ϕ is the functional defined by $(g.\phi)(v) = \phi(-g.v)$ for all $v \in V$. Now we take advantage of the presentation of V^* afforded by our bilinear form. We wish to find the action of the element $g \in \mathfrak{o}(V)$ on one of our basis vectors $v^j = \langle v_j, \cdot \rangle$ for V^*. The resulting functional pairs with a vector $v_i \in V$ by $g.v^j(v_i) = v^j(-g.v_i) = \langle v_j, -g.v_i \rangle = \langle g.v_j, v_i \rangle$, where the last equation uses the defining property of $\mathfrak{o}(V)$. Hence $g.v^j(v_i) = v^i(g.v_j)$. This implies that if we use our dual sets of coordinates to write matrices for endomorphisms of V and of V^*, then the (i,j) entries of the matrices for the dual representations of g are equal. Therefore to describe the dual representation we can use the same matrices as for the original representation.

Proof of Proposition 7.11. By Lemmas 7.9 and 7.10, any nonzero element in the image of $\Omega - 4r$

on V^* spans the submodule I. We compute $(\Omega - 4r).(v^{0^\sharp} v^{0^\flat})$ and check that it is nonzero.

Let S denote the set of nodes of the Dynkin diagram of $\mathfrak{g}$, and let $\{h^i\}_{i\in S}$ be the basis of the Cartan subalgebra that is dual in our normalized Killing form to the simple coroot basis $\{h_i\}_{i\in S}$ defined above. Since $\mathfrak{g}$ is simply laced, every root α satisfies $(\alpha,\alpha) = 2$. Hence the coroot h_α produces the root α via the Killing form by $\alpha = (h_\alpha, \cdot)$. Then in our normalized Killing form we have $(e_\alpha, f_\alpha) = 1$. Therefore the universal Casimir operator $\Omega \in \mathfrak{U}(\mathfrak{g})$ can be expressed as in [25] by the following sum of three terms: $\sum_{\alpha\in\Phi^+} h_\alpha + \sum_{i\in S} h^i h_i + 2\sum_{\alpha\in\Phi^+} f_\alpha e_\alpha$.

The first term $\sum_{\alpha\in\Phi^+} h_\alpha$ acts on a weight space of weight μ by the scalar $\mu(\sum_{\alpha\in\Phi^+} h_\alpha)$. Since the vector $v^{0^\sharp} v^{0^\flat}$ has weight 0, the result of this part of the action is the zero vector. By [24, Section 22.3], the second term $\sum_{i\in S} h^i h_i$ acts on a weight space of weight μ by the scalar (μ,μ). Again, the result of this part of the action on $v^{0^\sharp} v^{0^\flat}$ is the zero vector. We compute the action of the final term $2\sum_{\alpha\in\Phi^+} f_\alpha e_\alpha$ by using the Leibniz rule twice:

$$2\sum_{\alpha\in\Phi^+} f_\alpha e_\alpha (v^{0^\sharp} v^{0^\flat}) = 2\sum_{\alpha\in\Phi^+} \left[(f_\alpha e_\alpha v^{0^\sharp}) v^{0^\flat} + (e_\alpha v^{0^\sharp})(f_\alpha v^{0^\flat}) + (f_\alpha v^{0^\sharp})(e_\alpha v^{0^\flat}) + v^{0^\sharp}(f_\alpha e_\alpha v^{0^\flat}) \right].$$

When α is not of our second type, all four summands are 0. For $\star \in \{\sharp, \flat\}$, denote by $\Phi_\star$ the set of $r-1$ roots $\{\alpha_\star, \alpha_\star + \alpha_1, \ldots, \alpha_\star + \alpha_1 + \cdots + \alpha_{r-2}\}$ of our second type. Again take $\bar\star$ to be the opposite of $\star$ in $\{\sharp, \flat\}$. Fix a root $\alpha = \alpha_\star + \cdots + \alpha_i \in \Phi_\star$. For this root the first and last summand are $(f_\alpha e_\alpha v^{0^\star}) v^{0^{\bar\star}} = (E_{0^\star, i+1} E_{i+1, 0^\star} v^{0^\star}) v^{0^{\bar\star}} = v^{0^\star} v^{0^{\bar\star}}$ and $v^{0^\star}(f_\alpha e_\alpha v^{0^{\bar\star}}) = 0$. We have obtained $v^{0^\sharp} v^{0^\flat}$ once for each of the $2(r-1)$ roots of $\Phi_\sharp \cup \Phi_\flat$. For the second and third summands we have $(e_\alpha v^{0^\star})(f_\alpha v^{0^{\bar\star}}) = (E_{i+1,0^\star} v^{0^\star})((-1)^i E_{\overline{i+1}, 0^{\bar\star}} v^{0^{\bar\star}}) = (-1)^i v^{i+1} v^{\overline{i+1}}$ and $(f_\alpha v^{0^\star})(e_\alpha v^{0^{\bar\star}}) = 0$. We can see that we have obtained each of the $v^{i+1} v^{\overline{i+1}}$ terms for $0 \le i \le r-2$ twice across the roots $\Phi_\sharp \cup \Phi_\flat$.

Collecting like terms of $\Omega.(v^{0^\sharp} v^{0^\flat})$ we find:

$$\Omega.(v^{0^\sharp} v^{0^\flat}) = 4(r-1) v^{0^\sharp} v^{0^\flat} + 4 v^1 v^{\bar 1} - 4 v^2 v^{\bar 2} + \cdots + (-1)^r 4 v^{r-1} v^{\overline{r-1}}.$$

Hence the desired vector is:

$$(\Omega - 4r).(v^{0^\sharp} v^{0^\flat}) = -4 v^{0^\sharp} v^{0^\flat} + 4 v^1 v^{\bar 1} - 4 v^2 v^{\bar 2} + \cdots + (-1)^r 4 v^{r-1} v^{\overline{r-1}}.$$

□

CHAPTER 8

Minuscule posets

8.1 Introduction to minuscule posets

For a simply laced Lie algebra $\mathfrak{g}$, Wildberger constructed its minuscule representations using minuscule posets. We want to use this realization of the minuscule representations to produce some minuscule Plücker relations. Recall that the weights of a minuscule representation form a finite distributive lattice. It is well known that every finite distributive lattice is isomorphic to the lattice of "order filters" on its subposet of "meet irreducible" elements, as we recall below. In this fashion, a minuscule poset distills the lattice of weights of a minuscule representation into a smaller poset. These smaller posets are "d-complete" and are colored by the nodes of the Dynkin diagram so that they become "colored d-complete" posets [26].

An element of a lattice is *meet irreducible* if it is covered by exactly one other element of the lattice. Let L be a finite distributive lattice, and let P be its sub-poset of meet irreducible elements. An *order filter* of the poset P is a subset $J \subseteq P$ such that if $x \in J$ and $x \preceq y$, then $y \in J$. The lattice L is isomorphic to the lattice $J(P)$ of order filters of P ordered by reverse inclusion. The meet ($\wedge$) and join ($\vee$) operations on the lattice of order filters are the union and intersection operations on filters respectively. (Our order dualizations of the usual conventions have been chosen for Lie-theoretic considerations.)

Definition 8.1. A minuscule poset P_λ is the subposet of the meet irreducible elements in the distributive lattice L_λ of weights which occur in a minuscule representation V_λ.

The above definition of minuscule poset is the most appropriate for this dissertation. However, they can also be naturally defined as a certain subset of the poset of coroots [27, Theorem 11]. That alternative definition is more direct in that it does not use the above correspondence between a finite distributive lattice and its poset of meet irreducible elements.

The following notation was established in [27]. Let $\mathfrak{g}$ be a simple Lie algebra of Lie type X_r.

Let ω_j be a minuscule weight as in Definition 6.6. The minuscule poset arising from the $\mathfrak{g}$-module V_{ω_j} is denoted $x_r(j)$. See Figure 8.2 for the Hasse diagrams of the minuscule posets. When X_r is simply laced, the "top tree" of each poset is the corresponding Dynkin diagram and then the maximal element is the node of the minuscule weight in that diagram.

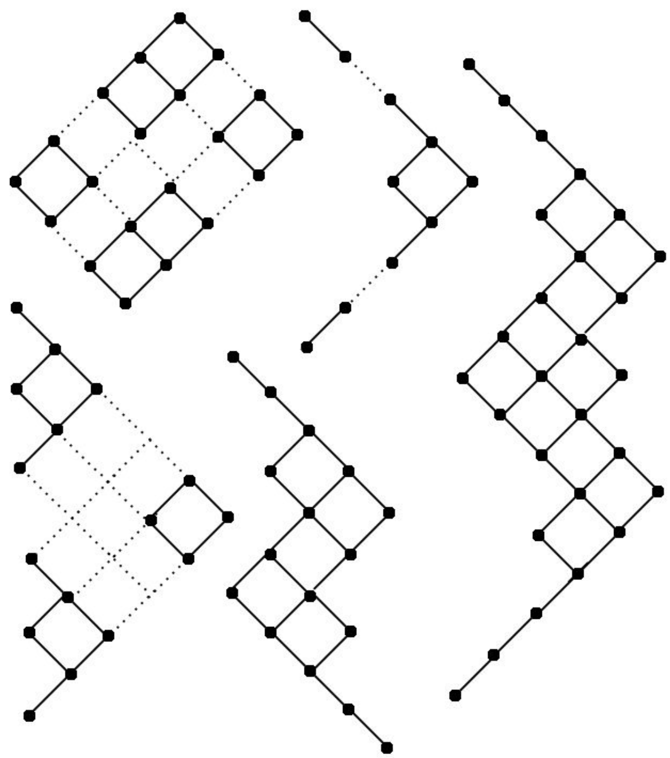


Figure 8.2: The Hasse diagrams of the minuscule posets. Clockwise from top left: $a_n(j)$, $d_n(1)$, $e_7(7)$, $e_6(1) \cong e_6(6)$, $b_n(n) \cong d_n(n-1) \cong d_n(n)$.

For the remainder of the chapter fix a simple Lie algebra $\mathfrak{g}$ and a minuscule weight λ. Let L_λ denote the lattice of weights in V_λ, and simply use P to denote the minuscule poset $P_\lambda \subseteq L_\lambda$. The properties of minuscule posets used in this section were established by Proctor for the simply laced cases [27, 28, 26] and (under the name "minuscule heaps") by Stembridge [29]. Let S be the set of nodes of the Dynkin diagram of $\mathfrak{g}$. The minuscule poset P is naturally colored by the function $\kappa : P \to S$ as follows: Let $\mu \in L_\lambda$ be a meet irreducible weight of V_λ. Then μ is covered by exactly one weight $\nu \in L_\lambda$. The difference $\nu - \mu$ is a simple root. The color $\kappa(\mu)$ is the index of this root. The elements of a given color in a minuscule poset form a chain. No element is covered by another element of the same color. The interval between consecutive elements of the same color contains every element covered in P by the maximal element of the interval. If an element $x \in P$ covers an element $y \in P$, then their colors $\kappa(x)$ and $\kappa(y)$ are adjacent. Elements whose colors are adjacent in the Dynkin diagram are comparable.

The Weyl group acts naturally on the lattice L_λ of weights in V_λ. This action can be described combinatorially with the lattice $J(P)$ of order filters in P. We first need some definitions: Fix a filter $J \subseteq P$ and a color i. Since the elements of color i form a chain, there is at most one element $x \in J$ with $\kappa(x) = i$ such that $J \setminus \{x\}$ is a filter. We say that such an element x (which must be minimal in J) and its color i are *removable* from J. Similarly, there is at most one element $y \in P \setminus J$ with $\kappa(y) = i$ such that $J \cup \{y\}$ is a filter. We say that such an the element y (which must be maximal in $P \setminus J$) and its color i are *available* to J. Since no element in a minuscule poset is covered by another element of the same color, no color is both removable from and available to a given filter.

Proposition 8.3. *Let $P \subset L_\lambda$ denote the colored minuscule poset of meet irreducible weights of the minuscule representation V_λ. The action of a simple reflection $s_i \in W$ on the weight in $L_\lambda \cong J(P)$ specified by an order filter $J \subseteq P$ is given by the following:*

$$s_i.J = \begin{cases} J \setminus \{x\} & \textit{if there exists } x \textit{ removable from } J \textit{ with } \kappa(x) = i \\ J \cup \{y\} & \textit{if there exists } y \textit{ available to } J \textit{ with } \kappa(y) = i \\ J & \textit{otherwise.} \end{cases}$$

8.2 Wildberger's construction of minuscule representations

Recall that a weight basis for the minuscule representation V_λ is indexed by the elements of its finite distributive lattice L_λ of weights. This lattice in turn is realized by the lattice $J(P)$ of order filters in the corresponding minuscule poset. Hence each vector in a weight basis of V_λ corresponds to a filter in $J(P)$. Wildberger constructed the minuscule representations for simply laced Lie algebras combinatorially from minuscule posets [30]. We detail his construction below.

Assume that our simple Lie algebra $\mathfrak{g}$ is simply laced. Let S be the set of nodes of the Dynkin diagram of $\mathfrak{g}$, and let $\{h_i\}_{i \in S}$ denote the simple coroot basis of the Cartan subalgebra. For each $i \in S$, choose any positive simple root vector $e_i \in \mathfrak{g}_{\alpha_i}$. Let $f_i \in \mathfrak{g}_{-\alpha_i}$ be the negative simple root vector such that $[e_i, f_i] = h_i$. The collection of root vectors $\{e_i, f_i\}_{i \in S}$ generates $\mathfrak{g}$. Here we realize the minuscule representation on the vector space $V_{J(P)}$, which is spanned by linearly independent vectors $\{\mathcal{J} | \, J \subseteq P \text{ an order filter}\}$: We are labeling a basis vector by writing the name of its order filter in calligraphic font. It is understood that when we invoke a filter operation on a vector $\mathcal{J}$ associated to the order filter J, we are indicating the vector associated to the result of the filter operation on J.

Proposition 8.4. *For each basis vector $\mathcal{J} \in V_{J(P)}$ define the following actions:*

$$e_i.\mathcal{J} := \begin{cases} \mathcal{J} \setminus \{x\} & \textit{if there exists } x \textit{ removable in } J \textit{ with } \kappa(x) = i \\ 0 & \textit{otherwise} \end{cases}$$

$$f_i.\mathcal{J} := \begin{cases} \mathcal{J} \cup \{y\} & \textit{if there exists } y \textit{ available to } J \textit{ with } \kappa(y) = i \\ 0 & \textit{otherwise.} \end{cases}$$

These actions generate an irreducible representation of $\mathfrak{g}$ on $V_{J(P)}$ that is isomorphic to V_λ. Moreover, the vectors h_i act by

$$h_i.\mathcal{J} = \begin{cases} \mathcal{J} & \textit{if color } i \textit{ is available to } J \\ -\mathcal{J} & \textit{if color } i \textit{ is removable in } J \\ 0 & \textit{otherwise.} \end{cases}$$

Hence each vector $\mathcal{J}$ is a weight vector of $V_{J(P)}$.

There are two special order filters in every poset: the empty filter and the full poset. The vectors in $V_{J(P)}$ for these filters are denoted $\aleph$ and $\overline{\aleph}$ respectively. It is easy to see that $\aleph$ is a highest weight vector for $\mathfrak{g}$ with weight λ. The weight of a vector $\mathcal{J}$ is $\lambda - \sum\limits_{x \in J} \alpha_{\kappa(x)}$; this follows from the action of the vectors f_i.

With respect to the simple root vectors chosen above and our fixed minuscule weight λ, Wildberger chose a special basis of non-simple root vectors in $\mathfrak{g}$. The definition of this basis relies on a notion generalizing that of the color of a single element of P. Recall the following property of subposets: A *convex* subset $R \subset P$ is a subset such that $x, y \in R$ and $x \preceq z \preceq y$ implies $z \in R$. Equivalently, R is convex if it can be written as the difference of two filters of P.

Definition 8.5. For a positive root α, an *α-layer* is a convex subset $R \subseteq P$ such that its color census satisfies $\sum_{x \in R} \alpha_{\kappa(x)} = \alpha$.

Consider the following partial order on the set of α-layers: For two α-layers R_1, R_2 we say that $R_1 \preceq R_2$ if R_1 is contained in the filter generated by R_2. Wildberger showed that for each $\alpha \in \Phi^+$ there is a unique α-layer R_α in P that is minimal with respect to this order. Extend the definitions of removable and available elements for a filter to convex sets. Now orient the Dynkin diagram outwards from the node corresponding to the fundamental weight λ. For each edge $x \to y$ of the Hasse diagram of the minuscule poset P, the colors $\kappa(x)$ and $\kappa(y)$ are adjacent in the Dynkin diagram. Orient this edge of the Hasse diagram of P to agree with the orientation of the Dynkin diagram.

For a positive root $\alpha \in \Phi^+$, Wildberger's choice of root vector $e_\alpha \in \mathfrak{g}_\alpha$ acts by

$$e_\alpha.\mathcal{J} := \begin{cases} \epsilon(R)\mathcal{J} \setminus \mathcal{R} & \text{if there exists an } \alpha\text{-layer } R \text{ removable from } J \\ 0 & \text{otherwise,} \end{cases}$$

where $\epsilon(R)$ is the *parity* of the α-layer R as defined in [30]. Wildberger defined the parity in terms of a "heap" defining the α-layer. There the parity $\epsilon(R)$ is defined to be $(-1)^c$ where c counts the number of "X-switches" needed to transform the heap for R into the heap for R_α. In terms of the oriented Hasse diagram, the exponent c counts the edges in R oriented differently than the corresponding edge in R_α. In particular if the simple root expansion of α has coordinates only 0 or 1, then all edges of the minimal layer R_α are oriented downwards. In this case, the exponent c counts the number of upward oriented edges in R. The corresponding root vector $f_\alpha \in \mathfrak{g}_{-\alpha}$ where $[e_\alpha, f_\alpha] = h_\alpha$ acts by

$$f_\alpha.\mathcal{J} := \begin{cases} \epsilon(R)\mathcal{J} \cup \mathcal{R} & \text{if there exists an } \alpha\text{-layer } R \text{ available to } J \\ 0 & \text{otherwise.} \end{cases}$$

The case where the minuscule poset P is the double-tailed diamond $d_r(1)$ (of cardinality $2r-2$) corresponds to the representation of $\mathfrak{o}(V)$ from Section 7.2 on the space V. The double-tailed diamond lattice (of cardinality $2r$) in that section can be viewed as the lattice $J(P)$. We develop new notation for this realization of that important family of model cases. First, label the incomparable pair of elements in $d_r(1)$ as $z^\sharp$ and $z^\flat$. Then from the middle rank outward, label its tail elements $y^{+/-}, x^{+/-}, \ldots, a^{+/-}$ using $+$ for elements on the upper tail and $-$ for elements on the lower tail. Let $Z^{\sharp/\flat}$ denote the principal filters $\langle z^{\sharp/\flat}\rangle$. Let $Y^+, X^+, \ldots, A^+$ denote the $r-2$ principal filters $\langle y^+\rangle, \langle x^+\rangle, \ldots, \langle a^+\rangle$ contained in the upper tail. Let $Y^-, X^-, \ldots, A^-$ denote the $r-2$ filters $\langle y^-\rangle \setminus y^-, \langle x^-\rangle \setminus x^-, \ldots, \langle a^-\rangle \setminus a^-$. The remaining two filters are $\emptyset$ and $P = d_r(1)$ itself. Define a map $\phi : V \to V_{d_r(1)}$ by linearly extending the map $v_{0^{\sharp/\flat}} \mapsto \mathcal{Z}^{\sharp/\flat}$, $v_1 \mapsto \mathcal{Y}^+$, $\ldots$, $v_{r-2} \mapsto \mathcal{A}^+$, $v_{r-1} \mapsto \aleph$; $v_{\overline{1}} \mapsto \mathcal{Y}^-$, $\ldots$, $v_{\overline{r-2}} \mapsto \mathcal{A}^-$, $v_{\overline{r-1}} \mapsto \overline{\aleph}$. The map ϕ is an isomorphism of $\mathfrak{o}(2r)$-modules. Moreover, one can check that the respective actions of the root vectors chosen here and in Section 7.2 agree under this map.

CHAPTER 9

Extreme weight Plücker relations

9.1 Representation theory setting

By a theorem of Kostant, the Plücker relation problem for flag varieties can be translated entirely into the language of representation theory. This is the language we use in this chapter. A guide on how to interface the results of this chapter with the geometry of flag varieties is given in Section 9.7. In particular, there we address the need to use dual modules for the geometric structure of interest.

Fix any simply-laced simple Lie algebra $\mathfrak{g}$ of rank n and minuscule weight λ. Let $P := P_\lambda$ denote the minuscule poset within the corresponding lattice L_λ of weights. Using the filters of P, construct Wildberger's realization $V_{J(P)}$ of the minuscule representation V_λ of $\mathfrak{g}$. Recall that $\aleph$ denotes a highest weight vector of $V_{J(P)}$. The following is the representation theoretic formulation of the Plücker relation problem for the associated minuscule flag manifold:

Problem 9.1. *The $\mathfrak{g}$-module $Sym^2(V_{J(P)})$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 \oplus I$ of $\mathfrak{g}$-modules for a unique submodule I. The Plücker relations for the flag manifold are depicted by the nonzero vectors of I. Find a spanning set (or basis) for I.*

By setting one of these nonzero vectors to 0, we produce a corresponding quadratic relation for the coordinate ring of the flag manifold. So henceforth we refer to a nonzero vector of I itself as a Plücker relation. If the Plücker relation has only one product of incomparable elements of L_λ when it is written in the usual basis for $Sym^2(V_{J(P)})$, then the remaining terms are standard monomials. Therefore such a Plücker relation provides a straightening law for the coordinate ring. The Plücker relations that we find will give straightening laws; this is how we choose to display them (compare to Proposition 7.11).

Before we begin, we use Seshadri's theorem to make some expository comments. (Seshadri's theorem is not used for any of the theorems proved here.) First note that a basis for $Sym^2(V_{J(P)})$ is given by unordered pairs of basis vectors of $V_{J(P)}$. We indexed our basis of $V_{J(P)}$ with the filters

$J(P)$. These filters also model the weights L_λ. Hence $\dim[Sym^2(V_{J(P)})]$ is the number of unordered pairs of elements in L_λ. By Seshadri's theorem (and the geometric content of Section 9.7) the quadratic standard monomials on L_λ, which are products of two comparable elements, form a basis of $Sym^2(V_{J(P)})/I$. Hence $\dim[\mathfrak{U}(\mathfrak{g}).(\aleph)^2]$ is the number of comparable pairs of elements in L_λ, and $\dim(I)$ is the number of incomparable pairs. In particular, the straightening laws for these incomparable pairs form a basis for I. We want to produce the straightening law for as many of the incomparable pairs in L_λ as possible.

It can be seen directly that if the poset P (or equivalently the lattice L_λ) is a chain, then $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 = Sym^2(V_{J(P)})$. Hence in this case $I = 0$ and there are no Plücker relations. So we will assume that P is not a chain.

As in Section 7.2, we use the Casimir operator to study the submodule I. First, we obtain a highest weight vector of I for $\mathfrak{g}$. In the model cases $P = d_n(1)$, this vector alone forms a basis for I. (When $n = 3$ we have $a_3(2) \cong d_3(1)$, and when $n = 4$ we have $d_4(3) \cong d_4(4) \cong d_4(1)$. Hence we also already have a basis for I in these small cases.) The highest weight vector we find will give a straightening law in the corresponding coordinate ring. (So in these cases we also have an explicit presentation of the only straightening law.) Let η denote this highest weight of I. We will see how to use Seshadri's theorem to see that every highest weight of I is dominated by η. Hence we will know that η is in fact the unique maximal weight of I. The weights in the Weyl group orbit of η are also weights of I. We call these *extreme weights*.

Definition 9.2. An *extreme weight Plücker relation* is a nonzero weight vector of I that has weight $w.\eta$ for some $w \in W$.

We will see that there is a unique Plücker relation of weight η, up to scalar multiple. By the W-symmetry of weight space dimensions, we know that up to scalar multiple there is a unique extreme weight Plücker relation for each weight in the W-orbit of η. These extreme weight Plücker relations all belong to the same foremost irreducible $\mathfrak{g}$-submodule of I. We describe each of these $|W.\eta|$ extreme weight vectors for I in terms of the order structure of the elements of its "support" in L_λ. In particular, we will see directly that these extreme weight relations are straightening laws. Moreover, we will show that these straightening laws have the same double-tailed diamond form as the single straightening law for the model variety of isotropic lines that was obtained in Chapter 7.

If I itself happens to also be a minuscule $\mathfrak{g}$-module, then every weight of I is an extreme weight. In this case, the extreme weight Plücker relations form a basis of all of the Plücker relations. Here we are presenting the straightening law for every incomparable pair in L_λ. For $P = e_6(1)$ and $P = e_6(6)$, we will see that I is minuscule. It can be seen that I is also minuscule for $P = a_n(2)$ and $P = a_n(n-1)$. A "quasiminuscule" representation is an irreducible representation in which every nonzero weight lies in the Weyl group orbit of its highest weight. If I is quasiminuscule, then the extreme weight Plücker relations give the straightening law for all but the zero weight incomparable pairs in L_λ. For the remaining exceptional case $P = e_7(7)$, we will see that I is quasiminuscule. The zero weight space of I in this case is seven dimensional. So seven straightening laws are not given by the extreme weight Plücker relations. We compute these seven remaining straightening laws by hand to fill out a basis of Plücker relations. So in these cases we solve Problem 9.1 completely. As mentioned in Section 6.2, we could use these complete lists of straightening laws to prove that standard monomials span the coordinate rings in these cases. A counting argument proves their linear independence. So we can also verify Seshadri's theorem in these cases. We do not obtain a full spanning set for the submodule I for the remaining minuscule posets.

In type A we will find that these extreme weight straightening laws express the product of an extreme incomparable pair as the product of their meet and join minus only one other term. We have found a description for this last term that uses a method of specifying the two needed filters of the minuscule poset $a_n(j)$ with certain Young diagrams. We state this result in Section 9.6 without proof. A proof will be included in a future paper based upon Part II of this dissertation.

9.2 Highest weight relation

Let $P := P_\lambda \subset L_\lambda$ be a simply laced minuscule poset that is not a chain. Let S denote the set of nodes of the Dynkin diagram of $\mathfrak{g}$. Recall that the top tree of P is this Dynkin diagram, so its elements correspond to simple roots of $\mathfrak{g}$. Notice that at the top of P there is a double-tailed diamond subposet, *i.e.* an order filter that is isomorphic to the minuscule poset $d_r(1)$ for some $r \leq n$. Let $D \subseteq S$ denote the intersection of the top tree of P with this top double-tailed diamond. Then D is the Dynkin diagram for a type D_r root system. For the type D model case of $P = d_n(1)$, we have $r = n$ and this top double-tailed diamond is all of P. This case was already handled in Section 7.2. For type A we have $r = 3$, for the type D spin representations we have $r = 4$, for type E_6 we have $r = 5$, and for type E_7 we have $r = 6$. Following the indexing of type D roots from Chapter 7, use $\alpha_\sharp$ and $\alpha_\flat$ to denote the simple roots of Φ which correspond to the colors of the incomparable pair of $D \subset P$. Also use $\alpha_1, \ldots, \alpha_{r-2}$ to denote the simple roots of Φ which correspond to the colors of the tail of $D \subset P$, numbering upward from the incomparable pair. It is straightforward to check that in these cases where $\mathfrak{g}$ is simply laced, the roots $\alpha_\sharp, \alpha_\flat, \alpha_1, \ldots, \alpha_{r-2}$ are distinct. This also follows from the simply-colored property for these posets proved in [26].

Let $\Phi_D \subset \Phi$ be the subset of roots in the span of $\alpha_\sharp, \alpha_\flat, \alpha_1, \ldots, \alpha_{r-2}$. Let $\mathfrak{g}_D \subseteq \mathfrak{g}$ be the subalgebra generated by the root subspaces $\{\mathfrak{g}_\alpha\}_{\alpha \in \Phi_D}$. The subalgebra $\mathfrak{h}_D := \mathfrak{h} \cap \mathfrak{g}_D$ is a Cartan subalgebra of $\mathfrak{g}_D$. We have a map $\mathfrak{h}^* \to \mathfrak{h}_D^*$ that is induced by the inclusion $\mathfrak{h}_D \subseteq \mathfrak{h}$, which is given by restriction of the domain. A weight vector for $\mathfrak{h}$ with weight $\mu \in \mathfrak{h}^*$ is a weight vector for $\mathfrak{h}_D$ with weight $\mu|_{\mathfrak{h}_D}$. In particular, the roots of $\mathfrak{g}_D$ are the restrictions $\Phi_D|_{\mathfrak{h}_D}$. For a root $\alpha \in \Phi_D$, the coroot of $\mathfrak{g}$ is $h_\alpha = [e_\alpha, f_\alpha]$, which lies in $\mathfrak{g}_D$. In fact, we can see that it is the coroot of $\mathfrak{g}_D$ for its root $\alpha|_{\mathfrak{h}_D}$. Let Φ_D^+ be the subset of positive roots of $\mathfrak{g}$ in Φ_D. Then its restriction $\Phi_D^+|_{\mathfrak{h}_D}$ is the subset of positive roots of Φ_D for the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}_D$. Its simple roots are the restrictions $\alpha_\sharp|_{\mathfrak{h}_D}, \alpha_\flat|_{\mathfrak{h}_D}, \alpha_1|_{\mathfrak{h}_D}, \ldots, \alpha_{r-2}|_{\mathfrak{h}_D}$ It is clear that $\mathfrak{g}_D$ is simple of type D_r.

Any $\mathfrak{g}$-module is naturally a $\mathfrak{g}_D$-module, which can be decomposed into $\mathfrak{g}_D$-irreducible components. Let V_D denote the $\mathfrak{g}_D$-irreducible component of $V_{J(P)}$ given by $\mathfrak{U}(\mathfrak{g}_D).\aleph$. It has highest weight $\lambda|_{\mathfrak{h}_D}$. It is clear from comparing the actions given by Wildberger that V_D is isomorphic to $V_{J(d_r(1))}$ as $\mathfrak{g}_D$-modules. Use the notation established in Section 8.2 for $V_{J(d_r(1))}$ here for V_D, but use $\aleph_D^-$ to denote the full double-tailed diamond filter. Problem 9.1 for the $\mathfrak{g}_D$-submodule V_D was solved in Proposition 7.11; there is a single Plücker relation for $\mathfrak{g}_D$ in $Sym^2(V_D)$:

$$\mathcal{Z}^\sharp \mathcal{Z}^\flat = \mathcal{Y}^+\mathcal{Y}^- - \mathcal{X}^+\mathcal{X}^- + \cdots + (-1)^{r-1}\mathcal{A}^+\mathcal{A}^- + (-1)^r \aleph\aleph_D^-.$$

The inclusion $V_D \subseteq V_{J(P)}$ induces a natural inclusion $Sym^2(V_D) \hookrightarrow Sym^2(V_{J(P)})$. Under this inclusion the model relation above is the foremost of the Plücker relations we seek:

Proposition 9.3. *The inclusion of the above Plücker relation for $\mathfrak{g}_D$ under $Sym^2(V_D) \hookrightarrow Sym^2(V_{J(P)})$ is a Plücker relation for $\mathfrak{g}$. Moreover, this relation is a highest weight vector for $\mathfrak{g}$.*

We make some remarks and definitions before presenting the proof. Recall that the universal Casimir element $\Omega \in \mathfrak{U}(\mathfrak{g})$ acts on each irreducible $\mathfrak{g}$-module by a scalar multiplication that depends only on the module's highest weight. On a module of highest weight 2λ this scalar is $\epsilon := (2\lambda + \rho, 2\lambda + \rho) - (\rho, \rho) = 4[(\lambda, \lambda) + (\lambda, \rho)]$ by [24, Section 22.3]. The submodule $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 \subset Sym^2(V_{J(P)})$ is $\mathfrak{g}$-irreducible of highest weight 2λ. So the image of the operator $(\Omega - \epsilon)$ on $Sym^2(V_{J(P)})$ is contained in the submodule I. As in Proposition 7.11, we will compute the image of the vector $\mathcal{Z}^\sharp \mathcal{Z}^\flat \in Sym^2(V_{J(P)})$ to find that the quadratic relation above also holds as a Plücker relation for $\mathfrak{g}$. Note that this relation is again a straightening law and the leading term on the right hand side is again the product of the meet and join of the incomparable pair.

The computation below is performed by decomposing both the Casimir operator Ω and the scalar ϵ into two pieces: one each associated to the roots Φ_D and one each associated to the roots $\Phi \setminus \Phi_D$. Since the root system Φ is simply laced, the standard $\mathfrak{sl}_2$ pairs $\{f_\alpha, e_\alpha\}$ for each $\alpha \in \Phi$ have the pairing $(f_\alpha, e_\alpha) = 1$ in the normalized Killing form for $\mathfrak{g}$. Let $\{h^i\}_{i \in S}$ be the basis of the Cartan subalgebra $\mathfrak{h}$ dual in the normalized Killing form for $\mathfrak{g}$ to the simple coroot basis $\{h_i\}_{i \in S}$. Then as in Section 7.3 the Casimir operator $\Omega \in \mathfrak{U}(\mathfrak{g})$ can be expressed by the following sum: $\sum_{\alpha \in \Phi^+} h_\alpha + \sum_{i \in S} h^i h_i + 2 \sum_{\alpha \in \Phi^+} f_\alpha e_\alpha$. Similarly since $\mathfrak{g}_D$ is simply laced, the standard $\mathfrak{sl}_2$ pairs $\{f_\alpha, e_\alpha\}$ for each $\alpha \in \Phi_D$ have the pairing $(f_\alpha, e_\alpha) = 1$ in the normalized Killing form for $\mathfrak{g}_D$. Let $\{k^j\}_{j \in D}$ be the basis of $\mathfrak{h}_D$ dual in the normalized Killing form for $\mathfrak{g}_D$ to the simple coroot basis $\{h_j\}_{j \in D}$. Then the Casimir operator $\Omega'_D \in \mathfrak{U}(\mathfrak{g}_D)$ can be expressed as the sum $\sum_{\alpha \in \Phi_D^+} h_\alpha + \sum_{j \in D} k^j h_j + 2 \sum_{\alpha \in \Phi_D^+} f_\alpha e_\alpha$. Let $\Omega_D \in \mathfrak{U}(\mathfrak{g})$ be the image of Ω'_D under the inclusion $\mathfrak{U}(\mathfrak{g}_D) \hookrightarrow \mathfrak{U}(\mathfrak{g})$.

Proof. The minuscule weight λ is the fundamental weight in $\mathfrak{h}^*_{\mathbb{R}}$ that corresponds to the simple root $\alpha_{r-2} \in \Phi$. Define the weight $\theta := \frac{1}{2}\alpha_\sharp + \frac{1}{2}\alpha_\flat + \alpha_1 + \cdots + \alpha_{r-2}$ of $\mathfrak{g}$. We can see that $(\theta, \theta) = 1$. Its restriction to $\mathfrak{h}_D$ is $\theta|_{\mathfrak{h}_D} = \frac{1}{2}\alpha_\sharp|_{\mathfrak{h}_D} + \frac{1}{2}\alpha_\flat|_{\mathfrak{h}_D} + \alpha_1|_{\mathfrak{h}_D} + \cdots + \alpha_{r-2}|_{\mathfrak{h}_D}$, which we recognize as the fundamental weight in $\mathfrak{h}^*_D$ that corresponds to $\alpha_{r-2}|_{\mathfrak{h}_D}$. For simple roots $\alpha \in \Phi_D$, we have $\langle \lambda, \alpha \rangle = \lambda(h_\alpha)$. Since λ is fundamental, this evaluates to 1 for $\alpha = \alpha_{r-2}$ and to 0 otherwise. We also have $\langle \theta, \alpha \rangle = \theta(h_\alpha) = \theta|_{\mathfrak{h}_D}(h_\alpha)$, since $h_\alpha \in \mathfrak{h}_D$. Since $\theta|_{\mathfrak{h}_D}$ is fundamental in $\mathfrak{h}^*_D$, this evaluates to 1 for $\alpha = \alpha_{r-2}$ and to 0 otherwise. Since these $\{h_\alpha\}$ span $\mathfrak{h}_D$, we have $\lambda|_{\mathfrak{h}_D} = \theta|_{\mathfrak{h}_D}$. Since Φ is simply laced we have $(\lambda, \alpha) = \langle \lambda, \alpha \rangle = \lambda(h_\alpha)$ for any root $\alpha \in \Phi_D$. Similarly for $\alpha \in \Phi_D$, and since $h_\alpha \in \mathfrak{h}_D$, we have $(\theta, \alpha) = \theta(h_\alpha) = \lambda(h_\alpha) = (\lambda, \alpha)$.

Let η denote the weight of the vector $\mathcal{Z}^\sharp \mathcal{Z}^\flat \in Sym^2(V_{J(P)})$ for $\mathfrak{h}$. From the expression of this weight using the color censuses of the filters $Z^\sharp$ and $Z^\flat$, one obtains $\eta = 2\lambda - 2\theta$. Then the weight of $\mathcal{Z}^\sharp\mathcal{Z}^\flat$ for $\mathfrak{h}_D$ is $\eta|_{\mathfrak{h}_D} = 2\lambda|_{\mathfrak{h}_\mathfrak{D}} - 2\theta|_{\mathfrak{h}_D} = 0$. Since Φ is simply laced, we have $(\eta, \alpha) = \langle \eta, \alpha \rangle$ for any root $\alpha \in \Phi$. For any root $\alpha \in \Phi_D$, we have $\langle \eta, \alpha \rangle = 2[\langle \lambda, \alpha \rangle - \langle \theta, \alpha \rangle] = 0$. Then by linearity, we have $(\eta, \theta) = \frac{1}{2}(\eta, \alpha_\sharp) + \frac{1}{2}(\eta, \alpha_\flat) + (\eta, \alpha_1) + \cdots + (\eta, \alpha_{r-2}) = 0 + \cdots + 0 = 0$.

Define the weight $\rho_D = \frac{1}{2}\sum_{\alpha \in \Phi_D^+} \alpha$ of $\mathfrak{g}$. Then by linearity, we have that $(\lambda, \rho_D) = (\theta, \rho_D)$. Note that $\rho_D|_{\mathfrak{h}_D} = \frac{1}{2}\sum_{\alpha \in \Phi_D^+} \alpha|_{\mathfrak{h}_D}$ is the Weyl vector of $\mathfrak{h}_D^*$. Then for simple roots $\alpha \in \Phi_D$, we have $\langle \rho_D, \alpha \rangle = \rho_D(h_\alpha) = \rho_D|_{\mathfrak{h}_D}(h_\alpha) = 1$. For the Weyl vector ρ of $\mathfrak{h}^*$, we also have $\langle \rho, \alpha \rangle = 1$ for all simple roots. Since Φ is simply laced we have for simple $\alpha \in \Phi_D$ that $([\rho - \rho_D], \alpha) = (\rho, \alpha) - (\rho_D, \alpha) = \langle \rho, \alpha \rangle - \langle \rho_D, \alpha \rangle = 0$. Then by linearity we have $([\rho - \rho_D], \theta) = \frac{1}{2}([\rho - \rho_D], \alpha_\sharp) + \frac{1}{2}([\rho - \rho_D], \alpha_\flat) + ([\rho - \rho_D], \alpha_1) + \cdots + ([\rho - \rho_D], \alpha_{r-2}) = 0 + \cdots + 0 = 0$. Solving for λ in $\eta = 2\lambda - 2\theta$ we obtain $\lambda = \theta + \frac{1}{2}\eta$. Again by linearity, we have $(\lambda, \rho - \rho_D) = \frac{1}{2}(\eta, \rho - \rho_D)$. By the root definition of θ and since Φ is simply laced, we see that $(\rho, \theta) = r - 1$. Hence we also have that $(\rho_D, \theta) = r - 1$.

We are prepared to decompose the scalar ϵ and the Casimir operator:

$$\begin{aligned}
\epsilon &= 4[(\lambda, \lambda) + (\lambda, \rho)] \\
&= 4[(\theta + \frac{1}{2}\eta, \theta + \frac{1}{2}\eta) + (\lambda, \rho - \rho_D + \rho_D)] \\
&= 4[(\theta, \theta) + \frac{1}{4}(\eta, \eta) + (\lambda, \rho_D) + (\lambda, \rho - \rho_D)] \\
&= 4[(\theta, \theta) + (\theta, \rho_D)] + (\eta, \eta) + 2(\eta, \rho - \rho_D) \\
&= 4r + (\eta, \eta) + 2(\eta, \rho - \rho_D).
\end{aligned}$$

The summations which appear first in their brackets below combine to form a copy of Ω_D:

$$\begin{aligned}
\Omega &= \sum_{\alpha \in \Phi^+} h_\alpha + \sum_{i \in S} h^i h_i + 2 \sum_{\alpha \in \Phi^+} f_\alpha e_\alpha \\
&= \left(\sum_{\alpha \in \Phi_D^+} h_\alpha + \sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} h_\alpha \right) + \left(\sum_{j \in D} k^j h_j - \sum_{j \in D} k^j h_j + \sum_{i \in S} h^i h_i \right) \\
&\qquad + 2 \left(\sum_{\alpha \in \Phi_D^+} f_\alpha e_\alpha + \sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} f_\alpha e_\alpha \right) \\
&= \Omega_D + \sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} h_\alpha + \sum_{i \in S} h^i h_i - \sum_{j \in D} k^j h_j + 2 \sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} f_\alpha e_\alpha.
\end{aligned}$$

Now we want to show that $(\Omega - \epsilon).(\mathcal{Z}^\sharp \mathcal{Z}^\flat) = (\Omega_D - 4r).\mathcal{Z}^\sharp \mathcal{Z}^\flat$: The term $\sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} h_\alpha$ of Ω acts on the η-weight space as multiplication by the scalar $\sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} \langle \eta, \alpha \rangle$. Since Φ is simply laced, this scalar is $\left(\eta, \sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} \alpha\right) = (\eta, 2[\rho - \rho_D])$. This is canceled by the last part of our decomposition of the scalar ϵ. As we saw in the proof of Proposition 7.11, the part $\sum_{i \in S} h^i h_i$ of Ω acts on the η-weight space for $\mathfrak{h}$ as multiplication by the scalar (η, η). This is canceled by the second part of our decomposition of the scalar ϵ. Similarly, we have that $\sum_{j \in D} k^j h_j$ acts on an $\eta|_{\mathfrak{h}_D}$-weight space for $\mathfrak{h}_D$ as multiplication by the scalar $(\eta|_{\mathfrak{h}_D}, \eta|_{\mathfrak{h}_D})_D$, where this bilinear form $(\cdot, \cdot)_D$ is the form on $\mathfrak{h}_D^*$ induced by the normalized Killing form on $\mathfrak{g}_D$. Since $\eta|_{\mathfrak{h}_D} = 0$, this scalar is 0. For the final part, note that each term in the expansion of $\sum_{\alpha \in \Phi^+ \setminus \Phi_D^+} f_\alpha e_\alpha.(\mathcal{Z}^\sharp \mathcal{Z}^\flat)$ involves acting on one of the vectors $\mathcal{Z}^\sharp$ or $\mathcal{Z}^\flat$ with a raising operator e_α. It is easy to see that for $\alpha \in \Phi^+ \setminus \Phi_D^+$ all of these terms are zero. Therefore $(\Omega - \epsilon).(\mathcal{Z}^\sharp \mathcal{Z}^\flat) = (\Omega_D - 4r).(\mathcal{Z}^\sharp \mathcal{Z}^\flat)$. Since Ω_D is the image of the Casimir operator Ω'_D, the calculation of $(\Omega_D - 4r).(\mathcal{Z}^\sharp \mathcal{Z}^\flat)$ was carried out in Proposition 7.11. We can use that result to obtain the claimed Plücker relation.

The raising operators e_α for roots $\alpha \in \Phi^+$ act on our basis of $V_{J(P)}$ by removing α-layers from the corresponding filters. The filters involved in our Plücker relation are contained in the top double-tailed diamond, and their elements are colored from the subset $D \subseteq S$. It is clear that for $\alpha \in \Phi^+ \setminus \Phi_D^+$, the operator e_α annihilates the relation. The relation is also a highest weight vector of $\mathfrak{g}_D$ since it is a basis for the Plücker submodule of $Sym^2(V_D)$. Hence $(\Omega_D - 4r).(\mathcal{Z}^\sharp \mathcal{Z}^\flat)$ is a highest weight vector of $\mathfrak{g}$. □

A quick examination of the minuscule posets reveals that every vector of $Sym^2(V_{J(P)})$ of the same weight η as $\mathcal{Z}^\sharp \mathcal{Z}^\flat$ lies in the image of $Sym^2(V_D)$. In fact, it can be seen that the Plücker relation we obtained is the unique Plücker relation of weight η (up to a scalar multiple). We now indicate how to use Seshadri's theorem to see that *every* weight of the Plücker submodule I is dominated by η. This will confirm that the Plücker relation above is truly the most prominent relation. To see this, recall that Seshadri's theorem implies that the straightening laws give a basis of I. It is easy to see that a straightening law on L_λ is given by a weight vector of I for $\mathfrak{g}$. Each straightening law involves one incomparable pair of filters in P. The sum of their corresponding weights is the weight of the straightening law. It is also easy to see that for any incomparable pair of filters in P, one filter must contain $Z^\sharp$ and the other must contain $Z^\flat$. Hence the weight of the straightening law is dominated by η, which is the sum of the weights corresponding to the pair $Z^\sharp$ and $Z^\flat$.

9.3 Rotation by Weyl group

When combined with the action of the Weyl group, the technique used in Proposition 9.3 to find a highest weight Plücker relation can be used to produce more Plücker relations. Recall that η denotes the weight of the vector $\mathcal{Z}^\sharp\mathcal{Z}^\flat \in Sym^2(V_{J(P)})$ for Proposition 9.3; it is a highest weight for the Plücker submodule I. Recall that an extreme weight Plücker relation is a nonzero vector in $Sym^2(V_{J(P)})$ of weight $w.\eta$ for some $w \in W$. Here we describe these $|W.\eta|$ extreme weight Plücker relations.

Definition 9.4. Let P be any poset. A *double-tailed diamond subposet* of P is a subset $P_D \subseteq P$ with the following properties:

- There are exactly two incomparable elements of P_D,
- half of the other elements of P_D form a chain that lies above the incomparable pair, and
- the remaining elements form a chain that lies below the incomparable pair.

Let L be any lattice. A *double-tailed diamond sublattice* of L is a double-tailed diamond subposet $L_D \subseteq L$ with the following properties:

- The join of its incomparable pair in L is the minimal element in the upper chain of L_D, and
- the meet of its incomparable pair in L is the maximal element in the lower chain of L_D.

Suppose that L generates a ring over $\mathbb{C}$. A double-tailed diamond sublattice $L_D \subseteq L$ is order isomorphic to one of the model lattices studied in Chapter 7. In this ring, we call a relation of the alternating sum form obtained in Proposition 7.11 the *standard straightening law* on L_D.

The following theorem is the foremost result of Part II of this dissertation:

Theorem 9.5. *Let λ be a minuscule weight of a simple Lie algebra $\mathfrak{g}$. Each extreme weight Plücker relation is the standard straightening law on a double-tailed diamond sublattice of L_λ.*

So an extreme weight Plücker relation is a straightening law for an "extreme" incomparable pair of L_λ, *i.e.* one whose weights sum to an extreme weight. The standard monomial expression for such a pair begins with the product of their meet and join, and continues with an alternating sum of standard monomials in a double-tailed diamond. These double-tailed diamonds will all be the same size as the one found in Section 9.2. Hence there will be r terms in the straightening law (including the incomparable pair), where r is the rank of the subalgebra $\mathfrak{g}_D$ from the previous section. In addition, Corollary 9.10 below indicates that the difference between the filters for two adjacent elements of this sublattice will be a root layer. For now we continue to assume that $\mathfrak{g}$ is simply laced. Here this theorem is obtained by combining Propositions 9.7, 9.9, 9.12, and 9.14 below. It will be extended to non-simply laced algebras in Section 9.4.

Let $W_\eta \subseteq W$ be the parabolic subgroup which stabilizes the weight η. Each coset in W/W_η is known to have a shortest length representative. Let W^η be the set of such representatives. The set of extreme weights is in bijection with W^η. Since W^η is defined in terms of coset representatives, it is a subset of the Weyl group. We will apply elements of W^η in settings where η is not a highest weight. We need some facts about W^η; they follow from [17, Theorem 2.5.5] and [31, Proposition 3]:

Lemma 9.6. *Let η be a dominant integral weight and let $w \in W^\eta$.*

1. *Let ℓ denote the length of w. There exists a reduced decomposition $w = s_{i_\ell} \dots s_{i_2} s_{i_1}$ such that for each $1 \leq k \leq \ell$ the element $w_k := s_{i_k} \dots s_{i_2} s_{i_1}$ of W belongs to W^η.*

2. *In such a reduced decomposition, we have $w.\eta \prec w_{\ell-1}.\eta$ and $\langle w.\eta, \alpha_{i_\ell} \rangle < 0$.*

3. *Let $s_i \in W$ be a simple reflection. If $s_i w.\eta \prec w.\eta$, then $s_i w \in W^\eta$.*

Fix an element $w \in W^\eta$. We now "rotate" the entire setup for Proposition 9.3 using w (which is actually a rotation/reflection of $\mathfrak{h}^*_{\mathbb{R}}$). Recall that the setup began by defining the root subsystem $\Phi_D \subseteq \Phi$. Now analogously create the root subsystem $w.\Phi_D$. All of the "sub-D" objects are analogously created by first applying w to the corresponding roots: Let $\mathfrak{g}_{D,w} \subseteq \mathfrak{g}$ be the subalgebra generated by the root subspaces $\{\mathfrak{g}_\alpha\}_{\alpha \in w.\Phi_D}$. Then $\mathfrak{g}_{D,w}$ is simple of type D_r, its coroots are $\{h_{w.\alpha} = [e_{w.\alpha}, f_{w.\alpha}]\}_{\alpha \in w.\Phi}$, and so on. Again, any $\mathfrak{g}$-module is naturally a $\mathfrak{g}_{D,w}$-module which can be decomposed into $\mathfrak{g}_{D,w}$-irreducible components. Let $V_{D,w}$ denote the $\mathfrak{g}_{D,w}$-irreducible component of $V_{J(P)}$ given by $\mathfrak{U}(\mathfrak{g}_{D,w}).(w.\aleph)$. Again we have that $V_{D,w}$ is isomorphic as a $\mathfrak{g}_{D,w}$-module to $V_{J(d_r(1))}$. However, our basis vectors for $V_{D,w}$ are not exactly the same as Wildberger's preferred basis vectors for $V_{J(d_r(1))}$. The preferred basis of $V_{D,w}$ which is analogous to Wildberger's basis for $V_{J(d_r(1))}$ is generated from a highest weight vector (say $w.\aleph$) with the nonnegative actions of the simple root vectors of $\mathfrak{g}_{D,w}$, which are not necessarily simple root vectors of $\mathfrak{g}$. We will later see that some of our basis vectors are preferred basis vectors, while the rest are the negatives of the preferred basis vectors. It is nonetheless straightforward to apply the techniques of Proposition 9.3 to $V_{D,w} \subseteq V_{J(P)}$, even though the signs in our expression for the resulting relation could potentially change:

Proposition 9.7. *Let $w \in W^\eta$. The inclusion of the Plücker relation for $\mathfrak{g}_{D,w}$ under $Sym^2(V_{D,w}) \hookrightarrow Sym^2(V_{J(P)})$ is an extreme weight Plücker relation for $\mathfrak{g}$ of weight $w.\eta$.*

Proof. The Casimir operator can be similarly decomposed using the choice of positive roots $w.\Phi^+$, and the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ is invariant under the action of w. Hence the proof of Proposition 9.3 also applies here, by replacing $\mathfrak{g}_D$ and V_D with $\mathfrak{g}_{D,w}$ and $V_{D,w}$. □

From this proposition we obtain an extreme weight relation which is a signed sum of r products of pairs of filters as in Proposition 9.3. We will later see that this signed sum is in fact still an alternating sum. We want to understand the order structure of these filters to prove that this relation produces a standard straightening law on a double-tailed diamond sublattice of L_λ. Let L_D denote the lattice of weights of V_D. The lattice L_D is a sublattice of L_λ. The corresponding extreme Plücker relation at $w = id$ of Proposition 9.3 was then the standard straightening law on this sublattice. Let $L_{D,w}$ denote the poset of weights of $V_{D,w}$; again we have $L_{D,w} \subseteq L_\lambda$. For an arbitrary $w \in W^\eta$ there is cause for concern that the rotation by w changes the order structure of $L_{D,w}$. Fortunately, it can be shown that the order structure of $L_{D,w}$ in the rotated setup is also a double-tailed diamond sublattice of L_λ. To begin our analysis of $L_{D,w}$, we need:

Lemma 9.8. *Let J, K be filters of the minuscule poset P in the subposet $L_D \subseteq L_\lambda \cong J(P)$ such that $\mathcal{J}\mathcal{K}$ has weight η. Let $w = s_{i_\ell} \ldots s_{i_2} s_{i_1} \in W^\eta$ be a reduced decomposition as in Lemma 9.6. Then s_{i_ℓ} either adds an element to each of the filters $w_{\ell-1}.J$ and $w_{\ell-1}.K$, or it adds an element to one and stabilizes the other.*

Proof. Let μ denote the weight of $\mathcal{J}$ and let ν denote the weight of $\mathcal{K}$, so that $\mu + \nu = \eta$. By Lemma 9.6 we have $w.\eta \prec w_{\ell-1}.\eta$. Recall from Proposition 8.3 that the simple reflection $s_{i_\ell} \in W$ acts on a filter of P by adding or removing a single element or doing nothing. All other possible actions of s_{i_ℓ} on these filters result in a contradiction: Suppose s_{i_ℓ} removes an element from one of the filters, say $w_{\ell-1}.J$. Then $w.\mu = s_{i_\ell} w_{\ell-1}.\mu = w_{\ell-1}.\mu + \alpha_{i_\ell}$. We would have $w.(\mu+\nu) \prec w_{\ell-1}.(\nu+\mu) = w_{\ell-1}.\nu + (w.\mu - \alpha_{i_\ell})$. By cancelling $w.\mu$ we obtain $w.\nu \prec w_{\ell-1}.\nu - \alpha_{i_\ell}$. This is impossible, since by the three possible actions of s_{i_ℓ} on $w.K$ we have $w_{\ell-1}.\nu \preceq w.\nu + \alpha_{i_\ell}$. On the other hand if s_{i_ℓ} stabilized both filters, we would have the contradiction $w.\eta = w_{\ell-1}.\eta$. $\square$

We now show that $L_{D,w}$ is a double-tailed diamond subposet:

Proposition 9.9. *Let $\mu, \nu \in L_D$, and let $w \in W^\eta$. Then $w.\mu \preceq w.\nu$ if and only if $\mu \preceq w$. Hence $L_{D,w} = w.L_D$ is order isomorphic to L_D and so is a double-tailed diamond subposet of L_λ.*

Therefore the Plücker relation obtained in Proposition 9.7 is again a straightening law on L_λ. This proposition comes with a caveat: Rotation by the Weyl group does not preserve covering relations in L_λ.

Proof. We first show that if $\mu \preceq \nu$, then we have $w.\mu \preceq w.\nu$. Since the weights of V_D form an interval of L_λ, we may reduce to the case where ν covers μ. The difference $\nu - \mu$ is a simple root $\alpha \in \Phi_D^+$. Since Φ is W-invariant, the difference $w.\nu - w.\mu = w.\alpha$ is a root. We must show that the root $w.\alpha$ lies in Φ^+. Let $s_{i_\ell} \dots s_{i_2} s_{i_1}$ be a reduced decomposition for $w \in W^\eta$ as in Lemma 9.6. We use induction on the length ℓ of w. The base case of $\ell = 0$ where $w = id$ is trivial. Note that $w_{\ell-1} \in W^\eta$ has length $\ell - 1$. By Lemma 9.6 we have $\langle w.\eta, \alpha_{i_\ell} \rangle < 0$, so that $\langle w_{\ell-1}.\eta, \alpha_{i_\ell} \rangle = \langle s_{i_\ell} w.\eta, \alpha_{i_\ell} \rangle = \langle w.\eta, -\alpha_{i_\ell} \rangle > 0$. Note that $\langle w_{\ell-1}.\eta, w_{\ell-1}.\alpha \rangle = \langle \eta, \alpha \rangle = 0$, since $\eta|_{\mathfrak{h}_D}$ is zero. Hence $w_{\ell-1}.\alpha \neq \alpha_{i_\ell}$. Recall that the simple reflection s_{i_ℓ} permutes the roots $\Phi^+ \setminus \{\alpha_{i_\ell}\}$. By induction we have $w_{\ell-1}.\alpha \in \Phi^+$. Hence $s_{i_\ell}(w_{\ell-1}).\alpha = w.\alpha \in \Phi^+$.

For the converse, suppose that μ and ν are incomparable. The only incomparable pair of weights μ and ν in V_D are the weights in L_λ that correspond to the filters $Z^\sharp$ and $Z^\flat$ of the minuscule poset P. Recall that $\mu + \nu = \eta$. We once again use induction on the length ℓ of w by taking a reduced decomposition $s_{i_\ell} \dots s_{i_1}$ for $w \in W^\eta$ as in Lemma 9.6. The base case of $\ell = 0$ where $w = id$ is again trivial. Again we have $w_{\ell-1} \in W^\eta$ has length $\ell - 1$. Our inductive hypothesis is that the filters $w_{\ell-1}.Z^\sharp$ and $w_{\ell-1}.Z^\flat$ are incomparable, so there is at least one element $x \in P$ in $w_{\ell-1}.Z^\sharp$ but not in $w_{\ell-1}.Z^\flat$. Suppose x is available to $w_{\ell-1}.Z^\flat$ and has color i_ℓ. Suppose further that x is the only element of $w_{\ell-1}.Z^\sharp \setminus w_{\ell-1}.Z^\flat$. Then x must be minimal in $w_{\ell-1}.Z^\sharp$. Hence the reflection s_{i_ℓ} removes x from $w_{\ell-1}.Z^\sharp$; this contradicts the result of Lemma 9.8 applied to the filters $Z^\sharp, Z^\flat$

and our decomposition of $w \in W^\eta$. If instead there is another element $y \in w_{\ell-1}.Z^\sharp \setminus w_{\ell-1}.Z^\flat$, then we will still have $y \in w.Z^\sharp$ while $y \notin w.Z^\flat$. Hence in this case we have $w.Z^\flat \not\preceq w.Z^\sharp$. Otherwise, we have that either x is not available to $w_{\ell-1}.Z^\flat$ or its color is not i_ℓ. In this case we will still have $x \in w.Z^\sharp$ and $x \notin w.Z^\flat$. Hence in all cases we have $w.Z^\flat \not\preceq w.Z^\sharp$. By symmetry we also have $w.Z^\sharp \not\preceq w.Z^\flat$. Therefore $w.\mu$ and $w.\nu$ are incomparable weights. □

As a corollary to this proof, we can describe the covering relations in $L_{D,w}$ using the language of filters:

Corollary 9.10. *Let $w \in W^\eta$. Let J, K be filters of the minuscule poset P such that J covers K in the subposet $L_{D,w} \subseteq L_\lambda \cong J(P)$. Then the subset $K - J$ is an α-layer for the following root $\alpha \in w.\Phi_D^+ \subseteq \Phi^+$: If one of J or K is one of the incomparable pair of elements of $L_{D,w}$, we have $\alpha = w.\alpha_\sharp$ or $\alpha = w.\alpha_\flat$. Otherwise, moving outward along the tails from the incomparable pair we have $\alpha = w.\alpha_1, \ldots, \alpha = w.\alpha_{r-2}$.*

Proof. As a difference of filters, the subset $K - J$ of P is a convex subset. The weights in $L_{D,w}$ that correspond to J and K differ by $\sum_{p \in K-J} \kappa(p)$, the sum of simple roots coloring the elements of $K - J$. From the previous proof, these weights also differ by $w.\nu - w.\mu$ where ν covers μ in L_D. In particular, $\nu - \mu = \alpha_i$ for the claimed index $i \in D$. Therefore we have the color census $\sum_{p \in K-J} \kappa(p) = w.\alpha_i$. □

In order to prove that the straightening law given by Proposition 9.7 is the standard straightening law on $L_{D,w}$, we need more information about these root layers:

Lemma 9.11. *Let $w \in W^\eta$. Let J_1, J_2 be filters of the minuscule poset P such that J_1 covers J_2 in the subposet $L_{D,w} \subseteq L_\lambda \cong J(P)$. Let K_1, K_2 be the filters in $L_{D,w}$ such that the vectors $\mathcal{J}_1\mathcal{K}_1$ and $\mathcal{J}_2\mathcal{K}_2$ have weight $w.\eta$. Then K_2 covers K_1, and the two root layers $J_2 - J_1$ and $K_1 - K_2$ are isomorphic as colored posets.*

Proof. Proposition 9.9 implies that K_2 covers K_1 in $L_{D,w}$. We prove that the root layers $J_2 - J_1$ and $K_1 - K_2$ are isomorphic by induction on the length of w. When the length of w is zero and $w = id$, each of the root layers is a single element of the same color from D.

Let $w = s_{i_\ell} \ldots s_{i_2} s_{i_1}$ be a reduced decomposition for w as in Lemma 9.6. By the proof of Proposition 9.9, the filters $A_1 := s_{i_\ell} J_1$, $A_2 := s_{i_\ell} J_2$, $B_1 := s_{i_\ell} K_1$, and $B_2 := s_{i_\ell} K_2$ satisfy our hypotheses for $w_{\ell-1} \in W^\eta$ of length $\ell - 1$. By induction, we have that the root layers $A_2 - A_1$ and $B_1 - B_2$ are isomorphic as colored posets. Denote this isomorphism $\phi : A_2 - A_1 \to B_1 - B_2$. Begin to construct a map $\phi' : J_2 - J_1 \to K_1 - K_2$ by defining $\phi'(p) = \phi(p)$ for all $p \in (J_2 - J_1) \cap (A_2 - A_1)$. We will finish constructing ϕ' and see that it is a colored poset isomorphism by showing that $A_2 - A_1$ deforms into $J_2 - J_1$ in the same fashion that $B_1 - B_2$ deforms into $K_1 - K_2$. Both $\mathcal{J}_1\mathcal{K}_1$ and $\mathcal{J}_2\mathcal{K}_2$ have weight $w.\eta$, and both $\mathcal{A}_1\mathcal{B}_1$ and $\mathcal{A}_2\mathcal{B}_2$ have weight $w_{\ell-1}.\eta$. So we have the cardinality fact $|(J_1 - A_1) \cup (K_1 - B_1)| = |(J_2 - A_2) \cup (K_2 - B_2)|$. By Lemma 9.8, the only possible change from each of the filters A_1, A_2, B_1, B_2 to its corresponding filter J_1, J_2, K_1, K_2 is the addition of a single element of color i_ℓ. Such are the only elements that will appear or disappear from the set differences $A_2 - A_1$ and $B_1 - B_2$ to the set differences $J_2 - J_1$ and $K_1 - K_2$.

Suppose there was an element $x \in A_2 - A_1$ that is no longer in $J_2 - J_1$. By Lemma 9.8, we must have that $x \in J_1 - A_1$. Since x was i_ℓ-available to A_1, it was maximal in $A_2 - A_1$. Since ϕ is an isomorphism, the corresponding element $\phi(x) \in B_1 - B_2$ was also maximal of color i_ℓ. So $\phi(x)$ was i_ℓ-available to B_2 and hence is no longer in $K_1 - K_2$. A similar argument shows that if there exists an element $x \in B_1 - B_2$ that is no longer in $K_1 - K_2$, then $\phi^{-1}(x) \in A_2 - A_1$ is no longer in $J_2 - J_1$. Either way an element in the domain of ϕ is no longer in the domain for ϕ', and its image under ϕ is no longer in the codomain for ϕ'.

Suppose there exists an element $y \in J_2 - J_1$ that was not in $A_2 - A_1$. By Lemma 9.8, we must have that $y \in J_2 - A_2$. We claim that $K_1 - B_1$ is nonempty and that its single element is in $K_1 - K_2$. First suppose that $J_1 - A_1$ is nonempty. Since $J_1 \subset J_2$, this set $J_1 - A_1$ consists of an element $x \in J_2$. The only element of $J_2 - A_2$ is y, which by hypothesis is not in J_1 and hence not in $J_1 - A_1$. So the element x is actually in A_2. In this case, the element x was in $A_2 - A_1$ but is no longer in $J_2 - J_1$. We have seen above that $\phi(x) \in B_1 - B_2$ is no longer in $K_1 - K_2$, and hence it is in $K_2 - B_2$. Here both $J_2 - A_2$ and $K_2 - B_2$ are nonempty. By the cardinality fact above, we must have that $K_1 - B_1$ is nonempty as well. In this case, $K_2 = B_2 \cup \{\phi(x)\}$ where $\phi(x) \in B_1$. We can conclude that $K_1 - B_1$ is contained in $K_1 - K_2$. Otherwise $J_1 - A_1$ is empty. But $J_2 - A_2$ is nonempty. By the cardinality fact above, we must also have that $K_1 - B_1$ is nonempty and $K_2 - B_2$ is empty. Then in this case too, we have that $K_1 - B_1$ is nonempty and contained in $K_1 - K_2$, proving the claim.

We continue to construct ϕ' by defining $\phi'(y)$ to be the single element of $K_1 - B_1$. It remains show that $\phi' : J_2 - J_1 \to K_1 - K_2$ preserves the poset structure. The element y is minimal in $J_2 - J_1$, and $\phi'(y)$ is minimal in $K_1 - K_2$. We now must show that ϕ' maps the elements that cover y in $J_2 - J_1$ to the elements that cover $\phi'(y)$ in $K_1 - K_2$. Since $y \notin J_1$ was not also i_ℓ-available to A_1, there was at least one element $z_1 \in A_2 - A_1$ that covers y in P. In the layer $A_2 - A_1$, there was no element of color i_ℓ that was less than the element z_1, since such an element would be comparable to y. Since ϕ was an isomorphism, there was no element of color i_ℓ in the layer $B_1 - B_2$ that was less than the element $\phi'(z_1) = \phi(z_1)$. Recall that every element covered by z_1 (including y) must lie in the interval between z_1 and the next smallest element of color $\kappa(z_1)$ in P, if such an element exists. Hence z_1 was the minimal element of color $\kappa(z_1)$ in $A_2 - A_1$. Since ϕ was an isomorphism, the element $\phi'(z_1)$ was the minimal element of color $\kappa(\phi'(z_1))$ in the layer $B_1 - B_2$. Since z_1 covers y in P, the color $\kappa(z_1)$ is adjacent in the Dynkin diagram to the color $\kappa(y) = i_\ell$. Recall that elements of a minuscule poset with adjacent colors are comparable. Since $\kappa(\phi'(z_1)) = \kappa(z_1)$ and $\kappa(\phi'(y)) = i_\ell$, the element $\phi'(y)$ is comparable to $\phi'(z_1)$. And since $\phi'(y)$ was not in the filter B_1, we have $\phi'(y) \prec \phi'(z_1)$. To see that $\phi'(z_1)$ must cover $\phi'(y)$: Recall that all covering relations in P involve two elements of adjacent colors. Since the Dynkin diagram is acyclic, there is a covering relation somewhere in the interval between $\phi'(y)$ and $\phi'(z_1)$ involving elements of the colors i_ℓ and $\kappa(\phi'(z_1))$. But there was no element of color i_ℓ in the layer $B_1 - B_2$ that was less than $\phi'(z_1)$, and $\phi'(z_1)$ was the minimal element of the layer $B_1 - B_2$ of color $\kappa(\phi'(z_1))$. So $\phi'(z_1)$ must now cover $\phi'(y)$ in $K_1 - K_2$. By the properties of minuscule posets, there was at most one other element $z_2 \in A_2 - A_1$ that covers y in P. If such an element exists, the arguments above can also be applied to z_2 and $\phi'(z_2)$. Conversely if there exists another element $z' \in B_1 - B_2$ which covers $\phi'(y)$, then an analogous argument shows that $\phi^{-1}(z')$ covers y in $J_2 - J_1$.

Finally, suppose there exists an element $y' \in K_1 - K_2$ that was not in $B_1 - B_2$. By arguments similar to those above, it can be seen that there exists an element y of color i_ℓ in $J_2 - J_1$ that was not in $A_2 - A_1$. After defining $\phi'(y) = y'$, the argument above that showed that ϕ' maps the elements that cover y in $J_2 - J_1$ to the elements that cover y' in $K_1 - K_2$ can be repeated. $\square$

We can now show that the extreme weight Plücker relations give standard straightening laws:

Proposition 9.12. *Let $w \in W^\eta$. The extreme weight Plücker relation of weight $w.\eta$ produced by Proposition 9.7 gives the standard straightening law on $L_{D,w}$.*

Proof. The preferred basis of $V_{D,w}$ is generated from $w.\aleph$ by the actions of simple root vectors of $\mathfrak{g}_{D,w}$. The simple roots for $\mathfrak{g}_{D,w}$ are the restrictions of the roots $\{w.\alpha_i\}_{i \in D} \subseteq \Phi$ to its Cartan subalgebra $\mathfrak{h}_{D,w}$. By Proposition 9.9 these roots $\{w.\alpha_i\}_{i \in D}$ are positive roots. Then we can take the nonsimple root vectors $\{e_{w.\alpha_i}, f_{w.\alpha_i}\}_{i \in D}$ for $\mathfrak{g}$ as the simple root vectors for $\mathfrak{g}_{D,w}$. These vectors transition within our $L_{D,w}$ basis with a sign given by the parity of corresponding root layers in the minuscule poset P. So our $L_{D,w}$ basis vectors for $V_{D,w}$ are each either a preferred basis vector, or the negative of the preferred basis vector of its weight. The Plücker relation is an alternating sum in the preferred basis as in Proposition 9.3. Hence this alternating sum in the preferred basis is a signed sum when expressed in our $L_{D,w}$ basis.

First suppose that $w.\mathcal{Z}^\sharp$ and $w.\mathcal{Z}^\flat$ are either both preferred basis vectors, or both are negatives of preferred basis vectors. We claim that any two of our $L_{D,w}$ basis vectors of $V_{D,w}$ that are paired in the signed sum are either both preferred basis vectors of $V_{D,w}$, or both are negatives of preferred basis vectors. Then the signed sum in our $L_{D,w}$ basis is an alternating sum, giving the standard straightening law on $L_{D,w}$. To see this, we work outward from the middle: By the definition of the preferred basis, the vectors $e_{w.\alpha_\sharp}(w.\mathcal{Z}^\sharp)$ and $f_{w.\alpha_\sharp}(w.\mathcal{Z}^\flat)$ obtained by the action of simple root vectors for $\mathfrak{g}_{D,w}$ are either both preferred basis vectors or both negatives of preferred basis vectors. These simple root vectors for $\mathfrak{g}_{D,w}$ are nothing more than root vectors for $\mathfrak{g}$ whose actions on our L_λ basis were defined in Section 8.2. By Corollary 9.10 we have that $w.Z^\sharp - w.Y^+$ and $w.Y^- - w.Z^\flat$ are $\alpha_\sharp$-layers. Then we have that $e_{w.\alpha_\sharp}(w.\mathcal{Z}^\sharp) = \epsilon(w.Z^\sharp - w.Y^+)w.\mathcal{Y}^+$ and $f_{w.\alpha_\sharp}(w.\mathcal{Z}^\flat) = \epsilon(w.Y^- - Z^\flat)w.\mathcal{Y}^-$. By Lemma 9.11, the parities $\epsilon(w.Z^\sharp - w.Y^+)$ and $\epsilon(w.Y^- - w.Z^\flat)$ are equal. Hence $w.\mathcal{Y}^+$ and $w.\mathcal{Y}^-$ are either both the preferred basis vectors $e_{w.\alpha_\sharp}(w.\mathcal{Z}^\sharp)$ and $f_{w.\alpha_\sharp}(w.\mathcal{Z}^\flat)$ or both negatives of these preferred basis vectors. By the same reasoning applied to $e_{w.\alpha_1}(w.\mathcal{Y}^+) = \epsilon(w.Y^+ - w.X^+)w.\mathcal{X}^+$ and $f_{w.\alpha_1}(w.\mathcal{Y}^-) = \epsilon(w.X^- - w.Y^-)w.\mathcal{X}^-$, we have that $w.\mathcal{X}^+$ and $w.\mathcal{X}^-$ are either both preferred vectors or both negatives of preferred vectors. Iterate this process outward along the tails to finish the proof of the claim.

Otherwise, exactly one of $w.\mathcal{Z}^\sharp$ and $w.\mathcal{Z}^\flat$ is a preferred basis vector. The argument above can be applied to show that given any two of our $L_{D,w}$ basis vectors of $V_{D,w}$ that are paired in the signed sum, exactly one is a preferred basis vector. In this case, we again have that the signed sum in our $L_{D,w}$ basis is an alternating sum giving the standard straightening law on $L_{D,w}$. $\square$

We have demonstrated that $L_{D,w}$ is a double-tailed diamond subposet of L_λ. The next proposition states that it is a double-tailed diamond sublattice as well. We first need:

Lemma 9.13. *Let J and K be filters of a minuscule poset P, and let $s_i \in W$ be a simple reflection such that $s_i.J \neq J$ or $s_i.K \neq K$. Suppose that there is no element in P of this color i that is both removable from J and available to K or vice versa. Then we have $s_i.(J \vee K) = s_i.J \vee s_i.K$ and $s_i.(J \wedge K) = s_i.J \wedge s_i.K$.*

Proof. There are three possibilities for the action of s_i on J and three possibilities for its action on K. By symmetry, we only need to consider five of the eight cases that satisfy our hypotheses. We may assume that $s_i.J \neq J$. For three of these cases, first suppose that the reflection s_i adds an available element x to the filter J. We consider the possibilities for the action of s_i on K:

Suppose that s_i also adds an available element y to the filter K. If $x = y$, then s_i adds x to the four filters $J, K, J \vee K, J \wedge K$. Here $s_i.(J \vee K) = s_i.J \vee s_i.K$ and similarly for $\wedge$. Otherwise since $\kappa(x) = \kappa(y)$, the elements x and y are comparable. By symmetry, we may assume $y \prec x$. We have $x \in K$, while $y \notin J$ and hence $y \notin J \vee K$. Then x is i-available to the intersection filter $J \vee K$ and y is i-available to the union filter $J \wedge K$. Here $s_i.(J \vee K) = (J \vee [K \cup \{y\}]) \cup \{x\} = (J \cup \{x\}) \vee (K \cup \{y\}) = s_i.J \vee s_i.K$ and $s_i.(J \wedge K) = ([J \cup \{x\}] \wedge K) \cup \{y\} = (J \cup \{x\}) \wedge (K \cup \{y\})$.

Suppose that $s_i.K = K$. Consider the case that $x \in K$. Then the intersection $s_i.J \vee s_i.K = (J \vee K) \cup \{x\}$. Hence x is i-available to $J \vee K$, and we have $s_i.(J \vee K) = (J \vee K) \cup \{x\} = s_i.J \vee s_i.K$. We have the union $s_iJ \wedge s_iK = J \wedge K$. We would like to show that no element y of color i is available to the union filter $J \wedge K$. Indeed, such an element would satisfy $y \preceq x$. Since $x \in J \wedge K$ already, this order is strict. But an element of a minuscule poset cannot be covered by another element of the same color. Hence x is strictly greater than the elements which cover y. These covering elements do not belong to J, and therefore belong to K. We conclude that the element y is i-available to K, contradicting the assumption $s_i.K = K$. There is also no element y of color i which is removable from the union filter $J \wedge K$, since y would be removable from J or K. Therefore $s_i.(J \wedge K) = J \wedge K = s_i.J \wedge s_i.K$. If instead $x \notin K$, switch the arguments for $J \vee K$ and $J \wedge K$ by dualizing $\vee, \wedge$, "intersection", and "union" above, as well as the availability/removability of the hypothetical element y and its order with respect to x.

Suppose that s_i removes an element y from the filter K. By hypothesis $y \neq x$. The elements x and

y are comparable. If $x \prec y$, then $x \notin K$ and $y \in J$. We have that x is i-available to the union $J \wedge K$ and that y is i-removable from the intersection $J \vee K$. Then we have $s_i.(J \vee K) = (J \vee K) \setminus \{y\} = J \vee (K \setminus \{y\}) = s_i.J \vee s_i.K$ and $s_i.(J \wedge K) = (J \wedge K) \cup \{x\} = (J \cup \{x\}) \wedge K = s_i.J \wedge s_i.K$. Otherwise $y \prec x$. Hence $x \in K$, and $y \notin J$. It is clear that x is i-available to the intersection $J \vee K$ and that y is i-removable from the union $J \wedge K$. Then we have $s_i.(J \vee K) = (J \vee K) \cup \{x\} = (J \cup \{x\}) \vee K = s_i.J \vee s_i.K$ and $s_i.(J \wedge K) = (J \wedge K) \setminus \{y\} = J \wedge (K \setminus \{y\}) = s_i.J \wedge s_i.K$.

For the last two cases, suppose that s_i instead removes an element x from J. Consider the order dual P' of P. The poset P' is again a colored minuscule poset. The filters J and K of P become order ideals of P'. Let $\overline{J}, \overline{K}$ denote their complementary filters of P'. We have that s_i adds the element x to $\overline{J}$. Since the first two cases above did not use the simultaneous removable/available hypothesis, those arguments apply to $\overline{J}$ and $\overline{K}$. Therefore $s_i.(\overline{J} \wedge \overline{K}) = s_i.\overline{J} \wedge s_i.\overline{K}$ and $s_i.(\overline{J} \vee \overline{K}) = s_i.\overline{J} \vee s_i.\overline{K}$. This in turn implies $s_i.(J \vee K) = s_i.J \vee s_i.K$ and $s_i.(J \wedge K) = s_i.J \wedge s_i.K$. □

Proposition 9.14. *Let $w \in W^\eta$, and let $Z^\sharp$ and $Z^\flat$ be the incomparable filters defined for Proposition 9.3. Then $w.Z^\sharp \vee w.Z^\flat = w.(Z^\sharp \vee Z^\flat)$ and $w.Z^\sharp \wedge w.Z^\flat = w.(Z^\sharp \wedge Z^\flat)$. Hence the double-tailed diamond subposet $L_{D,w} \subseteq L_\lambda$ is a double-tailed diamond sublattice.*

Proof. Let $w = s_{i_\ell} \dots s_{i_2} s_{i_1}$ be a reduced decomposition for $w \in W^\eta$ as in Lemma 9.6. We use induction on the length ℓ of w. The base case of $\ell = 0$ where $w = id$ is trivial. We have that $w_{\ell-1} \in W^\eta$ has length $\ell - 1$ and $w.\eta \prec w_{\ell-1}.\eta$. By Lemma 9.8, the simple reflection s_{i_ℓ} either adds an element to each of the filters $w_{\ell-1}.Z^\sharp$ and $w_{\ell-1}.Z^\flat$, or it adds an element to one and stabilizes the other. Hence no element is both removable from $w_{\ell-1}.Z^\sharp$ and available to $w_{\ell-1}.Z^\flat$ or vice versa. Using Lemma 9.13 followed by the inductive hypothesis, we have $s_{i_\ell}.(w_{\ell-1}.Z^\sharp) \vee s_{i_\ell}.(w_{\ell-1}.Z^\flat) = s_{i_\ell}.(w_{\ell-1}.Z^\sharp \vee w_{\ell-1}.Z^\flat) = s_{i_\ell} w_{\ell-1}.(Z^\sharp \vee Z^\flat)$ and similarly $w.Z^\sharp \wedge w.Z^\flat = w.(Z^\sharp \wedge Z^\flat)$. □

The proof of Theorem 9.5 is now complete for simply laced algebras $\mathfrak{g}$. We have a collection of $|W^\eta|$ straightening laws. We can now show that there are no other incomparable pairs of filters that have the layer property of Corollary 9.10; this characterizes the incomparable pairs of elements of L_λ for which we have obtained a straightening law in terms of filters:

Proposition 9.15. *Let J, K be incomparable filters of a minuscule poset P. If there exist roots $\alpha, \beta \in \Phi^+$ such that the subset $J - (J \vee K)$ is an α-layer and $K - (J \vee K)$ is a β-layer, then there* *exists a $w \in W^\eta$ such that the vector $\mathcal{J}\mathcal{K}$ has weight $w.\eta$ and the relation obtained by Proposition 9.7 with this w gives the straightening law for J and K.*

Proof. Let μ denote the weight of the vector $\mathcal{J}\mathcal{K} \in Sym^2(V_{J(P)})$. Suppose there exist roots $\alpha, \beta \in \Phi^+$ as in the proposition statement. As mentioned at the conclusion of Section 9.2 it is easy to see that for incomparable pair of filters J, K in P, one filter must contain $Z^\sharp$ and the other must contain $Z^\flat$. Hence $\mu \preceq \eta$. We proceed by induction on the weight μ, increasing in the order $\preceq$ to the base case weight η. The base case is only attained by the incomparable pair $\{J, K\} = \{Z^\sharp, Z^\flat\}$. Here the conclusion is true with $w = id$.

Now we further assume that α and β are simple roots, say $\alpha = \alpha_j$ and $\beta = \alpha_k$. Then $J - (J \vee K)$ is a single element x of color j and $K - (J \vee K)$ is a single element y of color k. Choose a minimal element p in the intersection filter $J \vee K$, and let i denote the color $\kappa(p)$. Of the elements in J or K, only x and y can be covered by p. We proceed based on the order relationship between p and the elements x and y. If p covers neither x nor y, then p is minimal in both filters J and K. The simple reflection $s_i \in W$ will remove p from both filters and their intersection. Then we have $\mu \prec s_i.\mu$. We also have $s_i.J - (s_i.J \vee s_i.K) = J - (J \vee K)$ is an α_j-layer and $s_i.K - (s_i.J \vee s_i.K) = K - (J \vee K)$ is a α_k-layer. Then by induction there exists an element $w' \in W^\eta$ such that $s_i.\mathcal{J} s_i.\mathcal{K}$ has weight $w'.\eta$. This vector $s_i.\mathcal{J} s_i.\mathcal{K}$ has weight $s_i.\mu$, and so $s_i.\mu = w'.\eta$. Therefore $s_i w'.\eta = s_i(s_i.\mu) = \mu \prec s_i.\mu = w'.\eta$. Hence by Lemma 9.6 we have $s_i w' \in W^\eta$, and so we may take $w := s_i w'$: Here $w.\eta = \mu$ is the weight of $s_i.(s_i.\mathcal{J} s_i.\mathcal{K}) = \mathcal{J}\mathcal{K}$. Next suppose p covers only x. No element of a minuscule poset can be covered by an element of the same color, so the colors i and j are distinct. The reflection s_i will remove p from the filter K. The element p is not removable from J since it covers x. Suppose for the sake of contradiction that an element z of color i were available to J. Then we have $z \prec p$, and x is the only element between them. But by a result of Proctor in [26], between any two elements of the same color in a minuscule poset there are at least two other elements. Therefore s_i indeed leaves J unaffected. We again have that $\mu \prec s_i.\mu$. We also have that $s_i.K - (s_i.J \vee s_i.K) = K - (J \vee K)$ is a α_k-layer. By Lemma 9.13 we have $s_i.J - (s_i.J \vee s_i.K) = s_i.J - s_i.(J \vee K)$. By taking color censuses and using the linearity of the simple reflections on $\mathfrak{h}^*_{\mathbb{R}}$, we obtain $s_i.(\lambda - \sum_{z \in J} \alpha_{\kappa(z)}) - s_i.(\lambda - \sum_{z \in J \vee K} \alpha_{\kappa(z)}) = s_i.(\sum_{z \in J - J \vee K} \alpha_{\kappa(z)}) = s_i.\alpha_j$. Hence $s_i.J - s_i.(J \vee K)$ is an $s_i.\alpha_j$-layer. By induction as above there exists an element

$w' \in W^\eta$ such that $\mathcal{JK}$ has weight $w.\eta$ where $w := s_i w' \in W^\eta$. By symmetry, the case where p covers only y also holds. Suppose on the other hand that all minimal elements $p \in J \vee K$ cover both x and y. By another result in [26], no two elements of a minuscule poset can cover each of two other elements such as x and y. We conclude that there is a unique minimal $p \in J \vee K$. By inspection of the minuscule posets, we see that this is the base case $\{J, K\} = \{Z^\sharp, Z^\flat\}$ above.

Otherwise, at least one of the roots α or β is not simple. Suppose that α is not a simple root. Choose a minimal element x of the α-layer $J - (J \vee K)$, and let i denote the color $\kappa(x)$. Act on both filters by s_i; we have $s_i.J = J \setminus \{x\}$ while $s_i.K$ is unknown. We claim that x is not available to K. Since α is not simple, we have that x is not the only element in $J - (J \vee K)$. It is known that the simple root expansion of any root (such as α) has connected support in the Dynkin diagram. Hence $J - (J \vee K)$ has at least one element z of a color j adjacent to i in the Dynkin diagram. Recall that color adjacent elements such as z and x are comparable. Since x is minimal, we must have $x \prec z$. Then x is not maximal in $P - K$, and so is not available to K. By Lemma 9.13 we have $s_i.J - (s_i.J \vee s_i.K) = s_i.J - s_i.(J \vee K)$. By taking color censuses we obtain $s_i.(\lambda - \sum_{z \in J} \alpha_{\kappa(z)}) - s_i.(\lambda - \sum_{z \in J \vee K} \alpha_{\kappa(z)}) = s_i.(\sum_{z \in J - J \vee K} \alpha_{\kappa(z)}) = s_i.\alpha$. Hence $s_i.J - s_i.(J \vee K)$ is an $s_i.\alpha$-layer. The root β has no α_i component, since $x \in J - (J \vee K)$ has color i and elements of color i are comparable. Then similarly we have $s_i.K - (s_i.J \vee s_i.K) = s_i.K - s_i.(J \vee K))$ is a $s_i.\beta$-layer. Suppose further that $\mu \prec s_i.\mu$. Then by induction as above there exists an element $w' \in W^\eta$ such that $\mathcal{JK}$ has weight $w.\eta$ where $w := s_i w' \in W^\eta$. Otherwise, we have $\mu = s_i.\mu$ and the reflection s_i added an i-available element $y \neq x$ to the filter K. The elements of color i form a chain and we have that $y \not\preceq x$. Hence we have that $x \prec y$ and so $y \in J$. Then we have that $s_i.J - (s_i.J \vee s_i.K) \subsetneq J - (J \vee K)$ and that $s_i.K - (s_i.J \vee s_i.K) = K - (J \vee K)$. We can continue this process to reduce to the case where $|J - (J \vee K)| = 1$, *i.e.* where α is simple. By symmetry we can also reduce to the case where β is simple, where we have already proved the existence of the desired $w \in W^\eta$. □

9.4 Exceptional cases

There are only three minuscule weights for the exceptional root systems. In this section we provide all of the straightening laws for the coordinate rings of the corresponding minuscule flag manifolds.

Let us determine how close Theorem 9.5 is to solving Problem 9.1 for the three exceptional minuscule weights. We give the dimensions of the Plücker submodule $I \subset Sym^2(V_{J(P)})$ in these cases. The dimensions of irreducible modules cited in this section can be found in [32].

Lemma 9.16. *For both minuscule weights in type E_6, the dimension of the Plücker submodule $I \subset Sym^2(V_{J(P)})$ is 27. For the minuscule weight in type E_7, the dimension of I is 133.*

Proof. For the minuscule posets $P := e_6(1)$ and $P := e_6(6)$, we have that $\dim(V_{J(P)}) = 27$ and hence that $\dim(Sym^2(V_{J(P)})) = \binom{28}{2} = 378$. The submodule $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 \subseteq Sym^2(V_{J(P)})$ is irreducible with highest weight 2λ; it has dimension 351. Hence, the complementary submodule $I \subset Sym^2(V_{J(P)})$ has dimension 27. For the poset $P := e_7(7)$, we have that $\dim(V_{J(P)}) = 56$ and hence that $\dim(Sym^2(V_{J(P)})) = \binom{57}{2} = 1596$. The submodule $\mathfrak{U}(\mathfrak{g}).(\aleph)^2 \subseteq Sym^2(V_{J(P)})$ has dimension 1463. Hence, the submodule I has dimension 133. $\square$

In the E_6 cases, the weight η of the Plücker relation from Proposition 9.3 is the minuscule weight opposite from the fixed minuscule λ. In the E_7 case, the weight η is the adjoint weight which is quasiminuscule. The irreducible modules with these highest weights have dimensions 27 and 133 respectively. So by Lemma 9.16, the Plücker submodule I is irreducible and either minuscule or quasiminuscule in these three cases. Theorem 9.5 describes the extreme weight vectors of I. Since we now know the isomorphism class of I in the exceptional cases, we can deduce:

Theorem 9.17. *For the two type E_6 cases, the Plücker relations described by Theorem 9.5 form a basis of Plücker relations. For the type E_7 case, they combine with the seven relations of zero weight in Figure 9.20 to form a basis of Plücker relations.*

Proof. In the E_6 cases, the 27 dimensional minuscule $\mathfrak{g}$-module generated from the weight η Plücker relation forms all of I. Since every weight of a minuscule representation is an extreme weight, the relations from Theorem 9.5 form a basis. In the E_7 case, the 133 dimensional adjoint $\mathfrak{g}$-module generated from the weight η Plücker relation forms all of I. In the adjoint representation, there are 126 extreme weights. Only the 7-dimensional zero weight space does not have extreme weight. It is not difficult to generate a spanning set for this zero weight space by hand: For each of the seven simple roots α_i, use Theorem 9.5 to write down a Plücker relation whose weight is α_i. Then, act on each of these with the corresponding negative simple root vector f_{α_i}. One can check that the resulting seven vectors are linearly independent. We then formed linear combinations of these vectors to obtain the relations presented in Figure 9.20. □

As mentioned in Section 9.1, using Seshadri's theorem we can see for all minuscule cases that the straightening laws for the coordinate ring give a basis for I. In fact, for the E_6 cases we can also see that this is true using our results. There are exactly 27 incomparable pairs of elements in L_λ and we have obtained all of their straightening laws. We list these 27 straightening laws in Figure 9.4. Similarly in the type E_7 case, there are exactly 133 incomparable pairs of elements in L_λ. Theorem 9.5 described the straightening laws for 126 of these. There are 7 incomparable pairs remaining; by process of elimination each of these has total weight zero. We list straightening laws for them in Figure 9.20. Unlike our extreme weight relations, these seven straightening laws were not obtained immediately. Instead, we generated enough weight zero Plücker relations (which were not straightening laws) so that we could solve for the straightening laws with Gaussian elimination. (The ability to do so was guaranteed in advance by Seshadri's theorem.) One nice feature of these weight zero straightening laws is that they are integral. However none of these seven straightening laws are supported on double-tailed diamond subposets of L_λ. In summary, we have obtained for all exceptional cases the straightening law for every incomparable pair of elements in L_λ. Moreover, our bases for I in these cases show that the standard monomials are linearly independent. So we have verified Seshadri's theorem in these cases.

Before we present the straightening laws, we establish some notation. First, we label the Dynkin diagram of E_6 with letters $\{a, b, c, d, e, o\}$ and the diagram of E_7 with letters $\{a, b, c, d, e, f, o\}$. This leads to the coloring of the minuscule posets shown in Figure 9.18. Recall that elements of a minuscule poset with a given color form a chain. We will name an element with its color and a subscript that indicates its position in this chain, counting from the top. We name a filter by the capitalized string of its minimal elements. For example, in both posets, the filter A_2 is the top double-tailed diamond. In $e_7(7)$, the filter A_2E_2 also includes the elements f_1 and e_2. We keep our usual convention of naming a basis vector of Wildberger's $\mathfrak{g}$-module $V_{J(P)}$ by writing the corresponding filter in calligraphic font. Since we are using strings of letters to name filters, we place a dot between two vectors of $V_{J(P)}$ to indicate their product in $Sym^2(V_{J(P)})$.

Since we are listing only the zero weight relations in the E_7 case, every filter will appear paired

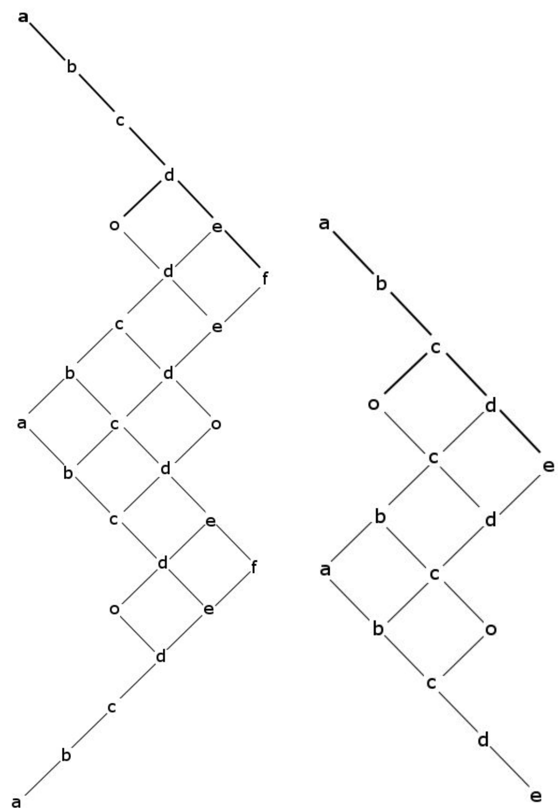


Figure 9.18: The colored Hasse diagrams for $e_7(7)$ and $e_6(1) \cong e_6(6)$.

with its "complementary" filter corresponding to the negative of its weight. So we use our usual naming convention for a filter that is contained in the top "half" of the poset $e_7(7)$, that is to say in its top nine ranks. For each filter which is not contained in the top half of $e_7(7)$, we take advantage of the symmetry of the poset. Flip the Hasse diagram upside-down. The set of elements which formed our filter now form an order ideal. The complement of this ideal is an order filter which is contained in the top half of the Hasse diagram. Name the original filter by placing a bar over the name of this filter. For example the principal filter genereated by f_2 is named $\overline{A_2}$. This way, the weight of the vector $\mathcal{A}_2 \cdot \overline{\mathcal{A}_2}$ is zero. There are 28 zero weight pairs of filters. Seven of these are the incomparable pairs, while the other 21 are the standard monomials. To display the relations in matrix form, we must fix some total ordering of the monomials. We use the following arbitrary order for the incomparable pairs of monomials:

$$\mathcal{A}_2 \cdot \mathcal{F}_2, \quad \mathcal{E}_3 \cdot \mathcal{A}_2\mathcal{F}_1, \quad \mathcal{D}_4 \cdot \mathcal{A}_2\mathcal{E}_2, \quad \mathcal{A}_2\mathcal{D}_3 \cdot \mathcal{C}_3\mathcal{O}_2, \quad \mathcal{B}_2\mathcal{O}_2 \cdot \mathcal{A}_2\mathcal{C}_3, \quad \mathcal{B}_3 \cdot \mathcal{O}_2, \quad \mathcal{C}_3 \cdot \mathcal{A}_2\mathcal{O}_2$$

We order the standard monomials by the following reverse lexicographic order of a certain total order extension on P:

$$\mathcal{B}_2\mathcal{D}_3 \cdot \overline{\mathcal{B}_2\mathcal{D}_3},\ \mathcal{B}_2\mathcal{E}_2 \cdot \overline{\mathcal{B}_2\mathcal{E}_2},\ \mathcal{B}_2\mathcal{F}_1 \cdot \overline{\mathcal{B}_2\mathcal{F}_1},\ \mathcal{B}_2 \cdot \overline{\mathcal{B}_2},\ \mathcal{D}_3 \cdot \overline{\mathcal{D}_3},\ \mathcal{C}_2\mathcal{E}_2 \cdot \overline{\mathcal{C}_2\mathcal{E}_2},\ \mathcal{C}_2\mathcal{F}_1 \cdot \overline{\mathcal{C}_2\mathcal{F}_1},$$
$$\mathcal{C}_2 \cdot \overline{\mathcal{C}_2},\ \mathcal{E}_2 \cdot \overline{\mathcal{E}_2},\ \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1},\ \mathcal{D}_2 \cdot \overline{\mathcal{D}_2},\ \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1},\ \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1},\ \mathcal{O}_1 \cdot \overline{\mathcal{O}_1},$$
$$\mathcal{F}_1 \cdot \overline{\mathcal{F}_1},\ \mathcal{E}_1 \cdot \overline{\mathcal{E}_1},\ \mathcal{D}_1 \cdot \overline{\mathcal{D}_1},\ \mathcal{C}_1 \cdot \overline{\mathcal{C}_1},\ \mathcal{B}_1 \cdot \overline{\mathcal{B}_1},\ \mathcal{A}_1 \cdot \overline{\mathcal{A}_1},\ \aleph \cdot \overline{\aleph}$$

After presenting the 7 zero weight straightening laws in Figure 9.20, we display in Figure 9.21 a matrix which lists the 21 standard monomial cooordinates for each of the 7 products of incomparable pairs, with respect to the two total orders above. For legibility, negative coordinates are presented with bars.

9.5 Non-simply laced cases

Beginning with Section 9.2, we assumed that our algebra $\mathfrak{g}$ was simply laced. However, there are minuscule weights in the non-simply laced type B and C root systems. There is a single minuscule weight for each type C system. There a dimension calculation shows that the corresponding Plücker module $I = 0$, and so there are no Plücker relations for its flag variety. There is also a single minuscule weight for each type B system. For this case we deduce results about its Plücker relations from our results for a simply-laced type D case through the strategy of "diagram folding." This uses an embedding of a type B_{n-1} Lie algebra into one of type D_n. This strategy has been used for example in [33, 34].

We describe this embedding concretely. Fix $n \geq 3$. Let V be the $2n$-dimensional complex vector space with nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined in Section 7.1, and consider the orthogonal Lie algebra $\mathfrak{g} := \mathfrak{o}(V)$ which is simple of type D_n. Let $L \subset V$ be the line spanned by the vector $v_{0^\sharp} - v_{0^\flat}$ and let $\mathfrak{g}' \subset \mathfrak{g}$ be the subalgebra of endomorphisms that annihilate L. Then $\mathfrak{g}'$ stabilizes the orthogonal complement $W := L^\perp$; it is the hyperplane spanned by the vectors $\{v_n, \ldots, v_1, v_{0^\sharp} + v_{0^\flat}, v_{\overline{1}}, \ldots, v_{\overline{n}}\}$. It is simple to check that the restriction $\langle \cdot, \cdot \rangle_W$ of the bilinear form to W is again nondegenerate. Every endomorphism of W that fixes $\langle \cdot, \cdot \rangle_W$ is faithfully represented in the restriction of the endomorphisms in $\mathfrak{g}'$ to W. Hence $\mathfrak{g}'$ is isomorphic to an orthogonal Lie algebra for a $2n-1$ dimensional vector space with nondegenerate symmetric bilinear form. Therefore the subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is simple of type B_{n-1}, and so we rename it $\mathfrak{g}_B$. Label the type B_{n-1} and type D_n Dynkin diagrams as in Figure 9.22. The subalgebra $\mathfrak{h}_B := \mathfrak{h} \cap \mathfrak{g}_B$ is a Cartan subalgebra for $\mathfrak{g}_B$. We have a map $\mathfrak{h}^* \to \mathfrak{h}_B^*$ induced by the inclusion $\mathfrak{h}_B \subset \mathfrak{h}$ given by restriction of the domain.

In particular, the minuscule weight ω_0 of $\mathfrak{h}_B^*$ is the restriction $\omega_\sharp|_{\mathfrak{h}_B}$ (or $\omega_\flat|_{\mathfrak{h}_B}$) of a spin minuscule weight of $\mathfrak{h}^*$.

Let λ be one of the two spin weights $\omega_\sharp$ or $\omega_\flat$ of $\mathfrak{g}$. Let $P := P_\lambda$ be the corresponding minuscule poset, and construct Wildberger's representation $V_{J(P)}$ of $\mathfrak{g}$. Define the unique $\mathfrak{g}$-submodule $I \subset Sym^2(V_{J(P)})$ as for Problem 9.1. The subalgebra $\mathfrak{g}_B \subset \mathfrak{g}$ acts naturally on $V_{J(P)}$. Then the $\mathfrak{g}_B$ submodule $V_B := \mathfrak{U}(\mathfrak{g}_B).\aleph$ of $V_{J(P)}$ is an irreducible representation of $\mathfrak{g}_B$ whose highest weight $\lambda_B := \omega_0$ is its unique minuscule weight. Now one can pose Problem 9.1 for the $\mathfrak{g}_B$-module V_B: The $\mathfrak{g}_B$-module $Sym^2(V_B)$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}_B).(\aleph)^2 \oplus I_B$ of $\mathfrak{g}_B$-submodules for a unique submodule I_B. Find a spanning set (or basis) for I_B.

Proposition 9.23. *The $\mathfrak{g}_B$-modules V_B and $V_{J(P)}$ are equal. Moreover, the subspaces I and I_B of $Sym^2(V_{J(P)}) = Sym^2(V_B)$ are equal.*

Proof. We have that $V_B \subseteq V_{J(P)}$. The first assertion follows from the standard dimension fact $\dim(V_B) = 2^{n-1} = \dim(V_{J(P)})$. We have $\mathfrak{U}(\mathfrak{g}_B).(\aleph)^2 \subseteq \mathfrak{U}(\mathfrak{g}).(\aleph)^2$. Since $\mathfrak{g}$-modules are naturally $\mathfrak{g}_B$-modules, the $\mathfrak{g}$-module decomposition $Sym^2(V_{J(P)}) = \mathfrak{U}(\mathfrak{g}).(\aleph)^2 \oplus I$ is also a $\mathfrak{g}_B$-module decomposition. Since the complementary $\mathfrak{g}_B$-module I_B and $\mathfrak{g}$-module I are each unique, we have $I \subseteq I_B$.

We now show that $\dim(\mathfrak{U}(\mathfrak{g}_B).(\aleph)^2) = \dim(\mathfrak{U}(\mathfrak{g}).(\aleph)^2)$. We use the usual notation for the root system of $\mathfrak{g}$. Let $\Phi_B \subset \mathfrak{h}_B^*$ denote the root system of $\mathfrak{g}_B$. Let $\rho_B \in \mathfrak{h}_B^*$ denote the Weyl vector for the root system Φ_B. By the Weyl dimension formula, we have that $\dim(V_B) = \prod_{\alpha \in \Phi_B^+} \frac{\langle 2\lambda_B + \rho_B, \alpha\rangle}{\langle \rho_B, \alpha\rangle}$ and $\dim(V_{J(P)}) = \prod_{\alpha\in\Phi^+} \frac{\langle 2\lambda + \rho, \alpha\rangle}{\langle \rho, \alpha\rangle}$. We take $\lambda = \omega_\sharp$. For roots $\alpha \in \Phi_B$ (resp. $\alpha \in \Phi$) with no α_0 (resp. $\alpha_\sharp$) component, we have that $\langle 2\lambda_B + \rho_B, \alpha\rangle = \langle \rho_B, \alpha\rangle$ (resp. without the subscript B) and so the corresponding factor in the Weyl dimension formula above is 1. The remaining roots in Φ_B are the short roots $\{\alpha_0 + \alpha_1 + \cdots + \alpha_i\}$ for $0 \leq i \leq n-2$ and the long roots $\{2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j\}$ for $0 \leq i < j \leq n-2$. The remaining roots in Φ are the roots $\{\alpha_\sharp + \alpha_1 + \cdots + \alpha_i\}$ for $0 \leq i \leq n-2$ and $\{\alpha_\sharp + \alpha_\flat + 2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j\}$ for $0 \leq i < j \leq n-2$. These two sets of roots match up in the obvious fashion from Φ to Φ_B when restricting the domain to $\mathfrak{h}_B$. By considering short and long roots of Φ_B separately, it is easy to see that for matching roots $\alpha \in \Phi_B$ and $\beta \in \Phi$ as above we have $\langle \rho_B, \alpha\rangle = \langle \rho, \beta\rangle$. For both the short and long roots $\alpha \in \Phi_B$ above, we have that $\langle 2\lambda_B, \alpha\rangle = 2$. Similarly for the roots $\beta \in \Phi$ above, we have that $\langle 2\lambda, \beta \rangle = 2$. Hence, all of the corresponding factors in the Weyl dimension formulae are equal.

From $\dim(\mathfrak{U}(\mathfrak{g}_B).(\aleph)^2) = \dim(\mathfrak{U}(\mathfrak{g}).(\aleph)^2)$, it follows that $\dim(I_B) = \dim(I)$. Therefore $I_B = I$ as vector spaces. $\square$

This proposition allows us to obtain some Plücker relations for $\mathfrak{g}_B$ by first applying Section 9.3 with the simply laced algebra $\mathfrak{g}$ to obtain extreme weight relations in I, and then recognizing those as Plücker relations in I_B. From this we obtain the final ingredient for Theorem 9.5:

Corollary 9.24. *Each extreme weight Plücker relation for $\mathfrak{g}_B$ is the standard straightening law on a double-tailed diamond sublattice of L_{λ_B}. Moreover, Corollary 9.10 also holds here.*

Proof. We claim that the lattices L_λ and L_{λ_B} are equal. We use $\preceq$ to denote the usual order on $\mathfrak{h}^*_{\mathbb{R}}$ and $\preceq_B$ to denote the usual order on $(\mathfrak{h}_B)^*_{\mathbb{R}}$. The weights of L_{λ_B} are merely the weights of L_λ when restricted to $\mathfrak{h}_B$. We claim that $\mu \preceq \nu$ in L_λ if and only if $\mu|_{\mathfrak{g}_B} \preceq_B \nu|_{\mathfrak{g}_B}$ in L_{λ_B}. Indeed, the restriction of the simple roots of Φ to $\mathfrak{g}_B$ are again simple roots of Φ_B: $\alpha_\sharp \mapsto \alpha_0, \alpha_\flat \mapsto \alpha_0, \alpha_1 \mapsto \alpha_1, \ldots, \alpha_{n-2} \mapsto \alpha_{n-2}$. It is then straightforward to see that any positive root of Φ^+ restricts to a positive root of Φ^+_B, and that every root of Φ^+_B is the restriction of some root of Φ^+. Since the orders $\preceq, \preceq_B$ are defined in terms of positive roots, we have $L_\lambda \cong L_{\lambda_B}$ as lattices.

Then the Plücker relations for $\mathfrak{g}_B$ obtained by Proposition 9.23 retain in L_{λ_B} whatever order structure they may have in L_λ. In particular, each extreme weight Plücker relation is again the standard straightening law on a double-tailed diamond sublattice of L_{λ_B}. $\square$

Note that since the restriction of weights gives $L_\lambda \cong L_{\lambda_B}$ as lattices it also gives an order isomorphism of the uncolored minuscule posets of meet irreducible weights $P_\lambda \cong P_{\lambda_B}$. However the coloring functions κ and κ_B on P_λ and P_{λ_B} are not the same: elements $x \in P_\lambda$ with either $\kappa(x) = \sharp$ or $\kappa(x) = \flat$ now have the B-color $\kappa_B(x) = 0$. The coloring κ_B of P_{λ_B} was not used in the proof of the proposition or its corollary.

9.6 Extreme relations in type A

This section is an informal discussion of the extreme relations in type A without formal proofs. We relate the combinatorial notions for minuscule posets in this case to established combinatorial notions for Young diagrams. Interestingly, these combinatorial notions of "content" and "rim hooks" usually arise from the representation theory of the symmetric group. Use the labelling of type A_{n-1} roots and weights from Section 5.2. Fix a minuscule weight $\lambda = \omega_j$.

Rotate the Hasse diagram for the poset $P := a_{n-1}(j)$ as pictured in Figure 8.2 clockwise by 45° to obtain a rectangular array of dots with j rows and $n-j$ columns. A rotated filter of the poset is a left justified subarray of dots such that the number of dots in each of its rows is weakly decreasing. Fix a filter $J \subset P$. Let $\mu_1, \mu_2, \ldots, \mu_j$ be the number of dots in the 1st, 2nd, ..., jth rows of the rotated depiction of J. Then $\mu := (\mu_1, \mu_2, \ldots, \mu_{n-j})$ is a partition. The dots in the rotated depiction of J is the *Ferrer's diagram* of μ. If we replace each dot with a box, then we obtain the Young diagram of μ. This rotation process is a bijection from the filters of P (and hence a basis of the $\mathfrak{g}$-module $V_{J(P)}$) to the Young diagrams fitting inside a $j \times n-j$ rectangle.

In this depiction of filters, the boxes represent elements of P. It is not difficult to verify the following description of the coloring function κ on P. Each diagonal of a Young diagram is assigned a distinct color. The diagonal consisting of the location $(j, 1)$ at the southwest corner of the $j \times n-j$ rectangle is assigned color 1. The diagonal of locations $(j-1, 1)$ and $(j, 2)$ is assigned color 2, and so on. In particular, the main diagonal of locations $(1, 1), (2, 2), \ldots, (j, j)$ is assigned color j. This coloring of the locations is similar to the usual notion of the *content* of a location in a Young diagram, except that contents on such diagrams range from $1-j$ to $n-1-j$ instead of 1 to $n-1$.

We now show that α-layers for roots $\alpha \in \Phi^+$ are simply "rim hooks" in this formulation.

Definition 9.25. A *rim hook* is a connected subset of boxes in the $j \times n-j$ array that does not have two boxes on the same diagonal.

It is possible to show that α-layers must be connected. In type A_{n-1}, the simple root expansion of a root $\alpha \in \Phi^+$ is $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for some $1 \leq i \leq j \leq n-1$. Fix such an α, and suppose that $R \subset P$ is an α-layer. Then R has a single element of each of the colors $i, i+1, \ldots, j$. Hence when rotated, the subset R does not have two boxes on the same diagonal. Therefore the rotated depiction of R is a rim hook. On the other hand, suppose that H is a rim hook in the $j \times n-j$ array. It clearly corresponds to a convex subset of P. Let i denote the color of its southwesternmost box, and let j denote the color of its northeasternmost box. Since H is connected, it must have a box in each of the diagonals of color $i+1, \ldots, j-1$. Since H does not have two boxes on the same diagonal, it has exactly one box of each of these colors. Therefore we have $\sum_{x \in H} \kappa(x) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j =: \alpha$

and that H is an α-layer.

According to Theorem 9.5, the extreme weight Plücker relations are standard straightening laws on double-tailed diamond sublattices of L_λ. As we saw in Section 9.2, in type A these double-tailed diamond lattices are isomorphic to the one that appears in the model D_3 case. Hence there are only three terms in these straightening laws: the product of the incomparable pair, the product of their meet and join, and one other standard monomial. It is simple to describe this last standard monomial in type A without using the determinantal Plücker relations from Section 3.1. First we need some preliminary definitions:

Definition 9.26. Let $\alpha, \beta \in \Phi^+$ and write $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ and $\beta = \alpha_k + \alpha_{k+1} + \cdots + \alpha_\ell$. If $j < k-1$ or $\ell < i-1$, then α and β are said to be *separated*. In that case, without loss of generality suppose $j < k$ and define their *bridge root* $\gamma(\alpha, \beta)$ to be $\gamma(\alpha, \beta) := \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_{k-1}$.

Recall that for any root $\alpha \in \Phi^+$, we defined the actions of root vectors e_α, f_α of $\mathfrak{g}$ on $V_{J(P)}$ in Section 8.2.

Proposition 9.27. *Let $J, K \in J(P)$ be incomparable filters such that $J - (J \vee K)$ and $K - (J \vee K)$ are rim hooks. Then these rim hooks are root layers for separated roots $\alpha, \beta \in \Phi^+$ and we have the type A straightening law:*

$$\mathcal{J}\mathcal{K} = (\mathcal{J} \vee \mathcal{K})(\mathcal{J} \wedge \mathcal{K}) - (e_{\gamma(\alpha,\beta)} . [\mathcal{J} \vee \mathcal{K}])(f_{\gamma(\alpha,\beta)} . [\mathcal{J} \wedge \mathcal{K}]).$$

The filters of the two vectors in the final standard monomial are obtained from the meet and join by transferring a rim hook of color $\gamma(\alpha, \beta)$ from the join filter to the meet filter.

In Section 9.1, we claimed that the Plücker module I is minuscule in type A_{n-1} for the minuscule weights ω_2 and ω_{n-2}. In these cases, the extreme weight Plücker relations given by Theorem 9.5 and described in detail by Proposition 9.27 above form a basis of I. We now confirm that claim. First define ω_0 and ω_n to be the trivial weight 0. For $\lambda = \omega_j$ we can see from the diagrams of the top incomparable pair $Z^\sharp, Z^\flat$ that the highest weight η of the submodule I is $\omega_{j-2} + \omega_{j+2}$. So when $j = 2$, this weight is ω_4. And when $j = n-2$, this weight is ω_{n-4}. In both cases, a dimension count shows that $\dim(I) = \dim(V_\eta)$. Hence I is irreducible and isomorphic to V_η.

9.7 Geometry appendix

We give the geometric motivation behind Problem 9.1. This section outlines the general geometric construction of minuscule flag varieties, and the connection to decompositions of representations. In order to match the modules in the rest of Chapter 9 with the geometrically motivated modules here, one must dualize those modules. More details on the following constructions are available in [13, Chapter 10, Section 6.6].

Fix a minuscule weight λ of $\mathfrak{g}$. Let G be the connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. The representation of $\mathfrak{g}$ on V_λ determines a representation of G on V_λ. Let v be a highest weight vector of V_λ.

Definition 9.28. Let $P \subset G$ denote the parabolic subgroup that leaves the highest weight line $\mathbb{C}v \subset V_\lambda$ invariant. The homogeneous space G/P is called a *minuscule flag manifold.*

The *Plücker embedding* of G/P is the G-equivariant map $G/P \hookrightarrow \mathbb{P}(V_\lambda)$ from the coset space G/P to the projective space of V_λ given by $gP \mapsto \mathbb{C}gP.v$. The Plücker embedding realizes G/P as a projective variety. The *homogeneous coordinate ring* of the ambient space $\mathbb{P}(V_\lambda)$ is defined to be $Sym(V_\lambda^*)$. A weight basis of V_λ^* is called a set of *Plücker coordinates.* Let $\overline{v} \in V_\lambda^*$ be dual to the highest weight vector $v \in V_\lambda$. Since the Plücker embedding is G-equivariant, the coordinate ring of G/P is a $\mathfrak{g}$-module. The degree two submodule $Sym^2(V_\lambda^*)$ decomposes into a direct sum $\mathfrak{U}(\mathfrak{g}).(\overline{v})^2 \oplus I$ of $\mathfrak{g}$-submodules for a unique submodule I. The flag manifold $G/P \subseteq \mathbb{P}(V_\lambda)$ is the zero set of an ideal of $Sym(V_\lambda^*)$ called its vanishing ideal. Kostant proved the following result:

Proposition 9.29. *The quotient of $Sym(V_\lambda^*)$ by the ideal generated by I is the homogeneous coordinate ring of $G/P \subset \mathbb{P}(V)$. Hence, the vanishing ideal of $G/P \subset \mathbb{P}(V)$ is generated by quadratic relations.*

Kostant's result motivates the following definition:

Definition 9.30. A *Plücker relation* for the Plücker embedding of G/P is a a nonzero element of I.

Setting these elements of I to 0 produces the homogeneous coordinate ring $Sym(V_\lambda^*)/I$ for G/P. Note that the dual module V_λ^* of V_λ is minuscule if and only if V_λ is minuscule. So to apply our results of this chapter to this geometric setting, one needs to re-label the indexing minuscule weight λ with its dual minuscule weight $-w_0.\lambda$, where w_0 is the longest element of W. However we have that $-w_0.\lambda = \lambda$ for the type B_n minuscule weight, the type D_n natural weight, the type D_n spin weights when the rank n is even, and the type E_7 minuscule weight. In these cases, no relabeling is necessary.

Inc. Pair		Meet · Join						
$\mathcal{D}_1 \cdot \mathcal{O}_1$	=	$\mathcal{C}_1 \cdot \mathcal{D}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{C}_2$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2$	-	$\aleph \cdot \mathcal{A}_2$
$\mathcal{E}_1 \cdot \mathcal{O}_1$	=	$\mathcal{C}_1 \cdot \mathcal{E}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\aleph \cdot \mathcal{A}_2\mathcal{E}_1$
$\mathcal{E}_1 \cdot \mathcal{D}_1\mathcal{O}_1$	=	$\mathcal{D}_1 \cdot \mathcal{E}_1\mathcal{O}_1$	-	$\mathcal{B}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\aleph \cdot \mathcal{A}_2\mathcal{D}_2$
$\mathcal{C}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{C}_3$	-	$\aleph \cdot \mathcal{A}_2\mathcal{C}_3$
$\mathcal{C}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{C}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{D}_2$	+	$\mathcal{A}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{A}_2\mathcal{O}_2$
$\mathcal{B}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{C}_3$	-	$\aleph \cdot \mathcal{B}_3$
$\mathcal{B}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{B}_3\mathcal{O}_2$
$\mathcal{A}_2 \cdot \mathcal{E}_1$	=	$\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{C}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{A}_1 \cdot \mathcal{B}_3$
$\mathcal{B}_2 \cdot \mathcal{C}_2\mathcal{E}_1$	=	$\mathcal{C}_2 \cdot \mathcal{B}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{C}_3$	+	$\mathcal{C}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{C}_4$
$\mathcal{A}_2 \cdot \mathcal{E}_1\mathcal{O}_1$	=	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	+	$\mathcal{B}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{B}_3\mathcal{O}_2$
$\mathcal{B}_2 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{C}_3$	+	$\mathcal{D}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{D}_3$
$\mathcal{A}_2 \cdot \mathcal{C}_2\mathcal{E}_1$	=	$\mathcal{C}_2 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{C}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{C}_4$
$\mathcal{A}_2 \cdot \mathcal{B}_2\mathcal{E}_1$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{E}_1$	-	$\mathcal{O}_1 \cdot \mathcal{B}_3$	+	$\mathcal{C}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \mathcal{C}_4$
$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{C}_3$	+	$\mathcal{E}_1 \cdot \mathcal{O}_2$	-	$\aleph \cdot \mathcal{E}_2$
$\mathcal{A}_2 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{D}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2 \cdot \mathcal{B}_2\mathcal{D}_2$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{D}_1 \cdot \mathcal{B}_3$	+	$\mathcal{D}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{D}_2$	=	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	+	$\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{A}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{B}_2\mathcal{D}_2$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{D}_2$	-	$\mathcal{O}_1\mathcal{E}_1 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{C}_2 \cdot \mathcal{B}_3$	+	$\mathcal{D}_1 \cdot \mathcal{C}_4$	-	$\mathcal{C}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{C}_4$	-	$\mathcal{C}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{C}_2 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{C}_4$	-	$\mathcal{O}_1 \cdot \mathcal{D}_3$
$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{C}_3$	=	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{A}_2\mathcal{C}_3$	-	$\mathcal{D}_2 \cdot \mathcal{B}_3$	+	$\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{D}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{E}_1\mathcal{O}_1 \cdot \mathcal{C}_4$	-	$\mathcal{O}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{O}_2$	=	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{D}_2 \cdot \mathcal{B}_3\mathcal{O}_2$	+	$\mathcal{E}_1\mathcal{O}_1 \cdot \mathcal{D}_3$	-	$\mathcal{D}_1\mathcal{O}_1 \cdot \mathcal{E}_2$
$\mathcal{A}_2\mathcal{C}_3 \cdot \mathcal{O}_2$	=	$\mathcal{C}_3 \cdot \mathcal{A}_2\mathcal{O}_2$	-	$\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{C}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{C}_2 \cdot \mathcal{E}_2$
$\mathcal{B}_3 \cdot \mathcal{O}_2$	=	$\mathcal{C}_3 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{B}_2\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{B}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{B}_2 \cdot \mathcal{E}_2$
$\mathcal{B}_3 \cdot \mathcal{A}_2\mathcal{O}_2$	=	$\mathcal{A}_2\mathcal{C}_3 \cdot \mathcal{B}_3\mathcal{O}_2$	-	$\mathcal{A}_2\mathcal{D}_2 \cdot \mathcal{C}_4$	+	$\mathcal{A}_2\mathcal{E}_1 \cdot \mathcal{D}_3$	-	$\mathcal{A}_2 \cdot \mathcal{E}_2$

Figure 9.19: The 27 straightening laws for the complex Cayley plane on its Plücker coordinates.

$$
\begin{aligned}
\mathcal{A}_2 \cdot \mathcal{F}_2 = &\mathcal{B}_2 \cdot \overline{\mathcal{B}_2} - \mathcal{C}_2 \cdot \overline{\mathcal{C}_2} + \mathcal{D}_2 \cdot \overline{\mathcal{D}_2} - \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1} + \mathcal{O}_1 \cdot \overline{\mathcal{O}_1} \\
&+ \mathcal{E}_1 \cdot \overline{\mathcal{E}_1} - \mathcal{D}_1 \cdot \overline{\mathcal{D}_1} + \mathcal{C}_1 \cdot \overline{\mathcal{C}_1} - \mathcal{B}_1 \cdot \overline{\mathcal{B}_1} + \mathcal{A}_1 \cdot \overline{\mathcal{A}_1} - \aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.1}
$$

$$
\begin{aligned}
\mathcal{E}_3 \cdot \mathcal{A}_2\mathcal{F}_1 = &\mathcal{B}_2\mathcal{F}_1 \cdot \overline{\mathcal{B}_2\mathcal{F}_1} - \mathcal{C}_2\mathcal{F}_1 \cdot \overline{\mathcal{C}_2\mathcal{F}_1} + \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1} - \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1} - \mathcal{O}_1 \cdot \overline{\mathcal{O}_1} \\
&+ \mathcal{F}_1 \cdot \overline{\mathcal{F}_1} + \mathcal{D}_1 \cdot \overline{\mathcal{D}_1} - \mathcal{C}_1 \cdot \overline{\mathcal{C}_1} + \mathcal{B}_1 \cdot \overline{\mathcal{B}_1} - \mathcal{A}_1 \cdot \overline{\mathcal{A}_1} + \aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.2}
$$

$$
\begin{aligned}
\mathcal{D}_4 \cdot \mathcal{A}_2\mathcal{E}_2 = &\mathcal{B}_2\mathcal{E}_2 \cdot \overline{\mathcal{B}_2\mathcal{E}_2} - \mathcal{C}_2\mathcal{E}_2 \cdot \overline{\mathcal{C}_2\mathcal{E}_2} + \mathcal{E}_2 \cdot \overline{\mathcal{E}_2} + \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1} - \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1} \\
&- \mathcal{F}_1 \cdot \overline{\mathcal{F}_1} + \mathcal{E}_1 \cdot \overline{\mathcal{E}_1} + \mathcal{C}_1 \cdot \overline{\mathcal{C}_1} - \mathcal{B}_1 \cdot \overline{\mathcal{B}_1} + \mathcal{A}_1 \cdot \overline{\mathcal{A}_1} - \aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.3}
$$

$$
\begin{aligned}
\mathcal{A}_2\mathcal{D}_3 \cdot \mathcal{C}_3\mathcal{O}_2 = &\mathcal{B}_2\mathcal{D}_3 \cdot \overline{\mathcal{B}_2\mathcal{D}_3} - \mathcal{D}_3 \cdot \overline{\mathcal{D}_3} - \mathcal{E}_2 \cdot \overline{\mathcal{E}_2} + \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1} - \mathcal{D}_2 \cdot \overline{\mathcal{D}_2} \\
&+ \mathcal{F}_1 \cdot \overline{\mathcal{F}_1} - \mathcal{E}_1 \cdot \overline{\mathcal{E}_1} + \mathcal{D}_1 \cdot \overline{\mathcal{D}_1} + \mathcal{B}_1 \cdot \overline{\mathcal{B}_1} - \mathcal{A}_1 \cdot \overline{\mathcal{A}_1} + \aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.4}
$$

$$
\begin{aligned}
\mathcal{A}_2\mathcal{C}_3 \cdot \mathcal{B}_2\mathcal{O}_2 = &\mathcal{B}_2\mathcal{D}_3 \cdot \overline{\mathcal{B}_2\mathcal{D}_3} - \mathcal{B}_2\mathcal{E}_2 \cdot \overline{\mathcal{B}_2\mathcal{E}_2} + \mathcal{B}_2\mathcal{F}_1 \cdot \overline{\mathcal{B}_2\mathcal{F}_1} - \mathcal{B}_2 \cdot \overline{\mathcal{B}_2} - \mathcal{E}_2 \cdot \overline{\mathcal{E}_2} \\
&+ \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1} - \mathcal{D}_2 \cdot \overline{\mathcal{D}_2} - \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1} + \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1} - \mathcal{O}_1 \cdot \overline{\mathcal{O}_1} \\
&+ \mathcal{F}_1 \cdot \overline{\mathcal{F}_1} - \mathcal{E}_1 \cdot \overline{\mathcal{E}_1} + \mathcal{D}_1 \cdot \overline{\mathcal{D}_1} - \mathcal{C}_1 \cdot \overline{\mathcal{C}_1} + 2\mathcal{B}_1 \cdot \overline{\mathcal{B}_1} - \mathcal{A}_1 \cdot \overline{\mathcal{A}_1} + 2\aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.5}
$$

$$
\begin{aligned}
\mathcal{B}_3 \cdot \mathcal{O}_2 = &\mathcal{D}_3 \cdot \overline{\mathcal{D}_3} - \mathcal{C}_2\mathcal{E}_2 \cdot \overline{\mathcal{C}_2\mathcal{E}_2} + \mathcal{C}_2\mathcal{F}_1 \cdot \overline{\mathcal{C}_2\mathcal{F}_1} - \mathcal{C}_2 \cdot \overline{\mathcal{C}_2} + \mathcal{E}_2 \cdot \overline{\mathcal{E}_2} \\
&- \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1} + \mathcal{D}_2 \cdot \overline{\mathcal{D}_2} + \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1} - \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1} + \mathcal{O}_1 \cdot \overline{\mathcal{O}_1} \\
&- \mathcal{F}_1 \cdot \overline{\mathcal{F}_1} + \mathcal{E}_1 \cdot \overline{\mathcal{E}_1} - \mathcal{D}_1 \cdot \overline{\mathcal{D}_1} + \mathcal{C}_1 \cdot \overline{\mathcal{C}_1} - \mathcal{B}_1 \cdot \overline{\mathcal{B}_1} + 2\mathcal{A}_1 \cdot \overline{\mathcal{A}_1} - 2\aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.6}
$$

$$
\begin{aligned}
\mathcal{C}_3 \cdot \mathcal{A}_2\mathcal{O}_2 = &\mathcal{B}_2\mathcal{D}_3 \cdot \overline{\mathcal{B}_2\mathcal{D}_3} - \mathcal{B}_2\mathcal{E}_2 \cdot \overline{\mathcal{B}_2\mathcal{E}_2} + \mathcal{B}_2\mathcal{F}_1 \cdot \overline{\mathcal{B}_2\mathcal{F}_1} - \mathcal{B}_2 \cdot \overline{\mathcal{B}_2} - \mathcal{D}_3 \cdot \overline{\mathcal{D}_3} \\
&+ \mathcal{C}_2\mathcal{E}_2 \cdot \overline{\mathcal{C}_2\mathcal{E}_2} - \mathcal{C}_2\mathcal{F}_1 \cdot \overline{\mathcal{C}_2\mathcal{F}_1} + \mathcal{C}_2 \cdot \overline{\mathcal{C}_2} - \mathcal{E}_2 \cdot \overline{\mathcal{E}_2} + \mathcal{D}_2\mathcal{F}_1 \cdot \overline{\mathcal{D}_2\mathcal{F}_1} \\
&- \mathcal{D}_2 \cdot \overline{\mathcal{D}_2} - \mathcal{F}_1\mathcal{O}_1 \cdot \overline{\mathcal{F}_1\mathcal{O}_1} + \mathcal{E}_1\mathcal{O}_1 \cdot \overline{\mathcal{E}_1\mathcal{O}_1} - \mathcal{O}_1 \cdot \overline{\mathcal{O}_1} \\
&+ 2\mathcal{F}_1 \cdot \overline{\mathcal{F}_1} - 2\mathcal{E}_1 \cdot \overline{\mathcal{E}_1} + 2\mathcal{D}_1 \cdot \overline{\mathcal{D}_1} - 2\mathcal{C}_1 \cdot \overline{\mathcal{C}_1} + 2\mathcal{B}_1 \cdot \overline{\mathcal{B}_1} - 2\mathcal{A}_1 \cdot \overline{\mathcal{A}_1} + 3\aleph \cdot \overline{\aleph}
\end{aligned} \tag{9.7}
$$

Figure 9.20: The zero weight straightening laws for the Freudenthal variety on its Plücker coordinates.

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} \\
0 & 0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & \bar{1} & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 \\
0 & 1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 & 1 & \bar{1} & 0 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 & \bar{1} \\
1 & 0 & 0 & 0 & \bar{1} & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 1 & \bar{1} & 1 & 0 & 1 & \bar{1} & 1 \\
0 & 0 & 0 & 0 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 2 & \bar{2} \\
1 & \bar{1} & 1 & \bar{1} & 0 & 0 & 0 & 0 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 2 & \bar{1} & 2 \\
1 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & 1 & \bar{1} & \bar{1} & 1 & \bar{1} & 2 & \bar{2} & 2 & \bar{2} & 2 & \bar{2} & 3
\end{bmatrix}$$

Figure 9.21: The matrix which lists the 21 standard monomial coordinates of the 7 products of weight zero incomparable pairs of Plücker coordinates for the Freudenthal variety, in the total order of Section 9.4.

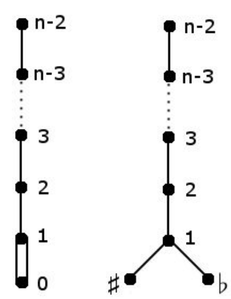


Figure 9.22: The type B_{n-1} and type D_n Dynkin diagrams. The $\sharp$ and $\flat$ type D nodes are "folded" together into the short type B node 0 upon restriction to the subalgebra $\mathfrak{g}_B \subset \mathfrak{g}$.

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