

Localized Energy Estimates for Wave Equations on Higher Dimensional Black Holes

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Honors Thesis

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April 9th, 2014

Approved:

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Abstract

A robust measure of decay and dispersion for the wave equation is provided by the localized energy estimates, which have been essential in proving, e.g. the Strichartz estimates on black hole backgrounds. We study localized energy estimates for the wave equation on (1+4)-dimensional Myers-Perry space-times, which represent a family of rotating asymptotically flat black holes with spherical horizon topology and generalize the well-known Kerr space-times to higher dimensions. Because of the extra dimension, the Myers-Perry family is parameterized by two angular momentum parameters, which we assume to be sufficiently small relative to the mass of the black hole, essentially allowing us to treat the space-time as a perturbation of the Schwarzschild black hole. This investigation is motivated by the nonlinear stability problem for the Kerr family of black holes, which may be easier to understand in higher dimensions.

Typically, the localized energy estimates are proved by commuting the wave operator with a suitable first-order differential operator and integrating by parts. However, the underlying black hole geometry introduces a number of difficulties related to the trapping phenomenon, which is a known obstruction to dispersion and necessitates a loss in decay. This phenomenon is manifest along the event horizon of the Schwarzschild/Kerr black holes, but its effect is rendered negligible due to the celebrated red-shift effect. More delicate analysis is required to deal with trapping that occurs along e.g., the so-called photon sphere in the Schwarzschild geometry. Localized energy estimates on higher dimensional Schwarzschild black holes were proved by Laul-Metcalf in [34] using a single differential multiplier, but their method relies fundamentally on the fact that the trapped null geodesics lie on a sphere. On the Myers-Perry space-time, the nature of the trapped set is much more complicated and must be described in phase space rather than by position alone, and consequently a single differential multiplier is insufficient to prove the desired result. Once it is determined that all trapped geodesics lie on surfaces of constant r , we can adapt the method of Tataru and Tohaneanu in [62], which perturbs off the Schwarzschild case by instead commuting with an appropriate pseudodifferential operator to generate a positive commutator near the trapped set. This describes joint work with Parul Laul, Jason Metcalfe, and Mihai Tohaneanu [35].

Acknowledgements

First and foremost, I am extremely grateful to my adviser Jason Metcalfe, whose support, encouragement, and endless patience have been essential to this thesis and to my success as a student. He taught me the importance of perseverance and has always believed in my abilities, especially when I was most frustrated. I am honored to have worked with him during my time at UNC.

I would like to thank a few other professors who have made an impact on my undergraduate career. I have learned so much from my coursework and reading with Professors Hans Christianson and Jeremy Marzuola, and I sincerely appreciate the time and effort each of them invested to teach me about PDE's. I am grateful for the many helpful conversations shared with Professor Mark Stern, with whom I have had the privilege of taking many fascinating and challenging courses at Duke. Professor Sue Goodman has been a valuable mentor and friend since my very first day at UNC, and I am thankful for her support and advice over the past four years.

I would like to thank my collaborators Parul Laul and Mihai Tohaneanu for the opportunity to work together on the research which appears in the final chapter of this thesis. I also thank my committee members, Professors Mark Stern and Michael Taylor, for their suggestions and feedback on my draft.

Finally, none of this would be possible without the love and support of my friends and family.

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Introduction and Mathematical Preliminaries

In recent years, interest has been growing in the study of partial differential equations in general relativity as a way to gain insight into the causal structure/stability of certain space-times. In loose terms, general relativity is the field of study pertaining to Einstein's theory of gravitation, in which the gravitational field is manifest as the curvature of a space-time manifold, and the dynamics are governed by the Einstein field equations, which relate the matter and energy fields in a space-time to its intrinsic geometric properties. The initial value formulation of the Einstein equations views these dynamics as the evolution of an initial space "slice" in time, and the notion of stability regards the behavior of perturbations within this framework. Roughly speaking, does a small perturbation of a known solution to Einstein equations remain "close" to the original solution? Of particular interest is to rigorously prove the stability of black holes, which are regions of space-time in which the gravitational field is so strong that even light cannot escape. The underlying geometry and inherent nonlinearity of the Einstein equations make this problem highly nontrivial and have motivated significant work on linear field equations on black hole backgrounds in the past decade.

A particularly robust measure of decay and dispersion for the wave equation is provided by the localized energy estimates, which have been essential in proving other types of dispersive estimates on black hole backgrounds such as Strichartz estimates [39], [64], [63] and pointwise decay estimates [61], [44], [19, 21], [37, 38]. In this thesis, we study localized energy estimates for the wave equation on $(1+4)$ -dimensional Myers-Perry space-times, which represent a family of rotating asymptotically flat black holes with spherical horizon topology¹. The Myers-Perry space-times are stationary and axially symmetric, and generalize the well-known Kerr space-times to higher dimensions. Because of the extra dimension, the Myers-Perry family is parameterized by two angular momentum parameters, which we assume to be sufficiently small relative to the mass of the black hole, allowing us to treat the space-time as a perturbation of the Schwarzschild black hole.

To describe the localized energy estimates in Minkowski space $\mathbb{R} \times \mathbb{R}^n$ for $n \geq 3$, we set $A_j = \{2^{j-1} \leq |x| \leq 2^j\}$ for $j \in \mathbb{Z}$, and define our local energy space LE_M by

$$\|u\|_{LE_M} = \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|u\|_{L^2_{t,x}(\mathbb{R} \times A_j)},$$

and its H^1 counterpart LE_M^1 ,

$$\|u\|_{LE_M^1} = \|\nabla_{t,x} u\|_{LE_M} + \||x|^{-1} u\|_{LE_M}.$$

To measure an inhomogeneous term, we define a dual norm by

$$\|F\|_{LE_M^*} = \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \|F\|_{L^2_{t,x}(\mathbb{R} \times A_k)}.$$

The localized energy estimates for the wave equation state that

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2} + \|u\|_{LE_M^1} \lesssim \|\nabla_{t,x} u(0, \cdot)\|_{L^2} + \|\square u\|_{LE_M^* + L_t^1 L_x^2} \quad (1.1)$$

when $n \geq 3$. Integrated decay estimates of this form originated in [47], and generalizations of the original estimate have appeared in, e.g. [58], [32], [54], [31], [45], [13], [28], [57], [40], [41], [43], [42], and have ultimately led to (1.1).

The localized energy estimates can be proved by commuting the wave operator with a carefully chosen first-order differential operator, integrating by parts, and analyzing the resulting space-time integral. Ideally,

¹Additional spatial dimensions permit asymptotically flat black holes with exotic horizon topologies; see, e.g. [24].

the multiplier is chosen so that all of the terms are positive, which yields a localized energy estimate by what is, in essence, a *positive commutator* argument. The rough idea is that within a finite region of space, waves will disperse and leave the region in a fixed amount of time, giving an average decay in energy over time. Such estimates are characteristic of dispersive equations² and are generally robust in studying perturbations of the space-time metric. For example, similar estimates have been proved for small, time-dependent, long-range perturbations of the Minkowski metric [1], [40, 41], [42, 43], and for time-independent, non-trapping, asymptotically flat perturbations [11], [12], [56].

However, the underlying black hole geometry introduces a number of difficulties related to the trapping phenomenon, which is a known obstruction to dispersion and necessitates a loss in decay (see [51] and the recent work [52]). The trapping phenomenon is manifest along the event horizon of the Schwarzschild and Kerr black holes, but its effect is negated due to the celebrated red-shift effect (see, e.g. [20], [39]). More delicate analysis is required to deal with trapping that occurs along the so-called *photon sphere* in the Schwarzschild geometry. Specifically, there is a loss in the localized energy estimates in the sense that the coefficients of the time and angular derivatives vanish to quadratic order at the photon sphere³. Nevertheless, by a careful choice of multiplier, localized energy estimates with a loss were proved in, e.g. [8, 9, 10, 7], [20, 19], [39], and analogous estimates on higher dimensional Schwarzschild black holes were proved independently in [34] and [53]. Moreover, each of these results rely fundamentally on the fact that the trapped null geodesics lie on a sphere.

On the Kerr space-time, the nature of the trapped set is much more complicated and must be described in phase space rather than by position alone, and consequently a single differential multiplier is insufficient to prove the desired result (see [2]). Nevertheless, localized energy estimates (with loss) have been proved on the Kerr space-time using three distinct approaches:

- (1) the existence of a nontrivial Killing tensor (derived in [14]) in [3];
- (2) separation of variables and a frequency decomposition in [22], [17, 18, 23]; and
- (3) a pseudodifferential multiplier in [62].

Our approach relies most heavily on the ideas from [62], which perturbs off the Schwarzschild case by instead commuting with an appropriate pseudodifferential operator to generate a positive commutator near the trapped set. As such, our proof depends heavily on the integrability of the geodesic flow, which was shown for $(1 + 4)$ -dimensional Myers-Perry space-times in [26].

As stated earlier, our investigation is motivated by the nonlinear stability problem for the Kerr family of black holes (described in, e.g. [22]), which may be easier to understand in higher dimensions. Heuristically, similarly posed quasilinear wave equations (which roughly model the more complicated Einstein equations) have global solutions in dimensions $1 + n$ for small initial data when $n > 3$, while the dimension $n = 3$ is, in a sense, “critical” because the global existence result is generically false and sensitive to the structure of the nonlinearity (see, e.g. [55]). This difficulty is demonstrated in the proofs [16], [36] of global stability for the $1 + 3$ -dimensional Minkowski space-time (the trivial solution to the Einstein equations), while an analogous result for dimensions $1 + n$ with $n > 5$ can be proved using the technique of [36] in just a few pages [15].

We briefly outline the structure of this thesis. At the end of this chapter, we state some basic facts about general relativity and the Minkowski space-time, which is the trivial solution to the Einstein equations. In the second chapter, we discuss the geometry of the Schwarzschild and Myers-Perry space-time; in particular, we prove that trapped null geodesics in the Myers-Perry space-time lie on surfaces of constant r . In the third chapter, we discuss the red-shift effect and its applications to the wave equation; in particular, the constructions in this chapter are used heavily in subsequent chapters.

The final three chapters regard localized energy estimates for the wave equation in the Minkowski, Schwarzschild, and Myers-Perry space-times, respectively. In particular, in the fourth chapter we prove localized energy estimates for small long-range perturbations of the Minkowski metric. In the fifth chapter, we prove a similar estimate on the higher dimensional Schwarzschild space-times, but we modify the proof in [34] by smoothing out the multiplier near the photon sphere, which will simplify the microlocal analysis in

²e.g. similar estimates are available for the Schrodinger equation

³With some, e.g. microlocal analysis, this can be improved to a logarithmic loss.

later chapters. Finally, in the sixth chapter, we prove localized energy estimates on the $(1 + 4)$ -dimensional Myers-Perry space-times for small angular momenta, which is the main content of this thesis [35]. We define our local energy spaces to account for the complicated nature of trapping and thereafter prove the main estimate by perturbing off the Schwarzschild result in the spirit of [62].

1. A Brief Introduction to Relativity and Gravitation

In general relativity, the gravitational field is manifest as the curvature of a Lorentzian space-time manifold $(\mathcal{M}, \mathbf{g})$, and the dynamics are governed by the *Einstein field equations*,

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad (1.2)$$

which relate the matter and energy fields in a space-time to its intrinsic geometric properties. In this equation, $R_{\alpha\beta}$ and R denotes the Ricci and scalar curvatures of \mathcal{M} , respectively, and $T_{\alpha\beta}$ is the *stress-energy-tensor* of the matter/energy fields associated to \mathcal{M} . When there are no matter/energy fields in a space-time, i.e. the stress-energy tensor $T_{\alpha\beta}$ vanishes, then (1.2) reduces to

$$R_{\alpha\beta} = 0, \quad (1.3)$$

which are called the *vacuum Einstein equations*.

A point $x \in \mathcal{M}$ is referred to as an *event*, and represents a point fixed in both time and space. Because the metric \mathbf{g} has indefinite sign, the tangent space $T_x\mathcal{M}$ is partitioned into three categories; a vector X is called

- (1) *timelike* if $g(X, X) < 0$,
- (2) *spacelike* if $g(X, X) > 0$, and
- (3) *null* if $g(X, X) = 0$,

and moreover U is called *causal* if it is timelike or null. Similarly, a curve γ is called *timelike*, *null*, or *spacelike* if its tangent vector $\dot{\gamma}$ is everywhere timelike, null, or spacelike, respectively. In this formulation, null curves represent the trajectories of photons and other massless particles, while timelike curves represent the paths of any particle that has mass (see, e.g. [27]). Such a classification is important because it generalizes the notion of “light cones” and defines a precise notion of causality.

A curve $x(\lambda)$ in \mathcal{M} is called *geodesic* if it solves the *geodesic equations*

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \quad (1.4)$$

In this notation, the $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})$ are the Christoffel symbols associated to the metric \mathbf{g} . In fact (1.4) are just the Euler-Lagrange equations for the path length functional

$$L(\gamma) = \int_0^1 |\langle \dot{\gamma}, \dot{\gamma} \rangle|^{1/2} dt \quad (1.5)$$

of a curve $\gamma : (0, 1) \rightarrow \mathcal{M}$. This is consistent with the fact that on a Riemannian manifold where the metric \mathbf{g} is positive definite, geodesics are locally length-minimizing curves on \mathcal{M} . When the metric has an indefinite signature, this is not exactly the case as geodesics are only critical points, not necessarily minima, of the path length functional; instead, one can think of geodesics as the generalization of “straight lines” to curved space-time and as the natural paths of inertial test particles in free-fall. As a result, understanding the nature of geodesics is extremely important in understanding the underlying geometry of a given space-time and the gravitational forces involved. In the following chapter, we will analyze the geodesic flow in the Schwarzschild and Myers-Perry space-times; in particular, we are concerned with the nature of *trapped* null geodesics that forever reside within a compact region of space.

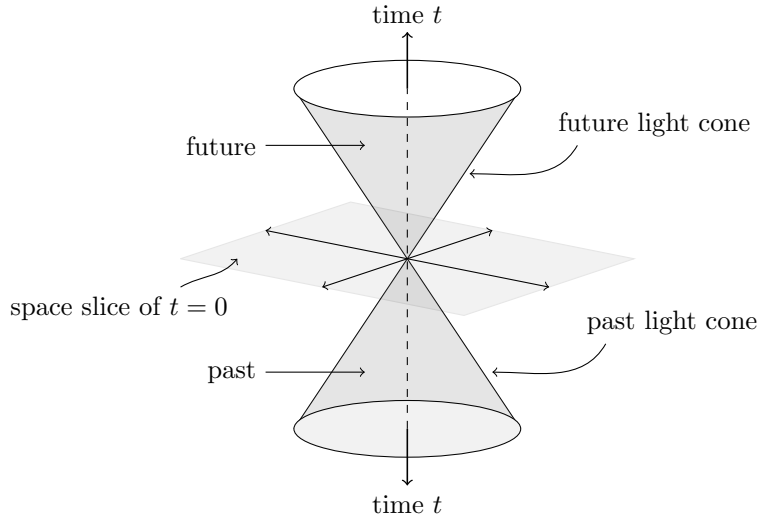


FIGURE 1.1. The light-cone structure of Minkowski space-time.

2. Minkowski space-time

The trivial solution to the vacuum Einstein equation (1.3) is called the *Minkowski space-time*, and is described by the manifold $\mathbb{R} \times \mathbb{R}^n$ equipped with the metric \mathbf{m} , whose line element is given by

$$ds^2 = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2. \quad (1.6)$$

We have conveniently chosen units in which the speed of light is equal to 1. The space-time is *flat* in the sense that all of the curvature components vanish identically. Historically, the Minkowski space-time describes the setting of *special relativity*, which was developed by Albert Einstein to resolve conflicts between electromagnetism and Newtonian mechanics (see, e.g. [46]).

Note that the geodesics of the Minkowski space-time are straight lines, as the metric components are all constant. As a result, one can easily visualize the causal structure of the space-time as in Figure 1.1. We imagine an observer in space-time and label his event of existence at one particular time as the “origin” of a Lorentzian coordinate system. We can draw this coordinate system by calling one axis the time t , and suppressing excess spatial variables. This given time $t = 0$ is a hyperplane of simultaneity or, in other words, is a space “slice” in space-time. The notion of causality pertains to the region of space-time to the future that an observer can reach or communicate with, with regard to the speed of light as an absolute limit. Thus, the observer’s *worldline* (t, x) is confined to the region in which $|x| < |t|$, which forms two cones corresponding to the future and the past. The boundary of the cone $|x| = |t|$ is called the *light cone* or *null cone* and represents the possible paths of light rays that propagate to and from the observer.

In practice, a more convenient way to represent the causal structure of space-times is by a *Penrose diagram*. We construct such a diagram for Minkowski space below by compactifying the space-time by a change of coordinates. Following the details in [27], we begin by defining *null coordinates* u, v by

$$u = t - r, \quad v = t + r, \quad (1.7)$$

so that the metric takes the form

$$ds^2 = -dudv + r^2 d\Omega_{\mathbb{S}^{n-1}}^2, \quad (1.8)$$

where

$$d\Omega_{\mathbb{S}^{n-1}}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} d\theta_{n-1}^2$$

is the metric on \mathbb{S}^{n-1} induced by the inclusion $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. In particular, surfaces of constant u, v represent paths of radial null geodesics. To compactify the space-time, we set

$$U = \arctan(u), \quad V = \arctan(v), \quad (1.9)$$

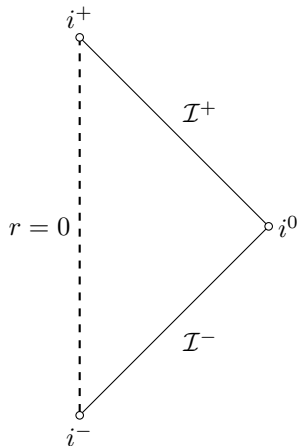


FIGURE 1.2. The Penrose diagram for the Minkowski space-time

where $-\frac{\pi}{2} < U, V < \frac{\pi}{2}$ now have finite range. The Minkowski line element now becomes

$$\frac{1}{4} \sec^2 U \sec^2 V (-4dUdV + \sin^2(V - U)d\Omega_{\mathbb{S}^{n-1}}^2). \quad (1.10)$$

Finally, we set

$$T = V + U, \quad R = V - U, \quad (1.11)$$

and the metric now takes the more standard form

$$ds^2 = \varphi^2 (-dT^2 + dR^2 + \sin^2 R d\Omega_{\mathbb{S}^{n-1}}^2), \quad (1.12)$$

where $\varphi^2 = \frac{1}{4} \sec^2(\frac{1}{2}(T + R)) \sec^2(\frac{1}{2}(T - R))$. Note that the T, R coordinates have finite range, since $-\pi < T < \pi$ and $0 < R < \pi$. Suppressing the spherical symmetries, we can represent the space-time in a two-dimensional diagram shown in Figure 1.2. Most importantly, radial null geodesics travel at 45° angles, which allows us to read off the causal structure of the space-time. We now address the boundary components introduced by compactifying the space-time. The boundaries \mathcal{I}^\pm denote *future and past null infinities* in the sense that they represent future and past endpoints, respectively, of future-directed null geodesics. The points i^\pm denote *future and past timelike infinities* in the sense that they represent future and past endpoints, respectively, of future-directed timelike geodesics. Finally, the point i^0 represents *spacelike infinity* and represents the endpoint of spacelike geodesics.

Black Holes

1. The Schwarzschild Solution and its Maximal Extension

The Schwarzschild solution is an exact solution to the vacuum Einstein equations that represents the gravitational field exterior to a spherical, non-rotating star or mass. It is a relatively simple model to study because it is spherically symmetric and static, which makes many calculations on a Schwarzschild background much easier to carry out. In fact, it is a classical result that every spherically symmetric solution to the Einstein field equations in vacuum is locally isometric to the Schwarzschild manifold (see, e.g. [27]):

THEOREM 2.1 (Birkhoff's Theorem). *Suppose $(\mathcal{M}, \mathbf{g})$ is a spherically symmetric solution to the vacuum Einstein equations. Then the solution is static and asymptotically flat.*

In this context, a space-time is called *static* if it has a timelike Killing vector field T which is orthogonal to a family of spacelike hypersurfaces. Often a space-time may have a timelike Killing vector, but fails to satisfy the second condition; in such a case, the space-time is called *stationary*. In particular, the Kerr (and more generally, Myers-Perry) family of black holes falls into the latter category. Given the assumption that a vacuum space-time is spherically symmetric and static, it is straightforward to show that it is necessarily isometric to the Schwarzschild manifold [46]. Hence Schwarzschild is the only nontrivial spherically symmetric solution to the vacuum Einstein equations.

The Schwarzschild black hole in $(1+n)$ dimensions, which was derived in [59], is described by the space-time manifold $\mathcal{M} = \mathbb{R} \times (r_s, \infty) \times \mathbb{S}^{d+2}$ and metric \mathbf{g} given by the line element field

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) dt^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} dr^2 + r^2 d\Omega_{\mathbb{S}^{d+2}}^2, \quad (2.1)$$

where we set $d = n - 3$ so that we may compare calculations to the $(1+3)$ -dimensional case. The quantity $r = r_s$ is called the *Schwarzschild radius* and represents the event horizon of the black hole. While the metric seemingly degenerates along this radius, the singularity is artificial and represents a breakdown of (t, r, ω) coordinates. To mediate the apparent singularity at $r = r_s$, we introduce the *Regge-Wheeler tortoise coordinate* $r_* = r_*(r)$ defined by

$$\frac{dr}{dr_*} = \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right), \quad r_*(r_{ps}) = 0, \quad (2.2)$$

which serves as a useful alternative to r because the metric components no longer blow up as $r \rightarrow r_s$. The radius $r = r_{ps} = \left(\frac{d+3}{2}\right)^{\frac{1}{d+1}} r_s$ is called the *photon sphere* and represents a surface generated by trapped null geodesics, which is significant in the considerations of localized energy estimates in later chapters; we will return to the topic of trapping later in the chapter. We can express r_* by:

$$\begin{aligned} r_*(r) &= \int_{r_{ps}}^r \frac{\rho^{d+1}}{\rho^{d+1} - r_s^{d+1}} d\rho = \int_{r_{ps}}^r \left(1 + \frac{r_s^{d+1}}{\rho^{d+1} - r_s^{d+1}}\right) d\rho \\ &= (r - r_{ps}) + \int_{r_{ps}}^r \frac{r_s^{d+1}}{\rho^{d+1} - r_s^{d+1}} d\rho^2. \end{aligned} \quad (2.3)$$

In particular, the integral on the right-hand-side is bounded for r large, but has a logarithmic blow-up near the event horizon, so that $r_* \approx r$ for r large and $r_* \approx \log(r - r_s)$ as $r \searrow r_s$.

To seek a smooth extension across the horizon, we define

$$v = t + r_*, \quad u = t - r_* \quad (2.4)$$

which are called the *ingoing and outgoing Eddington-Finkelstein null coordinates*, respectively. The coordinates (u, v) are directly analogous to the null coordinates in Minkowski space. In the *ingoing Eddington-Finkelstein coordinates* (v, r, ω) , the metric is expressed by

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) dv^2 + 2dvdr + r^2 d\Omega_{\mathbb{S}^{d+2}}^2, \quad (2.5)$$

and analytically extends the Schwarzschild manifold to the full range $0 < r < \infty$. In particular, every future-directed causal curve at the event horizon necessarily flows into the interior region $0 < r < r_s$; indeed, $x(\tau) = (v(\tau), r(\tau), \omega(\tau))$ is future directed if

$$0 \geq \langle \dot{x}, \partial_v \rangle = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \dot{v} + \dot{r},$$

which can be rewritten as

$$\dot{r} \leq \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \dot{v}.$$

The right hand side of this expression vanishes at $r = r_s$ so that $\dot{r}|_{r=r_s} \leq 0$, and hence $r = r_s$ is a one-way horizon that particles and light rays can only cross in one direction. For this reason, we call $\{(v, r, \omega) : r \leq r_s\}$ the *black hole* region.

However, the extension afforded by the ingoing coordinates (v, r, ω) is incomplete, which can be seen by choosing a different coordinate system. Indeed, we can instead consider the *outgoing null coordinates* (u, r, ω) and carry out a similar exercise to above. In this case, we instead find that future-directed causal curves flow *out* from the horizon $r = r_s$, which seems to contradict our previous exercise. However, while the exterior region $\{r \geq r_s\}$ in the ingoing and outgoing coordinates is the same, we remark that the interior region $r \leq r_s$ in the outgoing coordinates (u, r, ω) is *different* from that defined by the ingoing coordinates (v, r, ω) . The two sets of coordinates in fact describe different coordinate patches of the Schwarzschild manifold. In the Penrose diagram in Figure 2.1 (which we will explain later), the ingoing coordinates (v, r, ω) correspond to regions *I* and *II*, while the outgoing coordinates (u, r, ω) correspond to regions *I* and *IV*. We call $\{(u, r, \omega) : r \leq r_s\}$ the *white hole* region and distinguish it from the black hole region above. In particular, the outgoing coordinates yield a second analytic extension of the Schwarzschild manifold, which too is incomplete.

To obtain a maximal extension gluing these two together, we begin with the *double null coordinates* (u, v, ω) , which transforms the line element into the form

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) dudv + r^2 d\Omega_{d+2}^2. \quad (2.6)$$

Unfortunately, the double null coordinates are valid only on the range $r_s < r < \infty$. To remedy this, we consider a coordinate transformation of the form

$$V = e^{v/2r_s^{d+1}} = e^{(r_*+t)/2r_s^{d+1}}, \quad U = -e^{-u/2r_s^{d+1}} = -e^{(r_*-t)/2r_s^{d+1}}. \quad (2.7)$$

Under such a transformation, the line element field becomes

$$ds^2 = 4r_s^{2d+2} \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) e^{-r_*/r_s^{d+1}} dUdV + r^2 d\Omega_{d+2}^2. \quad (2.8)$$

We noted earlier that $r_* \approx \log(r - r_s)$ near $r = r_s$, so in fact the above metric is non-degenerate in (U, V, ω) coordinates in the full range $-\infty < U, V < \infty$. In particular, the event horizon $r = r_s$ is given by $V = \pm U$. We now define the higher-dimensional analogue of *Kruskal-Szekeres coordinates*,

$$T = \frac{1}{2}(V - U) = e^{r_*/2r_s^{d+1}} \cosh \left(\frac{t}{2r_s^{d+1}} \right), \quad R = \frac{1}{2}(V + U) = e^{r_*/2r_s^{d+1}} \sinh \left(\frac{t}{2r_s^{d+1}} \right). \quad (2.9)$$

The line element takes a more standard form, and is given by

$$ds^2 = 4r_s^{2d+2} \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) e^{-r_*/r_s^{d+1}} (-dT^2 + dR^2) + r^2 d\Omega_{\mathbb{S}^{d+2}}^2. \quad (2.10)$$

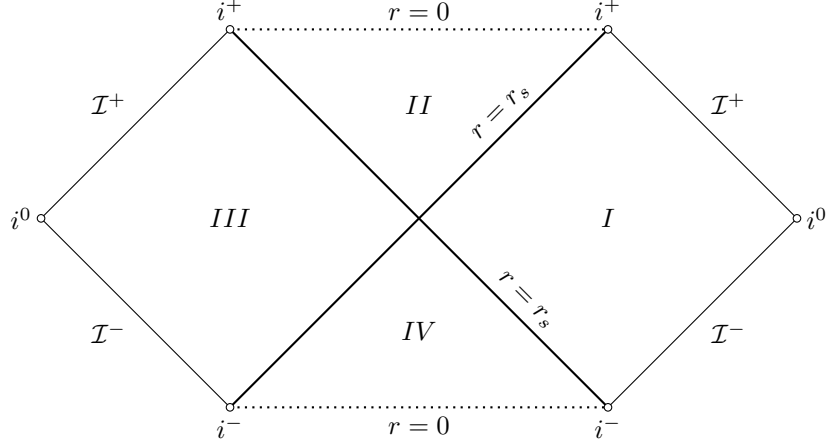


FIGURE 2.1. The Penrose diagram for maximally-extended Schwarzschild space-time

The resulting space-time $(\mathcal{M}^*, \mathbf{g}^*)$ is called the *maximal analytic extension* of the Schwarzschild manifold and is *inextendible* in the sense that there is no solution $(\mathcal{M}^{**}, \mathbf{g}^{**})$ to the Einstein equations into which $(\mathcal{M}^*, \mathbf{g}^*)$ is isometrically embedded.

To represent the causal structure of the Schwarzschild space-time, we compactify the space-time given by (U, V, ω) coordinates by

$$\tilde{U} = \arctan(U), \quad \tilde{V} = \arctan(V), \quad (2.11)$$

and correspondingly define

$$\tilde{T} = \tilde{V} + \tilde{U}, \quad \tilde{R} = \tilde{V} - \tilde{U}. \quad (2.12)$$

The $(\tilde{T}, \tilde{R}, \omega)$ coordinates have finite range, and by suppressing spherical symmetries we can conveniently represent the space-time in a two-dimensional diagram shown in Figure 2.1, which is called a *Penrose diagram*. Most importantly, radial null geodesics travel at 45° angles, which allows us to read off the causal structure of the space-time.

The space-time is naturally partitioned into four regions I, II, III, IV which represent an exterior region, black hole region, symmetric exterior region, and white hole region, respectively. The region I corresponds to the familiar exterior region of the black hole in Schwarzschild coordinates (t, r, ω) , while III denotes a symmetric exterior region to I . Note that any causal future-directed geodesic which enters the black hole region II is future-incomplete and terminates at the singularity at $r = 0$. Similarly, every future-directed causal geodesic in the white hole region IV is past-incomplete and originates from the singularity at $r = 0$. We now address the boundary components introduced by compactifying the space-time. The boundaries \mathcal{I}^\pm denote *future and past null infinities* in the sense that they represent “endpoints” of future-inextendible and past-inextendible null geodesics, respectively. The points i^\pm are denote *future and past timelike infinities* in the sense that they represent “endpoints” of future-inextendible and past-inextendible timelike geodesics, respectively. Finally, the points i^0 represent *spacelike infinity* and are endpoints of inextendible spacelike geodesics.

1.1. Geodesic Motion on Schwarzschild. We now turn to the topic of geodesic motion on the Schwarzschild space-time by using the time translation symmetry T and the full set of spherical symmetries $\Omega_1, \dots, \Omega_{d+2}$. For this thesis, we are concerned with the behavior of trapped null geodesics; see, e.g. [46] for a more thorough discussion of geodesic motion on the Schwarzschild space-time.

Recall that a vector field K is called *Killing* if its flow map generates an isometry of the metric tensor \mathbf{g} ; this condition is equivalent to

$$\frac{1}{2} \mathcal{L}_K \mathbf{g}_{\alpha\beta} = \frac{1}{2} (\mathbf{D}_\alpha K_\beta + \mathbf{D}_\beta K_\alpha) =: {}^{(K)}\pi_{\alpha\beta} = 0, \quad (2.13)$$

which is referred to as *Killing's equation*. The quantity ${}^{(K)}\pi_{\alpha\beta}$ is known as the *deformation tensor* of K .

Every Killing vector field K corresponds to a conserved quantity along geodesics, for if $X^\alpha = \frac{dx^\alpha}{d\lambda}$ is the tangent vector to a geodesic curve $\gamma(\lambda) = (x^\mu(\lambda)) = (t(\lambda), r(\lambda), \omega(\lambda))$, then

$$\frac{\mathbf{D}}{\partial\lambda} K_\alpha X^\alpha = X^\beta \mathbf{D}_\beta K_\alpha X^\alpha = X^\alpha X^\beta \mathbf{D}_\beta K_\alpha + K_\alpha X^\beta \mathbf{D}_\beta X^\alpha = 0 \quad (2.14)$$

by Killing's equation and the geodesic equations. Since the Schwarzschild metric is spherically symmetric, it suffices to consider geodesics γ that satisfy say, $\theta_1 = \dots = \theta_{d+1} = \pi/2$. As a result, we can use the Killing vector fields $T = \partial_t$ and $\Theta = \partial_{\theta_{d+2}}$ to derive conserved quantities for γ by

$$E := -T_\alpha \frac{dx^\alpha}{d\lambda} = \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \frac{dt}{d\lambda}, \quad L := \Theta_\alpha \frac{dx^\alpha}{d\lambda} = r^2 \frac{d\theta_{d+2}}{d\lambda}. \quad (2.15)$$

From the geodesic equations we also note that $X_\alpha X^\alpha$ is constant along γ , so for a null geodesic γ we have

$$\begin{aligned} 0 &= -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta_{d+2}}{d\lambda}\right)^2 \\ &= -\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \left(E^2 + \left(\frac{dr}{d\lambda}\right)^2\right) + \frac{L^2}{r^2}. \end{aligned} \quad (2.16)$$

We can rewrite this expression as

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{1}{r^2} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) L^2, \quad (2.17)$$

which allows us to easily characterize trapped rays. We note that $E > 0$ is a necessary condition for there to be motion, since the right hand side of (2.17) must be positive. By scaling the affine parameter $\lambda \rightarrow E\lambda$ accordingly, let us assume without loss of generality that $E = 1$. Now for $L > 0$, we can write

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 - V(r), \quad V(r) = \frac{1}{r^2} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) L^2. \quad (2.18)$$

The quantity $V(r)$ is called an *effective potential* and can be treated like a potential energy quantity; when $P(r) := 1 - V(r) > 0$ there is motion in the r coordinate, which increases or decreases monotonically until it hits a point where $P = 0$, at which point the particle or light ray either turns around or enters a trapped orbit. In particular, maxima and minima of V correspond to unstable and stable trapped orbits, respectively. To this end, let us consider

$$V'(r) = L^2 \left(-\frac{2}{r^3} + \frac{(d+3)r_s^{d+1}}{r^{d+4}}\right) = -2\frac{L^2}{r^3} \left(1 - \frac{d+3}{2} \frac{r_s^{d+1}}{r^{d+1}}\right) = 0, \quad (2.19)$$

which is solved by

$$r = r_{ps} := \left(\frac{d+3}{2}\right)^{\frac{1}{d+1}} r_s. \quad (2.20)$$

The radius $r = r_{ps}$ is called the *photon sphere* and represents a surface generated by trapped null geodesics. That is, a null geodesic that is initially tangent to $r = r_{ps}$ will forever remain on it. Since the value r_{ps} gives a local maximum of V , however, the photon sphere represents unstable trapping in the sense that small perturbations of trapped rays will not be trapped. We remark while that it is generated by trapped null geodesics, the photon sphere is *not* a null surface in the Schwarzschild geometry, as the induced metric is simply a scaling of the \mathbb{S}^{d+2} metric and hence is positive definite.

On the other hand, if $L = 0$, then γ is a radial null geodesic and we have

$$\left(\frac{dr}{d\lambda}\right)^2 = 1, \quad (2.21)$$

so $\langle X, X \rangle = 0$ implies

$$\frac{dt}{dr} = \frac{dt}{d\lambda} = \pm \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}, \quad (2.22)$$

and in particular the right hand side of this equation blows up as $r \rightarrow r_s$. Hence, an ingoing radial null geodesic reaches the event horizon $r = r_s$ in finite affine time λ , yet the coordinate t becomes infinite.

Here, the null cones at a given point appear to become vertical, which highlights the breakdown of (t, r, ω) coordinates and suggests that the event horizon is a trapped null surface (see, e.g. [46]).

1.2. Useful Calculations of the Schwarzschild Metric. It is useful to note certain calculations before we proceed further. The non-vanishing Christoffel symbols for the Schwarzschild metric in (t, r, ω) coordinates are given by

$$\begin{aligned}\Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{(d+1)r_s^{d+1}}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}, & \Gamma^r_{tt} &= \frac{(d+1)r_s^{d+1}}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right), \\ \Gamma^r_{rr} &= -\frac{(d+1)r_s^{d+1}}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}, & \Gamma^{\theta_i}_{r\theta_j} = \Gamma^{\theta_i}_{\theta_j r} &= \delta_{ij} \frac{1}{r}, \\ \Gamma^r_{\theta_i\theta_j} = \Gamma^r_{\theta_j\theta_i} &= \delta_{ij} r \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \sin^2 \theta_1 \cdots \sin^2 \theta_{j-1},\end{aligned}\tag{2.23}$$

and in (t, r_*, ω) coordinates by

$$\begin{aligned}\Gamma^t_{tr_*} = \Gamma^t_{r_*t} &= \frac{(d+1)r_s^{d+1}}{2r^{d+2}}, & \Gamma^{r_*}_{tt} &= \frac{(d+1)r_s^{d+1}}{2r^{d+2}} \\ \Gamma^{r_*}_{r_*r_*} &= \frac{(d+1)r_s^{d+1}}{2r^{d+2}}, & \Gamma^{\theta_i}_{r_*\theta_j} = \Gamma^{\theta_i}_{\theta_j r_*} &= \delta_{ij} \frac{1}{r} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right), \\ \Gamma^{r_*}_{\theta_i\theta_j} = \Gamma^{r_*}_{\theta_j\theta_i} &= \delta_{ij} (-r) \sin^2 \theta_1 \cdots \sin^2 \theta_{j-1}.\end{aligned}\tag{2.24}$$

The remaining terms are purely angular and are shared in both coordinate systems, and are

$$\Gamma^{\theta_i}_{\theta_j\theta_k} = \begin{cases} 0 & \text{if } i, j, k \text{ are distinct,} \\ \cot \theta_j & \text{if } i = k \text{ and } j < i, \\ 0 & \text{if } i = k \text{ and } j \geq i, \\ -(\sin \theta_i \cos \theta_i) \sin^2 \theta_{i+1} \cdots \sin^2 \theta_{j-1} & \text{if } j = k \text{ and } j > i, \\ 0 & \text{if } j = k \text{ and } j \leq i. \end{cases}\tag{2.25}$$

These are the only nontrivial symbols, i.e. the others are all equal to 0.

2. Myers-Perry Space-Times

In higher dimensions, the Schwarzschild and Kerr space-times are members of a more general family of black holes called the *Myers-Perry space-times* [49], which are axially symmetric solutions to the vacuum Einstein equations. With the additional dimensions, additional rotation parameters are introduced. We focus here on the $(1+4)$ -dimensional case and characterize the behavior of trapped null geodesics. The line element for the Myers-Perry metric in $(t, r, \theta, \phi, \psi)$ coordinates is given by

$$\begin{aligned}ds^2 &= -dt^2 + \frac{r_s^2}{\rho^2} (dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\psi)^2 + \frac{r^2 \rho^2}{\Delta} dr^2 \\ &\quad + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\psi^2,\end{aligned}\tag{2.26}$$

where r_s is the Schwarzschild radius and a, b are angular momentum parameters. We have denoted

$$\begin{aligned}\rho^2 &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta \\ \Delta &= (r^2 + a^2)(r^2 + b^2) - r_s^2 r^2.\end{aligned}\tag{2.27}$$

The components of the inverse metric g_{mp}^{-1} are given by:

$$\begin{aligned}g^{tt} &= \frac{1}{\rho^2} \left((a^2 - b^2) \sin^2 \theta - \frac{(r^2 + a^2)(\Delta + r_s^2(r^2 + b^2))}{\Delta} \right), \\ g^{t\phi} &= \frac{ar_s^2(r^2 + b^2)}{\rho^2 \Delta}, & g^{t\psi} &= \frac{br_s^2(r^2 + a^2)}{\rho^2 \Delta}, \\ g^{\phi\phi} &= \frac{1}{\rho^2} \left(\frac{1}{\sin^2 \theta} - \frac{(a^2 - b^2)(r^2 + b^2) + b^2 r_s^2}{\Delta} \right),\end{aligned}$$

$$g^{\psi\psi} = \frac{1}{\rho^2} \left(\frac{1}{\cos^2 \theta} - \frac{(a^2 - b^2)(r^2 + a^2) - a^2 r_s^2}{\Delta} \right),$$

$$g^{\phi\psi} = -\frac{abr_s^2}{\rho^2 \Delta}, \quad g^{rr} = \frac{\Delta}{r^2 \rho^2}, \quad g^{\theta\theta} = \frac{1}{\rho^2}.$$

The Myers-Perry metric shares many important features of the Kerr metric, including the structure of singularities, horizons, and ergo-regions [48], [49]. For example, the Myers-Perry black hole is stationary and axially symmetric, with Killing vector fields $\partial_t, \partial_\phi, \partial_\psi$. In the case that $a = b = 0$, the space-time reduces to the Schwarzschild metric. We detail some important properties of the Myers-Perry metric below.

2.1. Event Horizons. The Myers-Perry black hole has event horizons $r = r_\pm$ given by

$$r_\pm = \sqrt{\frac{1}{2}(r_s^2 - a^2 - b^2) \pm \frac{1}{2}\sqrt{(r_s^2 - a^2 - b^2)^2 - 4a^2 b^2}}, \quad (2.28)$$

which are roots of the equation $\Delta = 0$. We note that this equation has real roots if and only if

$$|a| + |b| \leq r_s.$$

If one or both of a, b is zero, then there is a single event horizon $r = r_+ > 0$, as the other root of the equation is $r_- = 0$. If both a, b are non-zero and $|a| + |b| < r_s$, then there are two distinct event horizons $0 < r_- < r_+$. In either case, the radius $r = r_+$ is a coordinate singularity which can be mediated by an appropriate change of coordinates. We refer to $r = r_+$ as the event horizon, as this radius corresponds to a one-way horizon which null geodesics can cross in only one direction. When $|a| + |b| = r_s$, the two horizons $r = r_\pm$ merge into a single event horizon. This is called the *extremal* case, and represents the values of a, b beyond which no event horizon exists. Indeed for $|a| + |b| > r_s$ the curvature singularity at $\rho^2 = 0$ is no longer hidden behind an event horizon and instead becomes a *naked singularity* (see [22] for a relevant discussion of the Kerr space-time). These singularities and extremal horizons are believed to be unstable to perturbations (see, e.g. [4, 5, 6]).

2.2. Ergo-region. Unlike the Schwarzschild black hole, the Killing vector field ∂_t is not timelike on the whole exterior region $r > r_+$ and becomes null when $g_{tt} = 0$; the latter equation is solved by

$$r_E(\theta) = \sqrt{r_s^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta}.$$

This defines an oblate surface ironically referred to as the *ergo-sphere* which lies outside the event horizon. Then the Killing vector field ∂_t is spacelike in the so-called *ergo-region* $r_+ \leq r < r_E$. Penrose discovered that black holes with an ergo-region permit a process by which rotational energy can be “extracted” from the black hole [50], which is referred to as *energy extraction* or the *Penrose process* (see [65] for a discussion). In the context of the wave equation $\square_g u = 0$, the ergo-region allows for the conserved energy density to be negative on a neighborhood of the event horizon, which makes proving even uniform energy bounds nontrivial (see, e.g. [22]).

2.3. Singularities and Ingoing Coordinates. The coordinates $(t, r, \phi, \psi, \theta)$ degenerate along the event horizon $r = r_+$. To extend beyond the horizon, we define a set of ingoing coordinates analogous to the ingoing Eddington-Finkelstein coordinates in the Schwarzschild space-time. Define v_+, ϕ_+, ψ_+ by:

$$dv_+ = dt + \frac{(r^2 + a^2)(r^2 + b^2)}{\Delta} dr,$$

$$d\phi_+ = d\phi - \frac{a(r^2 + b^2)}{\Delta} dr,$$

$$d\psi_+ = d\psi - \frac{b(r^2 + a^2)}{\Delta} dr.$$

The line element is expressed in $(v_+, r, \phi_+, \psi_+, \theta)$ coordinates as

$$ds^2 = -dv_+^2 + 2dv_+ dr + \frac{r_s^2}{\rho^2} (dv_+ + a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+)^2 + \rho^2 d\theta^2$$

$$+ 2(a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+) dr + (r^2 + a^2) \sin^2 \theta d\phi_+^2 + (r^2 + b^2) \cos^2 \theta d\psi_+^2. \quad (2.29)$$

In particular, the ingoing coordinates are valid for the full range $\rho^2 > 0$. On the other hand, $\rho^2 = 0$ is a true curvature singularity, but now represents a ring instead of a point [48].

2.4. Geodesic motion on Myers-Perry space-times. We now turn our attention to the behavior of geodesics and show trapped null geodesics in the space-time satisfy $r = \text{constant}$. We henceforth use the variable $x = r^2$ to simplify calculations as in [26]. With respect to the x coordinate, the Myers-Perry metric has components

$$g_{xx} = \frac{\rho^2}{4\Delta}, \quad g^{xx} = \frac{4\Delta}{\rho^2}. \quad (2.30)$$

Along any null geodesic, we have conserved constants of motion E, Φ, Ψ arising from the Killing vector fields $-\partial_t, \partial_\phi, \partial_\psi$. However, as in the (1 + 3)-dimensional Kerr space-time (see, e.g. [46]), there is in fact an additional conserved constant of motion K , which is used to prove that the geodesic flow is integrable [26]. In [26], it was shown that the geodesic flow equations can be written as:

$$\begin{aligned} \rho^4 \dot{\theta}^2 &= \Theta, \\ \rho^4 \dot{x}^2 &= 4\chi, \\ \rho^2 \dot{t} &= E\rho^2 + \frac{r_s^2(x+a^2)(x+b^2)}{\Delta} \mathcal{E}, \\ \rho^2 \dot{\phi} &= \frac{\Phi}{\sin^2 \theta} - \frac{ar_s^2(x+b^2)}{\Delta} \mathcal{E} - \frac{(a^2-b^2)\Phi}{x+a^2}, \\ \rho^2 \dot{\psi} &= \frac{\Psi}{\cos^2 \theta} - \frac{br_s^2(x+a^2)}{\Delta} \mathcal{E} + \frac{(a^2-b^2)\Psi}{x+b^2}, \end{aligned} \quad (2.31)$$

where we denote:

$$\mathcal{E} = E + \frac{a\Phi}{x+a^2} + \frac{b\Psi}{x+b^2} \quad (2.32)$$

$$\Theta = E^2(a^2 \cos^2 \theta + b^2 \sin^2 \theta) - \frac{1}{\sin^2 \theta} \Phi^2 - \frac{1}{\cos^2 \theta} \Psi^2 + K \quad (2.33)$$

$$\chi = \Delta \left(E^2 x + (a^2 - b^2) \left(\frac{\Phi^2}{x+a^2} - \frac{\Psi^2}{x+b^2} \right) - K \right) + r_s^2(x+a^2)(x+b^2)\mathcal{E}^2. \quad (2.34)$$

We remark that since $a^2 - b^2 = (x+a^2) - (x+b^2)$, we have:

$$-r_s^2 x(a^2 - b^2) \left(\frac{\Phi^2}{x+a^2} - \frac{\Psi^2}{x+b^2} \right) + r_s^2(x+a^2)(x+b^2) \left(\frac{a^2\Phi^2}{(x+a^2)^2} + \frac{b^2\Psi^2}{(x+b^2)^2} \right) = r_s^2(b^2\Phi^2 + a^2\Psi^2)^2.$$

In particular, χ is a cubic polynomial in x with leading order coefficient $E^2 \geq 0$. We will show every trapped null geodesic corresponds to a double root of χ , and hence satisfies $x = \text{constant}$. Our analysis splits into several cases, depending on the constants E, K, Φ, Ψ . In particular, at least one of E, K, Φ, Ψ must be non-zero in order to have equations of motion. From (2.31) it follows that $\Theta \geq 0$, which implies that we cannot have both $E = K = 0$, for otherwise $\Theta < 0$.

First suppose that $E = 0$ and $K \neq 0$. We rewrite (2.34) as

$$\Theta = (K + (\Phi - Ea)^2 - \Phi^2 + (\Psi - Eb)^2 - \Psi^2) - \frac{(\Phi - Ea \sin^2 \theta)^2}{\sin^2 \theta} - \frac{(\Psi - Eb \cos^2 \theta)^2}{\cos^2 \theta}, \quad (2.35)$$

so the condition $\Theta \geq 0$ implies that

$$K + (\Phi - Ea)^2 - \Phi^2 + (\Psi - Eb)^2 - \Psi^2 \geq 0.$$

In particular, the condition $E = 0$ forces $K \geq 0$, and so $K \neq 0$ implies that $K > 0$. In particular χ is a quadratic polynomial with leading order coefficient $-K < 0$. Furthermore, since $\Delta(x_+) = 0$, we must also have that $\chi(x_+) \geq 0$, and thus χ has exactly one simple root outside the event horizon, where χ changes sign from positive to negative. This root corresponds to a right turning point for the ODE, so no trapped geodesics exist.

Next, we examine the case when $E \neq 0$ so that χ is a cubic polynomial with leading order coefficient $E^2 > 0$. Since $\chi(x_+) > 0$, at most two roots of χ can lie outside x_+ . We examine each case separately:

- (1) No such root exists. Then $\chi > 0$ for all $x > x_+$, and so x must increase or decrease monotonically along every null geodesic. Hence no trapped null geodesics exist.
- (2) There are two distinct roots $x_+ < x_1 < x_2$. Then x_1 is a right turning point and x_2 is a left turning point for the ODE. In particular, this does not support a trapped null geodesic.
- (3) There is a double root $x_+ < x_1$. Then the orbit $x = x_1$ is stable on one side and unstable on the other, and thus any null geodesic either crosses the event horizon $x = x_+$, escapes to infinity $x = \infty$, or converges to x_0 . In particular, every trapped null geodesic must lie on the root $x = x_1$.

We conclude that every trapped null geodesic corresponds to a double root of χ and satisfies $x = \text{constant}$.

The Gravitational Redshift Effect

In this chapter, we detail the application of the gravitational red-shift effect to evolution equations on black hole space-times. In general, the presence of trapped rays obstructs dispersion and necessitates a loss in decay, but for the wave equation, for example, the red-shift effect allows one to derive a non-degenerate estimate near the event horizon even though it contains trapped null geodesics in the black hole geometry (see [20] and [39]).

In general relativity, the red-shift effect refers to the tendency of gravity to affect the frequency of electromagnetic radiation, “shifting” it to the red. In the black hole geometry, the key observation is that trapped light rays along the event horizon are red-shifted by an exponential factor depending on the *surface gravity* of the black hole. To illustrate the exponential decay in frequency, we work in the Schwarzschild geometry in the ingoing Eddington-Finkelstein coordinates (v, r, ω) , and consider two observers A, B who cross the event horizon at times v_A, v_B , respectively, where $v_A < v_B$. We consider a (trapped) radial null geodesic along the event horizon that is transmitted by A when he crosses the event horizon at time v_A . Let ν, ξ denote Fourier variables for v, r and let

$$p(r, \nu, \xi) = 2g^{vr} \nu \xi + g^{rr} \xi^2$$

be the Hamiltonian for the geodesic flow; then the Hamilton flow equations dictate that

$$\begin{aligned} \left. \frac{d\xi}{dv} \right|_{r=r_s} &= \left. \frac{\dot{\xi}}{\dot{v}} \right|_{r=r_s} = - \left. \frac{\partial p}{\partial r} \left(\frac{\partial p}{\partial \nu} \right)^{-1} \right|_{r=r_s} = \left. \frac{\partial}{\partial r} g^{rr} \xi^2 (2g^{vr} \xi)^{-1} \right|_{r=r_s} \\ &= - \left. \frac{1}{2} \frac{\partial}{\partial r} \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \xi \right|_{r=r_s} = - \frac{(d+1)}{2r_s} \xi. \end{aligned} \quad (3.1)$$

Here we have used the fact that $\dot{r} = 0$ and $r \equiv r_s$, i.e. the “outgoing” geodesic transmitted by A is trapped along the event horizon. If the initial frequency at $v = v_A$ is denoted by ξ_0 , then the solution to the above ODE is

$$\xi(v) = \xi_0 e^{-\kappa(v-v_A)}, \quad \kappa = \frac{d+1}{2r_s}.$$

Hence, the frequency at which B receives this signal is given by $\xi_0 e^{-\kappa(v_B-v_A)}$, which exhibits the exponential decay. The exponential factor κ is called the *surface gravity* of the black hole (see Lemma 3.2).

Let us now describe the setup we use in some detail. For now, we will work in the Schwarzschild geometry. The *domain of outer communications* is the region

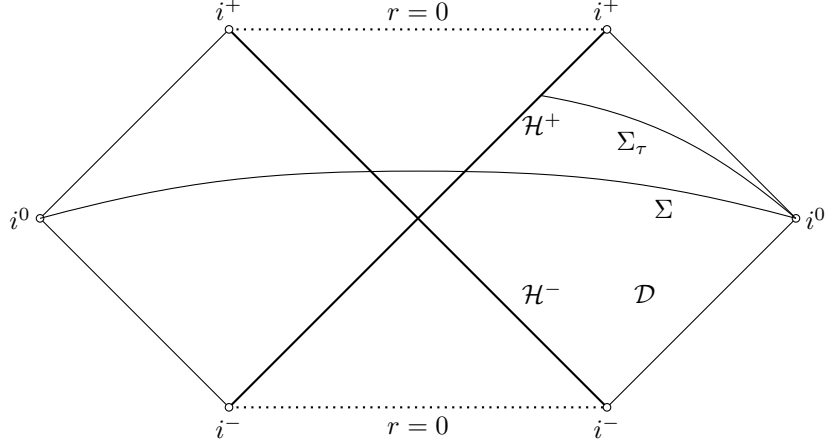
$$\mathcal{D} = \{-\infty < t < \infty, r \geq r_s\}, \quad (3.2)$$

which represents one of two exterior domains to the black/white hole regions. This is a slight abuse of notation, as the Schwarzschild coordinates (t, r, ω) in fact degenerate along $r = r_s$. The boundary of \mathcal{D} in \mathcal{M} has two components,

$$\mathcal{H}^+ = \{(v, r, \omega) : r = r_s\}, \quad \mathcal{H}^- = \{(u, r, \omega) : r = r_s\}, \quad (3.3)$$

which are called the *future* and *past* event horizons, respectively. Note that with respect to the Penrose diagram in Figure 2.1, \mathcal{H}^+ corresponds to the boundary between the regions I and II , while \mathcal{H}^- corresponds to the boundary between the regions I and IV .

Here u, v are the outgoing, ingoing Eddington-Finkelstein null coordinates, respectively, defined in (2.4). Let Σ be an arbitrary Cauchy hypersurface in \mathcal{M} which does not intersect \mathcal{H}^- . In this way, Σ does not

FIGURE 3.1. The Cauchy hypersurface Σ and the slices Σ_τ

intersect the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$. We set $\Sigma_0 = \Sigma \cap \mathcal{D}$ and $\Sigma_\tau = \varphi_\tau(\Sigma_0) \cap \mathcal{D}$, where φ_τ is the flow map of the Killing vector field $T = \partial_t$. We set

$$\mathcal{R} = \bigcup_{\tau \geq 0} \Sigma_\tau, \quad \mathcal{R}(\tau_0, \tau_1) = \bigcup_{\tau_0 \leq \tau \leq \tau_1} \Sigma_\tau, \quad \mathcal{H}^+(\tau_0, \tau_1) = \bigcup_{\tau_0 \leq \tau \leq \tau_1} \mathcal{R}(\tau_0, \tau_1) \cap \mathcal{H}^+. \quad (3.4)$$

In order to apply the divergence theorem, we need a well-defined volume element on the surfaces Σ_τ and on the event horizon \mathcal{H}^+ . For each $\tau \geq 0$, we let n_{Σ_τ} be the future-directed unit normal to Σ_τ . In particular, n_Σ is timelike, and projecting the space-time metric onto the orthogonal complement of n_Σ in each tangent space yields an induced Riemannian metric on each Σ_τ . On the other hand, the null hypersurface \mathcal{H}^+ has a null direction normal to itself, as well as a conjugate null direction transverse to it. The volume element is determined by the choice of a null generator $n_{\mathcal{H}^+}$ which is normal and tangent to \mathcal{H}^+ in the sense that $n_{\mathcal{H}^+} \in T_x \mathcal{H}^+ - \{0\}$ but $\langle n_{\mathcal{H}^+}, X \rangle = 0$ for every $X \in T_x \mathcal{H}^+$. A simple choice is $n_{\mathcal{H}^+} = T$, which is timelike in the interior of \mathcal{D} but becomes null on \mathcal{H}^+ .

We consider the Cauchy problem for the wave equation in \mathcal{D} , with initial data (u_0, u_1) along the slice Σ_0 . We are concerned with the behavior of functions u in the region \mathcal{D} , in which observers can communicate with each other freely. No signals from the black hole, however, can reach exterior observers.

The wave equation has an associated stress-energy tensor, defined by

$$Q_{\alpha\beta}(u) = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\gamma u \partial_\gamma u, \quad (3.5)$$

whose divergence equals

$$\mathbf{D}^\alpha Q_{\alpha\beta}(u) = (\square_g u) \partial_\beta u. \quad (3.6)$$

For a vector field X on \mathcal{M} , we define the corresponding momentum density,

$$P_\alpha^{(X)}(u) = Q_{\alpha\beta}(u) X^\beta, \quad (3.7)$$

whose divergence we compute as

$$\mathbf{D}^\alpha P_\alpha^{(X)}(u) = (Xu)(\square_g u) + Q_{\alpha\beta}(u)^{(X)} \pi_{\alpha\beta}. \quad (3.8)$$

With the Killing vector field $T = \partial_t$ and a simple application of the divergence theorem, we derive a positive energy for solutions to $\square_g u = 0$ along the slices Σ_τ which satisfies the following conservation law:

$$\int_{\mathcal{H}^+(\tau_0, \tau_1)} P_\alpha^{(T)}(u) n_{\mathcal{H}^+}^\alpha dS + \int_{\Sigma_{\tau_0}} P_\alpha^{(T)} n_{\Sigma_{\tau_0}}^\alpha dS_{\Sigma_{\tau_0}} = \int_{\Sigma_{\tau_1}} P_\alpha^{(T)} n_{\Sigma_{\tau_1}}^\alpha dS_{\Sigma_{\tau_1}}. \quad (3.9)$$

In particular, since all three terms are non-negative, we trivially obtain the energy inequality

$$\int_{\Sigma_{\tau_0}} P_\alpha^{(T)} n_{\Sigma_{\tau_0}}^\alpha dS_{\Sigma_{\tau_0}} \leq \int_{\Sigma_{\tau_1}} P_\alpha^{(T)} n_{\Sigma_{\tau_1}}^\alpha dS_{\Sigma_{\tau_1}}. \quad (3.10)$$

However, as T becomes null on the event horizon \mathcal{H}^+ , the energy, as defined, is degenerate, which makes proving even uniform energy bounds difficult (see [22] for a discussion). As a simple application of the red-shift effect, we will derive a non-degenerate energy defined by an everywhere timelike vector field N , for which the inequality (3.10) extends (see Proposition 3.3).

The main theorem is described below, following the statement in [20] and the generalization to higher dimensions in [53].

THEOREM 3.1. *There exists a future-directed timelike, φ_t -invariant smooth vector field N and radii $r_e < r_s < r_0^N < r_1^N < r_{ps}$ such that*

$$\begin{cases} \mathbf{D}^\alpha P_\alpha^{(N)}(u) \approx P_\alpha^{(N)}(u)n_{\Sigma_\tau}^\alpha, & r_e \leq r \leq r_0^N, \\ |\mathbf{D}^\alpha P_\alpha^{(N)}(u)| \lesssim P_\alpha^{(N)}(u)n_{\Sigma_\tau}^\alpha \approx P_\alpha^{(T)}(u)n_{\Sigma_\tau}^\alpha, & r_0^N \leq r \leq r_1^N, \\ N = T, \text{ i.e. } \mathbf{D}^\alpha P_\alpha^{(N)}(u) = 0, & r \geq r_1^N, \end{cases} \quad (3.11)$$

for all functions u that satisfy the wave equation $\square_g u = 0$.

We note that while we work with the Schwarzschild geometry, the theorem of Dafermos and Rodnianski [20] applies to a wider range of metrics, including the Kerr space-times. In our proof, we rely fundamentally on:

- (1) the existence of a Killing vector field T that is timelike outside the event horizon \mathcal{H}^+ and null and normal on \mathcal{H}^+ ;
- (2) that $\mathbf{D}_T T = \kappa T$ on \mathcal{H}^+ , where $\kappa > 0$ is a positive constant known as the *surface gravity* of the black hole.

The positivity of κ is extremely important in proving the aforementioned bounds. On the other hand, for the sub-extremal Kerr and Myers-Perry space-times, the Killing vector field $T = \partial_t$ fails to be timelike on a neighborhood of the event horizon called the ergo-region (see the discussion in the previous). As a result, the details our proof of Theorem 3.1 and Proposition 3.3 do not follow over trivially.

To prove the theorem, it is most convenient to work in the ingoing Eddington-Finkelstein coordinates (v, r, ω) , where $v = t + r_*$ and r_* is the Regge-Wheeler coordinate as before. The metric takes the form

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) dv^2 + 2dvdr + r^2 d\Omega_{\mathbb{S}^{d+2}}^2. \quad (3.12)$$

This coordinate system has the advantage that it is well defined on the full range $0 < r < \infty$, allowing us to make computations on the event horizon. Denote the vector fields

$$T = \frac{\partial}{\partial v}, \quad \hat{Y} = -2 \frac{\partial}{\partial r}, \quad (3.13)$$

expressed in the (v, r, ω) coordinates. In the more familiar Schwarzschild coordinates (t, r, ω) , this becomes:

$$T = \frac{\partial}{\partial t}, \quad \hat{Y} = -2 \left(\frac{\partial}{\partial r} - \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^{-1} \frac{\partial}{\partial t} \right) = -2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^{-1} \left(\frac{\partial}{\partial r_*} - \frac{\partial}{\partial t} \right). \quad (3.14)$$

The normalization is such that $\langle \hat{Y}, T \rangle = -2$ on \mathcal{H}^+ . Since \hat{Y} is a constant multiple of the vector field ∂_r expressed in (v, r, ω) coordinates, it is well defined up to \mathcal{H}^+ and into the interior region $0 < r < r_s$. We also let $\{E_A\}$ be an orthonormal set of vectors obtained by normalizing the coordinate vector fields $\left\{ \frac{\partial}{\partial \theta_j} \right\}$.

We remark that for $r > r_s$, by the calculations in (2.24) we have:

$$\begin{aligned}
\mathbf{D}_{\hat{Y}}\hat{Y} &= 4 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \mathbf{D}_{\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}} \left[\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}\right) \right] \\
&= 4 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-2} \left(\mathbf{D}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} - \mathbf{D}_{\frac{\partial}{\partial r_*}} \frac{\partial}{\partial t} - \mathbf{D}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial r_*} + \mathbf{D}_{\frac{\partial}{\partial r_*}} \frac{\partial}{\partial r_*} \right) \\
&\quad - 4 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-3} \frac{\partial}{\partial r_*} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}\right) \\
&= 4 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-2} \left[\frac{(d+1)r_s^{d+1}}{r^{d+2}} - \frac{\partial}{\partial r} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \right] \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*}\right) = 0.
\end{aligned} \tag{3.15}$$

Since the vector field \hat{Y} is geometrically well defined in the full range $r > 0$, we can analytically continue the above statement up to the horizon \mathcal{H}^+ . We also note that $[\hat{Y}, T] = 0$, since \hat{Y}, T are constant multiples of the coordinate vector fields ∂_v, ∂_r in (v, r, ω) coordinates.

LEMMA 3.2 (surface gravity). *There is a positive constant $\kappa_d > 0$ such that*

$$\mathbf{D}_T T = \kappa_d T \quad \text{on } \mathcal{H}^+. \tag{3.16}$$

PROOF. Since the calculation needs to be made on the event horizon itself, we need to take care to work in a geometrically invariant fashion. First, we see that

$$\langle T, \mathbf{D}_T T \rangle = \frac{1}{2} T \langle T, T \rangle = 0, \tag{3.17}$$

$$\langle E_A, \mathbf{D}_T T \rangle = T \langle E_A, T \rangle - \langle \mathbf{D}_T E_A, T \rangle = 0. \tag{3.18}$$

For the final component, we have, in (v, r, ω) coordinates:

$$\begin{aligned}
\langle \hat{Y}, \mathbf{D}_T T \rangle &= T \langle \hat{Y}, T \rangle - \langle \mathbf{D}_T \hat{Y}, T \rangle = -\langle \mathbf{D}_{\hat{Y}} T, T \rangle = -\frac{1}{2} \hat{Y} \langle T, T \rangle \\
&= -\frac{\partial}{\partial r} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) = -\frac{(d+1)r_s^{d+1}}{r^{d+2}} \Big|_{r=r_s} = -\frac{d+1}{r_s},
\end{aligned} \tag{3.19}$$

which implies that

$$\mathbf{D}_T T = \frac{d+1}{2r_s} T =: \kappa_d T, \tag{3.20}$$

and clearly $\kappa_d > 0$ as desired. \square

PROOF OF THEOREM 3.1. We first claim that given any $\sigma > 0$, there exists a vector field Y on \mathcal{M} satisfying the following properties:

- (1) Y is φ_t -invariant and spherically symmetric,
- (2) Y is future-directed null on \mathcal{H}^+ and transverse to \mathcal{H}^+ with $\langle Y, T \rangle = -2$, and
- (3) on \mathcal{H}^+ it holds that

$$\mathbf{D}_Y Y = -\sigma(Y + T). \tag{3.21}$$

Let us begin with the ansatz that

$$Y = \alpha(r)\hat{Y} + \beta(r)T, \tag{3.22}$$

where we require that $\alpha(r_s) = 1$ and $\beta(r_s) = 0$ so that Y satisfies the first two conditions. Now, on \mathcal{H}^+ we have

$$\begin{aligned}
\mathbf{D}_Y Y &= \mathbf{D}_{\hat{Y}} Y = \mathbf{D}_{\hat{Y}} (\alpha(r)\hat{Y} + \beta(r)T) \\
&= \alpha(r_s) \mathbf{D}_{\hat{Y}} \hat{Y} \Big|_{r=r_s} + \beta(r_s) \mathbf{D}_{\hat{Y}} T \Big|_{r=r_s} + \hat{Y} (\alpha(r)) \Big|_{r=r_s} \hat{Y} + \hat{Y} (\beta(r)) \Big|_{r=r_s} T \\
&= -2\alpha'(r_s)\hat{Y} - 2\beta'(r_s)T.
\end{aligned} \tag{3.23}$$

In other words, we require that

$$\frac{\partial}{\partial r} \alpha(r_s) = \frac{\partial}{\partial r} \beta(r_s) = \frac{\sigma}{2}. \tag{3.24}$$

This is easily satisfied by the functions

$$\alpha(r) = 1 + \frac{\sigma}{2}(r - r_s), \quad \beta(r) = \frac{\sigma}{2}(r - r_s), \quad (3.25)$$

which yields the desired vector field Y . Then defining $N = T + Y$, we see that $\mathbf{D}^\alpha P_\alpha^{(N)}(u) = \mathbf{D}^\alpha P_\alpha^{(Y)}(u)$.

We now make several calculations on the event horizon \mathcal{H}^+ . In particular, note that $\hat{Y} = Y$ on \mathcal{H}^+ . First, we compute:

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(Y)}(u) \Big|_{r=r_s} &= {}^{(Y)}\pi_{\alpha\beta} Q^{\alpha\beta} = \frac{1}{4} {}^{(Y)}\pi(T, Y) Q(T, T) + \frac{1}{4} {}^{(Y)}\pi(T, T) Q(Y, Y) + \frac{1}{2} {}^{(Y)}\pi(T, Y) Q(T, Y) \\ &\quad - \frac{1}{2} \sum_A {}^{(Y)}\pi(T, E_A) Q(Y, E_A) - \frac{1}{2} \sum_A {}^{(Y)}\pi(Y, E_A) Q(T, E_A) \\ &\quad + \sum_{A, B} {}^{(Y)}\pi(E_A, E_B) Q(E_A, E_B). \end{aligned} \quad (3.26)$$

We calculate the components of the stress energy tensor on \mathcal{H}^+ as follows:

$$\begin{aligned} Q(T, T) &= |Tu|^2, \\ Q(Y, Y) &= |Yu|^2, \\ Q(T, Y) &= (Tu)(Yu) - \frac{1}{2} \langle Y, T \rangle (-(Yu)(Tu) + |\nabla u|^2) = |\nabla u|^2, \\ Q(E_A, E_A) &= |E_A u|^2 - \frac{1}{2} (-(Tu)(Yu) + |\nabla u|^2) \\ &= \frac{1}{2} (Tu)(Yu) + |E_A|^2 - \frac{1}{2} |\nabla u|^2. \end{aligned} \quad (3.27)$$

Let us now calculate the deformation tensor of Y as follows, component by component. On \mathcal{H}^+ , we have

$$\begin{aligned} {}^{(Y)}\pi(T, T) &= \langle \mathbf{D}_T Y, T \rangle = T \langle T, Y \rangle - \langle Y, \mathbf{D}_T T \rangle = -\langle Y, \kappa_n T \rangle = 2\kappa_n, \\ {}^{(Y)}\pi(Y, Y) &= \langle \mathbf{D}_Y Y, Y \rangle = \langle -\sigma(T + Y), Y \rangle = 2\sigma, \\ {}^{(Y)}\pi(T, Y) &= \frac{1}{2} (\langle \mathbf{D}_T Y, Y \rangle + \langle \mathbf{D}_Y Y, T \rangle) \\ &= \frac{1}{4} T \langle Y, Y \rangle + \frac{1}{2} \langle -\sigma(T + Y), T \rangle = \sigma, \\ {}^{(Y)}\pi(T, E_A) &= \frac{1}{2} (\langle \mathbf{D}_T Y, E_A \rangle + \langle DD_{E_A} Y, T \rangle) \\ &= \frac{1}{2} T \langle Y, E_A \rangle - \langle Y, \mathbf{D}_T E_A \rangle = 0, \\ {}^{(Y)}\pi(Y, E_A) &= \frac{1}{2} (\langle \mathbf{D}_Y Y, E_A \rangle + \langle Y, \mathbf{D}_{E_A} Y \rangle) \\ &= \frac{1}{2} \langle -\sigma(T + Y), E_A \rangle + \frac{1}{4} E_A \langle Y, Y \rangle = 0. \end{aligned} \quad (3.28)$$

To calculate the angular components, let us note that $\mathbf{D}_{E_A} Y = \mathbf{D}_{E_A} \hat{Y}$. Thus, for $r > r_s$,

$$\begin{aligned} \mathbf{D}_{E_A} \hat{Y} &= \frac{2}{r \sin \theta_1 \cdots \sin \theta_{j-1}} \mathbf{D}_{\theta_j} \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^{-1} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r_*} \right) \\ &= -\frac{2}{r \sin \theta_1 \cdots \sin \theta_{j-1}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^{-1} \Gamma^{\theta_j}_{r_* \theta_j} \frac{\partial}{\partial \theta_j} = -\frac{2}{r} E_A. \end{aligned} \quad (3.29)$$

Because E_A, \hat{Y} are geometrically defined on the full range $0 < r < \infty$, this statement extends to \mathcal{H}^+ . Thus, on \mathcal{H}^+ we can write

$$\begin{aligned} {}^{(Y)}\pi(E_A, E_B) &= \frac{1}{2} (\langle \mathbf{D}_{E_A} Y, E_B \rangle + \langle \mathbf{D}_{E_B} Y, E_A \rangle) = \delta_{AB} \langle \mathbf{D}_{E_A} Y, E_A \rangle \\ &= \delta_{AB} \langle \mathbf{D}_{E_A} \hat{Y}, E_A \rangle = \delta_{AB} \left\langle -\frac{2}{r_s} E_A, E_A \right\rangle = -\frac{2}{r_s} \delta_{AB}. \end{aligned} \quad (3.30)$$

Thus on \mathcal{H}^+ , we compute that

$$\begin{aligned}
\mathbf{D}^\alpha P_\alpha^{(Y)}(u) &= \frac{\kappa_d}{2}|Yu|^2 + \frac{\sigma}{2}|Tu|^2 + \frac{\sigma}{2}|\nabla u|^2 \\
&\quad - \frac{2}{r_s} \sum_A \left(|E_A u|^2 + \frac{1}{2}(Tu)(Yu) - \frac{1}{2}|\nabla u|^2 \right) \\
&= \frac{\kappa_d}{2}|Yu|^2 + \frac{\sigma}{2}|Tu|^2 + \left(\frac{\sigma}{2} + \frac{d}{r_s} \right) |\nabla u|^2 - \frac{d+2}{r_s}(Tu)(Yu) \\
&\geq \frac{\kappa_d}{2}|Yu|^2 + \frac{\sigma}{2}|Tu|^2 + \left(\frac{\sigma}{2} + \frac{d}{r_s} \right) |\nabla u|^2 \\
&\quad - \left(\frac{d+2}{2r_s} \right) \epsilon |Yu|^2 - \left(\frac{d+2}{2r_s} \right) \frac{1}{\epsilon} |Tu|^2
\end{aligned} \tag{3.31}$$

for any $\epsilon > 0$. We then take $\epsilon \ll 1$ sufficiently small and subsequently $\sigma > 0$ sufficiently large that

$$\frac{\kappa_d}{2} > \left(\frac{d+2}{2r_s} \right) \epsilon, \quad \frac{\sigma}{2} > \left(\frac{d+2}{2r_s} \right) \frac{1}{\epsilon}. \tag{3.32}$$

This is enough to ensure that all of the resulting terms are positive, so we obtain

$$\mathbf{D}^\alpha P_\alpha^{(Y)}(u) \approx |Tu|^2 + |Yu|^2 + |\nabla u|^2 \approx P_\alpha^{(N)}(u)n^\alpha \tag{3.33}$$

on \mathcal{H}^+ , since N is timelike. By continuity, N must be timelike on a neighborhood of \mathcal{H}^+ , so the statement must hold for $r_e \leq r \leq r_0^N$ for some r_e, r_0^N with $r_e < r_s < r_0^N$. We simply cutoff N outside such a neighborhood to agree with T for some $r \geq r_1^N$ with $r_0^N < r_1^N < r_{ps}$, so that

$$|\mathbf{D}^\alpha P_\alpha^{(N)}(u)| \lesssim P_\alpha^{(N)}(u)n^\alpha \approx P_\alpha^{(T)}(u)n^\alpha \tag{3.34}$$

in the range $r_0^N < r < r_1^N$. This completes the construction of N . \square

Using the vector field N we can form a non-degenerate energy with density $P_\alpha^{(N)}(u)n_\Sigma^\alpha$. As a simple application, we prove that this non-degenerate energy remains uniformly bounded for a solution of the wave equation $\square_g u = 0$.

PROPOSITION 3.3. *Let u solve the wave equation $\square_g u = 0$ with appropriate initial data (u_0, u_1) on Σ_0 . Then u satisfies the following energy inequality:*

$$\sup_{\tau \geq 0} \int_{\Sigma_\tau} P_\alpha^{(N)}(u)n^\alpha dS \lesssim \int_{\Sigma_0} P_\alpha^{(N)}(u)n^\alpha dS. \tag{3.35}$$

PROOF. For $0 \leq \tau' < \tau$, the divergence theorem implies that:

$$\int_{\mathcal{R}(\tau', \tau)} \mathbf{D}^\alpha P_\alpha^{(N)}(u) dV_g + \int_{\mathcal{H}^+(\tau', \tau)} P_\alpha^{(N)}(u)n_{\mathcal{H}^+}^\alpha dS + \int_{\Sigma_\tau} P_\alpha^{(N)}(u)n^\alpha dS = \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u)n^\alpha dS. \tag{3.36}$$

Splitting up the integral over $\mathcal{R}(\tau', \tau)$ into pieces corresponding to $r_s \leq r \leq r_0^N$, $r_0^N \leq r \leq r_1^N$, and $r_1^N \leq r$, we can rewrite this expression as:

$$\begin{aligned}
&\int_{\Sigma_\tau} P_\alpha^{(N)}(u)n^\alpha dS + \int_{\mathcal{H}^+(\tau', \tau)} P_\alpha^{(N)}(u)n_{\mathcal{H}^+}^\alpha dS + \int_{\mathcal{R}(\tau', \tau) \cap \{r_s \leq r \leq r_0^N\}} \mathbf{D}^\alpha P_\alpha^{(N)}(u) dV_g \\
&= \int_{\mathcal{R}(\tau', \tau) \cap \{r_0^N \leq r \leq r_1^N\}} (-\mathbf{D}^\alpha P_\alpha^{(N)}(u)) dV_g + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u)n^\alpha dS.
\end{aligned} \tag{3.37}$$

Since it is non-negative, we may discard the integral along \mathcal{H}^+ , and furthermore add a large multiple of

$$\int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(N)}(u)n^\alpha dV_g \tag{3.38}$$

to both sides to obtain:

$$\begin{aligned} & \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS + \int_{\mathcal{R}(\tau', \tau) \cap \{r_s \leq r \leq r_0^N\}} \mathbf{D}^\alpha P_\alpha^{(N)}(u) dV_g + C \int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(N)}(u) n^\alpha dV_g \\ & \leq \int_{\mathcal{R}(\tau', \tau) \cap \{r_0^N \leq r \leq r_1^N\}} |\mathbf{D}^\alpha P_\alpha^{(N)}(u)| dV_g + C \int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(N)}(u) n^\alpha dV_g + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS. \end{aligned} \quad (3.39)$$

Noting that $N = T$ for $r \geq r_1^N$ and

$$P_\alpha^{(N)}(u) n^\alpha \approx P_\alpha^{(T)}(u) n^\alpha \quad (3.40)$$

for $r_0^N \leq r \leq r_1^N$, the right-hand-side of (3.39) is bounded by:

$$\begin{aligned} (RHS)_{(3.39)} & \leq C \int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(N)}(u) n^\alpha dV_g + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS \\ & \leq C \int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(T)}(u) n^\alpha dV_g + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS \\ & \leq C \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}} \cap \{r \geq r_0^N\}} P_\alpha^{(T)}(u) n^\alpha dS \right) d\bar{\tau} + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS \\ & \leq C \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}}} P_\alpha^{(T)}(u) n^\alpha dS \right) d\bar{\tau} + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS. \end{aligned} \quad (3.41)$$

Here we have used that the fact that

$$\int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(T)}(u) n^\alpha dS \approx_{N, r_0^N} \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}} \cap \{r \geq r_0^N\}} P_\alpha^{(T)}(u) n^\alpha dS \right) d\bar{\tau}. \quad (3.42)$$

Using the energy inequality (3.10) and the estimate

$$\int_{\Sigma_0} P_\alpha^{(T)}(u) n^\alpha dS \lesssim \int_{\Sigma_0} P_\alpha^{(N)}(u) n^\alpha dS, \quad (3.43)$$

we can further bound (3.41) by

$$\begin{aligned} (RHS)_{(3.39)} & \leq C(\tau - \tau') \left(\int_{\Sigma_0} P_\alpha^{(T)}(u) n^\alpha dS \right) + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS \\ & \leq C(\tau - \tau') \left(\int_{\Sigma_0} P_\alpha^{(N)}(u) n^\alpha dS \right) + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS. \end{aligned} \quad (3.44)$$

Now working with the left hand side of (3.39) and applying Theorem 3.1, we can bound from below:

$$\begin{aligned} & \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS + C \int_{\mathcal{R}(\tau', \tau) \cap \{r \geq r_0^N\}} P_\alpha^{(N)}(u) n^\alpha dV_g + \int_{\mathcal{R}(\tau', \tau) \cap \{r_s \leq r \leq r_0^N\}} \mathbf{D}^\alpha P_\alpha^{(N)}(u) dV_g \\ & \geq \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS + B \int_{\mathcal{R}(\tau', \tau)} P_\alpha^{(N)}(u) n^\alpha dV_g \\ & \geq \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS + B \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}}} P_\alpha^{(N)}(u) n^\alpha dS \right) d\bar{\tau}, \end{aligned} \quad (3.45)$$

for some constant $B > 0$. The last inequality follows from

$$\int_{\mathcal{R}(\tau', \tau)} P_\alpha^{(N)}(u) n^\alpha dV_g \approx \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}}} P_\alpha^{(N)}(u) n^\alpha dS \right) d\bar{\tau}, \quad (3.46)$$

which holds because N is timelike everywhere, including on \mathcal{H}^+ .

Combining (3.44) and (3.45), we have:

$$\begin{aligned} & \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS + B \int_{\tau'}^\tau \left(\int_{\Sigma_{\bar{\tau}}} P_\alpha^{(N)}(u) n^\alpha dS \right) d\bar{\tau} \\ & \leq C(\tau - \tau') \left(\int_{\Sigma_0} P_\alpha^{(N)}(u) n^\alpha dS \right) + \int_{\Sigma_{\tau'}} P_\alpha^{(N)}(u) n^\alpha dS. \end{aligned} \quad (3.47)$$

In other words, if we denote

$$g(\tau) = \int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS, \quad (3.48)$$

then we have

$$g(\tau) + B \int_{\tau'}^\tau g(\bar{\tau}) d\bar{\tau} \leq D(\tau - \tau')g(0) + g(\tau'). \quad (3.49)$$

Rearranging these terms gives

$$\frac{g(\tau) - g(\tau')}{\tau - \tau'} + \frac{B}{\tau - \tau'} \int_{\tau'}^\tau g(\bar{\tau}) d\bar{\tau} \leq Dg(0), \quad (3.50)$$

and taking the limit as $\tau' \rightarrow \tau$ gives

$$g'(\tau) + Bg(\tau) \leq Dg(0). \quad (3.51)$$

Multiply both sides of the equation by $e^{B\tau}$ to get

$$\frac{d}{d\tau} \left(e^{B\tau} g(\tau) - \frac{D}{B} e^{B\tau} g(0) \right) \leq 0, \quad (3.52)$$

and hence

$$g(\tau) \leq \left(\frac{D}{B} - \frac{D}{B} e^{-B\tau} + e^{-B\tau} \right) g(0) \lesssim g(0) \quad (3.53)$$

for all $\tau \geq 0$. Finally, we have obtained boundedness of the non-degenerate energy,

$$\int_{\Sigma_\tau} P_\alpha^{(N)}(u) n^\alpha dS \lesssim \int_{\Sigma_0} P_\alpha^{(N)}(u) n^\alpha dS, \quad (3.54)$$

as desired. \square

Localized Energy Estimates in Minkowski Spacetime

In this chapter, we consider the wave equation

$$\square_m u = 0$$

in Minkowski space $\mathbb{R} \times \mathbb{R}^n$ and prove various localized energy estimates. Here m stands for the Minkowski metric, which is expressed in polar coordinates (t, r, ω) by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{\mathbb{S}^{n-1}}^2.$$

In the following, we refer to the standard energy $E(t)$ as

$$E[u](t) := \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla u|^2 dx, \quad (4.1)$$

which is conserved along $t = \text{constant}$ slices if $\square_m u = 0$. The energy conservation is deduced by integrating $-(\partial_t u)(\square_m u)$ over a space-time slab $[0, T] \times \mathbb{R}^n$ and integrating by parts; indeed, we have:

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^n} (\partial_t u)(\partial_t^2 u - \Delta u) dx = \int_0^T \int_{\mathbb{R}^n} \partial_t u \partial_t^2 u + \nabla u \cdot \nabla \partial_t u dx \\ &= \int_0^T \int_{\mathbb{R}^n} \frac{1}{2} \partial_t (\partial_t u)^2 + \frac{1}{2} \partial_t |\nabla u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla u|^2 dx \Big|_0^T = E[u](T) - E[u](0), \end{aligned}$$

given that say, $u(t, \cdot)$ vanishes for $|x|$ large for each $t \in [0, T]$. The conservation of energy is closely related to the fact that ∂_t is a Killing vector field for the Minkowski metric. We will employ a related technique known as the *positive commutator method*. Essentially, one integrates $\square u$ against a carefully chosen multiplier and integrates by parts. By analyzing the resulting space-time integral, one ideally bounds the gradient $\nabla_{t,x} u$ in a weighted $L_t^2 L_x^2$ norm by the initial energy $E[u](0)$ to obtain a *localized energy estimate*.

In what follows, we will use the divergence theorem rather than integrating by parts directly, as this extends in a clean way to curved space-times. The wave operator \square_m has a naturally associated stress-energy tensor,

$$Q_{\alpha\beta}(u) := \partial_\alpha u \partial_\beta u - \frac{1}{2} m_{\alpha\beta} \partial^\gamma u \partial_\gamma u, \quad (4.2)$$

which is a symmetric 2-tensor satisfying the divergence identity

$$\mathbf{D}^\alpha Q_{\alpha\beta}(u) = (\square_m u)(\partial_\beta u).$$

For a vector field X , we define the corresponding momentum density,

$$P_\alpha^{(X)}(u) = Q_{\alpha\beta}(u) X^\beta,$$

whose divergence we calculate:

$$\mathbf{D}^\alpha P_\alpha^{(X)}(u) = X^\beta \mathbf{D}^\alpha Q_{\alpha\beta}(u) + Q_{\alpha\beta}(u) \mathbf{D}^\alpha X^\beta = (Xu)(\square_m u) + Q_{\alpha\beta}^{(X)} \pi^{\alpha\beta}.$$

In this notation, ${}^{(X)}\pi$ denotes the *deformation tensor* of the vector field X , defined by

$${}^{(X)}\pi_{\alpha\beta} = \frac{1}{2} (\mathbf{D}_\alpha X_\beta + \mathbf{D}_\beta X_\alpha),$$

and roughly measures the degree to which X fails to generate isometries of the space-time. To derive the localized energy estimates, we use a radial multiplier $X = f(r)\partial_r$. To this end, we briefly calculate the deformation tensor of $R = \partial_r$, and note that the only non-zero components are

$${}^{(R)}\pi(E_j, E_j) = \frac{1}{r}.$$

The E_1, \dots, E_{n-1} are an orthonormal basis tangent to spheres defined by normalizing the coordinate vector fields $\partial_{\theta_1}, \dots, \partial_{\theta_{n-1}}$, i.e. for each j ,

$$E_j := \frac{1}{r \sin \theta_1 \cdots \sin \theta_{j-1}} \frac{\partial}{\partial \theta_j}.$$

Hence, the divergence of the corresponding momentum density $P_\alpha^{(X)}(u)$ becomes:

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X)}(u) &= (Xu)(\square_m u) + f'(r)Q(R, R) + \frac{f(r)}{r} \sum_{j=1}^{n-1} Q(E_j, E_j) \\ &= (Xu)(\square_m u) + f'(\partial_r u)^2 + \frac{f}{r} |\nabla u|^2 - \left(\frac{f'}{2} + \frac{(n-1)}{2} f \right) \partial^\gamma u \partial_\gamma u. \end{aligned} \quad (4.3)$$

In this notation, $\nabla u = \nabla u - \frac{x}{r} \partial_r u$ denotes the angular derivative, so that $|\nabla u|^2 = (\partial_r u)^2 + |\nabla u|^2$.

Ideally, the function f is chosen so that the quadratic form $\mathbf{D}^\alpha P_\alpha^{(X)}$ is positive definite, though this is usually not possible without some modification. Instead, we seek a multiplier f such that the symbol of $\mathbf{D}^\alpha P_\alpha^{(X)}$ is positive on the characteristic set of \square_m , and then make the resulting terms positive by altering the energy-momentum vector by a lower order term. For a function q , we define a modified momentum density by

$$P_\alpha^{(X,q)}(u) = Q_{\alpha\beta}(u)X^\beta + qu\partial_\alpha u - \frac{1}{2}\partial_\alpha qu^2, \quad (4.4)$$

whose divergence is

$$\mathbf{D}^\alpha P_\alpha^{(X,q)}(u) = \mathbf{D}^\alpha P_\alpha^{(X)}(u) + qu(\square_m u) + q\partial^\gamma u \partial_\gamma u - \frac{1}{2}(\square_m q)u^2 =: (Xu + qu)(\square_m u) + Q^{(X,q)}(u). \quad (4.5)$$

As a simple application, we prove the classical result of Morawetz [47].

THEOREM 4.1. *Let u be a solution to the wave equation $\square_m u = 0$ such that $u(t, \cdot)$ vanishes for large $|x|$ for every $t \in [0, T]$. Then it holds that*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} \frac{1}{r} |\nabla u|^2 dx dt + \int_0^T u^2(t, 0) dt &\lesssim E(0) \quad \text{when } n = 3, \\ \int_0^T \int_{\mathbb{R}^n} \frac{1}{r} |\nabla u|^2 dx dt + \int_0^T \int_{\mathbb{R}^n} \frac{1}{r^3} u^2 dx dt &\lesssim E(0) \quad \text{when } n \geq 4, \end{aligned} \quad (4.6)$$

where the implicit constant is independent of $T > 0$.

PROOF. We set $f = 1$ and $q = \left(\frac{n-1}{2}\right) \frac{1}{r}$. When $n = 3$ we note that the function $1/r$ is, up to a constant, the fundamental solution to Laplace's equation $-\Delta u = 0$, so we obtain

$$-\Delta \frac{1}{r} = c\delta_0, \quad c > 0, \quad (4.7)$$

where δ_0 is the Dirac delta distribution centered at the origin. Integrating over a time slab $[0, T] \times \mathbb{R}^3$ and applying the divergence theorem yields

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^3} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) dx dt + \int_{\mathbb{R}^3} P_\alpha^{(X,q)}(u) T^\alpha dx dt \Big|_0^T \\ &= \int_0^T \int_{\mathbb{R}^3} \frac{1}{r} |\nabla u|^2 dx dt + c \int_0^T u^2(t, 0) dt + \int_{\mathbb{R}^3} \left(\partial_t u \partial_r u + \frac{u}{r} \partial_t u \right) dx \Big|_0^T \end{aligned} \quad (4.8)$$

In the case when $n > 3$, we get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^n} \frac{1}{r} |\nabla u|^2 dx + \frac{(n-1)(n-3)}{4} \int_0^T \int_{\mathbb{R}^n} \frac{u^2}{r^3} dx dt \\ &\quad + \int_{\mathbb{R}^n} \partial_t u \partial_r u dx \Big|_0^T + \frac{n-1}{2} \int_{\mathbb{R}^n} \frac{u}{r} \partial_t u dx \Big|_0^T. \end{aligned} \quad (4.9)$$

In either case, we can easily control:

$$\left| \int \partial_t u \partial_r u dx \right| \leq \frac{1}{2} \int ((\partial_t u)^2 + (\partial_r u)^2) dx \leq E[u](t) = E[u](0), \quad (4.10)$$

using the conservation of energy. This implies that

$$\left| \int_{\mathbb{R}^n} \partial_t u \partial_r u dx \Big|_0^T \right| \lesssim E[u](0). \quad (4.11)$$

The remaining part is estimated similarly using a Hardy inequality, which is stated below.

LEMMA 4.2 (Hardy inequality). *Let $u \in H^1(\mathbb{R}^n)$ with $n \geq 3$; then*

$$\int_{\mathbb{R}^n} \frac{u^2}{r^2} dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} (\partial_r u)^2 dx. \quad (4.12)$$

PROOF. Let us first assume that $u \in C_c^\infty(\mathbb{R}^n)$, and consider

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{u^2}{r^2} dx &= \int_0^\infty \int_{\mathbb{S}^{n-1}} u^2 \frac{\partial_r(r^{n-2})}{n-2} dr d\omega = -\frac{2}{n-2} \int_{\mathbb{R}^n} \frac{u}{r} \partial_r u dx \\ &\leq \frac{2}{n-2} \left(\int_{\mathbb{R}^n} \frac{u^2}{r^2} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (\partial_r u)^2 dx \right)^{\frac{1}{2}}, \end{aligned} \quad (4.13)$$

by the Cauchy-Schwarz inequality. While we integrated by parts in the first line, no boundary terms are picked up as we assume that $u \in C_c^\infty(\mathbb{R}^n)$, which is bounded near 0 and vanishes at infinity. Hence, we obtain

$$\int_{\mathbb{R}^n} \frac{u^2}{r^2} dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} (\partial_r u)^2 dx \quad (4.14)$$

for $u \in C_c^\infty(\mathbb{R}^n)$. Now for $u \in H^1(\mathbb{R}^n)$, take a sequence $\{u_j\} \subset C_c^\infty(\mathbb{R}^n)$ with $u_j \rightarrow u$ in the H^1 norm. The above inequality holds for each u_j , but passing the limit to u is non-trivial. Indeed, let $\epsilon > 0$ and consider now

$$\int_{\mathbb{R}^n} \frac{u_j^2}{r^2 + \epsilon^2} dx \leq \int_{\mathbb{R}^n} \frac{u_j^2}{r^2} dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} (\partial_r u_j)^2 dx. \quad (4.15)$$

For a fixed $\epsilon > 0$, we have $1/(r^2 + \epsilon^2) \lesssim 1$, so the convergence in H^1 norm gives us

$$\int_{\mathbb{R}^n} \frac{u^2}{r^2 + \epsilon^2} dx \leq \left(\frac{2}{n-2} \right)^2 \int_{\mathbb{R}^n} (\partial_r u)^2 dx \quad (4.16)$$

for each $\epsilon > 0$. In particular, the bounding constant is independent of ϵ , so we can pass the limit to $\epsilon \searrow 0$ to obtain the desired result. \square

Finally, the Hardy inequality allows us to bound:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \frac{u}{r} \partial_t u dx \right| &\leq \left(\int_{\mathbb{R}^n} \frac{u^2}{r^2} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (\partial_t u)^2 dx \right)^{\frac{1}{2}} \lesssim \left(\int_{\mathbb{R}^n} (\partial_r u)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (\partial_t u)^2 dx \right)^{\frac{1}{2}} \\ &\lesssim E[u](t) = E[u](0). \end{aligned}$$

which completes the proof of the estimate. \square

We now use a different radial multiplier that is used to derive a series of more general estimates. In particular, we can control the full space-time gradient $\nabla_{t,x} u$ in a bounded region.

THEOREM 4.3. *Let u be a solution to the wave equation $\square_m u = 0$ with initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$; then u satisfies the estimate*

$$\|\nabla_{t,x} u\|_{L_{t,x}^2([0,T] \times \{|x| \leq 1\})}^2 + \|u\|_{L_{t,x}^2([0,T] \times \{|x| \leq 1\})}^2 \lesssim E[u](0). \quad (4.17)$$

PROOF. Consider $f(r) = \frac{r}{r+1}$ and $q = \left(\frac{n-1}{2}\right) \frac{f}{r}$, and calculate

$$-\frac{1}{2}\square_m q = -\frac{n-1}{4}\Delta_r \frac{f}{r} = \frac{(n-1)(n-3)}{4} \frac{1}{(r+1)^3} + \frac{(n-1)^2}{4} \frac{1}{r(r+1)^3}. \quad (4.18)$$

This implies that $\mathbf{D}^\alpha P_\alpha^{(X,q)}(u)$ has the form

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) &= \frac{f'}{2}(\partial_t u)^2 + \frac{f'}{2}(\partial_r u)^2 + \left(\frac{f}{r} - \frac{f'}{2}\right)|\nabla u|^2 - \frac{1}{2}(\square_m q)u^2 \\ &= \frac{1}{2} \frac{1}{(r+1)^2}(\partial_t u)^2 + \frac{1}{2} \frac{1}{(r+1)^2}(\partial_r u)^2 + \left(\frac{1}{r+1} - \frac{1}{2} \frac{1}{(r+1)^2}\right)|\nabla u|^2 \\ &\quad + \left(\frac{(n-1)(n-3)}{4} \frac{1}{(r+1)^3} + \frac{(n-1)^2}{4} \frac{1}{r(r+1)^3}\right)u^2, \end{aligned} \quad (4.19)$$

wherein all of the coefficients are positive, so we can bound from below:

$$\int_0^T \int_{\mathbb{R}^n} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) dx dt \geq \int_0^T \int_{\{|x| \leq 1\}} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) dx dt \gtrsim \int_0^T \int_{\{|x| \leq 1\}} |\nabla_{t,x} u|^2 + u^2 dx dt. \quad (4.20)$$

We also have, by Cauchy's inequality and Hardy's inequality:

$$\begin{aligned} \int_{\mathbb{R}^n} P_\alpha^{(X,q)}(u) T^\alpha dx &= \int_{\mathbb{R}^n} \left(\frac{r}{r+1} \partial_t u \partial_r u + \left(\frac{n-1}{2}\right) \frac{1}{r+1} u \partial_t u \right) dx \\ &\lesssim \int_{\mathbb{R}^n} |\nabla_{t,x} u|^2 + \frac{u^2}{r^2} dx \lesssim \int_{\mathbb{R}^n} |\nabla_{t,x} u|^2 dx \approx E[u](t) = E[u](0). \end{aligned} \quad (4.21)$$

Applying this bound with the divergence theorem now yields

$$E[u](0) \gtrsim \int_0^T \int_{\mathbb{R}^n} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) dx dt \gtrsim \int_0^T \int_{\{|x| \leq 1\}} |\nabla_{t,x} u|^2 + u^2 dx dt, \quad (4.22)$$

which proves the desired estimate. \square

To derive a more precise estimate globally in space, we define the local energy space LE_M with norm

$$\|u\|_{LE_M} := \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|u\|_{L^2_{t,x}(\mathbb{R} \times A_j)}, \quad (4.23)$$

where $A_j = \{2^{j-1} \leq |x| \leq 2^j\}$ for $j \in \mathbb{Z}$. We define the H^1 counterpart LE_M^1 ,

$$\|u\|_{LE_M^1} := \|\nabla_{t,x} u\|_{LE_M} + \| |x|^{-1} u \|_{LE_M}. \quad (4.24)$$

To control an inhomogenous term $\square_m u = F$, we introduce a dual norm,

$$\|F\|_{LE_M^*} := \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|F\|_{L^2_{t,x}(\mathbb{R} \times A_j)}. \quad (4.25)$$

Such a framework was introduced in [43] and allows us to formulate a stronger and more precise estimate, though these types of bounds have appeared in, for example, [58], [32], [54], [31], [45], [13], [28], [57], [40], [41], [42].

THEOREM 4.4. *Let u be a solution to the wave equation $\square_m u = F$ with initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then it holds that*

$$\|\nabla_{t,x} u\|_{L^\infty_t L^2_x}^2 + \|u\|_{LE_M^1}^2 \lesssim E[u](0) + \|F\|_{LE_M^* + L^1_t L^2_x}^2. \quad (4.26)$$

PROOF. We will modify the proof of Theorem 4.3 by choosing $f(r) = \frac{r}{r+R}$ and $q = \left(\frac{n-1}{2}\right) \frac{f}{r}$ for a fixed radius $R > 0$. We record that

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) &= \frac{f'}{2}(\partial_t u)^2 + \frac{f'}{2}(\partial_r u)^2 + \left(\frac{f}{r} - \frac{f'}{2}\right)|\nabla u|^2 - \frac{n-1}{4}\square_m \left(\frac{f}{r}\right)u^2 \\ &\quad + f \partial_r u \square_m u + \frac{n-1}{2} \frac{f}{r} u \square_m u \\ &= Q^{(X,q)}(u) + f \partial_r u \square_m u + \frac{n-1}{2} \frac{f}{r} u \square_m u. \end{aligned} \quad (4.27)$$

Integrating over the space-time slab $[0, T] \times \mathbb{R}^n$ and applying the divergence theorem, we obtain:

$$0 = \int_0^T \int_{\mathbb{R}^n} Q^{(X,q)}(u) dx dt + \int_0^T \int_{\mathbb{R}^n} \left(f \partial_r u + \frac{n-1}{2} \frac{f}{r} u \right) \square_m u dx dt + \int_{\mathbb{R}^n} P_\alpha^{(X,q)}(u) T^\alpha dx \Big|_0^T. \quad (4.28)$$

Note that

$$f > 0, \quad f' > 0, \quad \frac{f}{r} - \frac{f'}{2} > 0, \\ -\frac{1}{2} \square_m q = \frac{(n-1)(n-3)}{4} \frac{1}{(r+R)^3} + \frac{(n-1)^2}{4} \frac{R}{r(r+R)^3} > 0,$$

and moreover that $f \leq 1$ independently of $R > 0$. Restricting to the annulus $R/2 \leq |x| \leq 2R$, it holds that

$$f' \gtrsim \frac{1}{r}, \quad \frac{f}{r} - \frac{f'}{2} \gtrsim \frac{1}{r}, \quad -\square_m q \gtrsim \frac{1}{r^3}, \quad (4.29)$$

independently of $R > 0$. Thus, we can estimate

$$\int_0^T \int_{\mathbb{R}^n} Q^{(X,q)}(u) dx dt \gtrsim \int_0^T \int_{\frac{R}{2} \leq |x| \leq 2R} |x|^{-1} |\nabla_{t,x} u|^2 + |x|^{-3} u^2 dx dt. \quad (4.30)$$

We next control the time boundary terms. Note that for the inhomogeneous equation $\square_m u = F$, the energy $E[u](t)$ is not conserved, so we consider the energy identity,

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla_{t,x} u(T, \cdot)|^2 dx + \int_0^T \int_{\mathbb{R}^n} \partial_t u \square_m u dx dt = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_{t,x} u(0, \cdot)|^2 dx. \quad (4.31)$$

To bound the inhomogeneous term in the appropriate $LE_M^* + L_t^1 L_x^2$ norm, we take an arbitrary decomposition $F = G + H$ with $G \in LE_M^*$ and $H \in L_t^1 L_x^2$, and write

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \partial_t u \square_m u dx dt &\leq \int_0^T \int_{\mathbb{R}^n} |\nabla_{t,x} u| |G| dx dt + \int_0^T \int_{\mathbb{R}^n} |\nabla_{t,x} u| |H| dx dt \\ &\leq \sum_{k \in \mathbb{Z}} \int_0^T \int_{A_k} |\nabla_{t,x} u| |G| dx dt + \int_0^T \|\nabla_{t,x} u\|_{L^2} \|H\|_{L^2} dt \\ &\leq \sum_{k \in \mathbb{Z}} \|\nabla_{t,x} u\|_{L_{t,x}^2(\mathbb{R} \times A_k)} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} \\ &= \sum_{k \in \mathbb{Z}} 2^{-\frac{k}{2}} \|\nabla_{t,x} u\|_{L_{t,x}^2(\mathbb{R} \times A_k)} 2^{\frac{k}{2}} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} \\ &\leq \left(\sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|\nabla_{t,x} u\|_{L_{t,x}^2(\mathbb{R} \times A_j)} \right) \left(\sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} \right) + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} \\ &= \|\nabla_{t,x} u\|_{LE_M} \|G\|_{LE_M^*} + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} \\ &\leq \epsilon \|u\|_{LE_M^1}^2 + \epsilon' \|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + C_{\epsilon, \epsilon'} \left(\|G\|_{LE_M^*}^2 + \|H\|_{L_t^1 L_x^2}^2 \right), \end{aligned} \quad (4.32)$$

for any $\epsilon, \epsilon' > 0$. The $\|G\|_{LE_M^*}^2 + \|H\|_{L_t^1 L_x^2}^2$ norms can be replaced with the $\|F\|_{LE_M^* + L_t^1 L_x^2}^2$ norm as the decomposition $F = G + H$ is arbitrary and the bounds are independent of G, H . Taking $\epsilon' \ll 1$ sufficiently small then yields a variant of the elementary energy inequality,

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 \lesssim E[u](0) + \epsilon \|u\|_{LE_M^1}^2 + C_\epsilon \|F\|_{LE_M^* + L_t^1 L_x^2}^2. \quad (4.33)$$

This allows us to estimate

$$\int_{\mathbb{R}^n} P_\alpha^{(X,q)}(u) T^\alpha dx \lesssim \|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 \lesssim E[u](0) + \epsilon \|u\|_{LE_M^1}^2 + C_\epsilon \|F\|_{LE_M^* + L_t^1 L_x^2}^2. \quad (4.34)$$

We now deal with the inhomogeneous terms. The boundedness of $0 < f \leq 1$ independently of $R > 0$ and Hardy's inequality allow us to write:

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^n} \frac{f}{r} u \square_m u dx dt &\leq \int_0^T \int_{\mathbb{R}^n} |x|^{-1} |u| |G| + |x|^{-1} |u| |H| dx dt \\
&\leq \sum_{k \in \mathbb{Z}} \int_0^T \int_{A_k} |x|^{-1} |u| |G| dx dt + \int_0^T \int_{\mathbb{R}^n} |x|^{-1} |u| |H| dx dt \\
&\leq \sum_{k \in \mathbb{Z}} \left\| |x|^{-1} u \right\|_{L_{t,x}^2(\mathbb{R} \times A_k)} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} + \int_0^T \left\| |x|^{-1} u \right\|_{L^2} \|H\|_{L^2} dt \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{-\frac{k}{2}} \left\| |x|^{-1} u \right\|_{L_{t,x}^2(\mathbb{R} \times A_k)} 2^{\frac{k}{2}} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} + \int_0^T \|\nabla_{t,x} u\|_{L^2} \|H\|_{L^2} dt \\
&\leq \left(\sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \left\| |x|^{-1} u \right\|_{L_{t,x}^2(\mathbb{R} \times A_j)} \right) \left(\sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \|G\|_{L_{t,x}^2(\mathbb{R} \times A_k)} \right) + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} dt \\
&= \left\| |x|^{-1} u \right\|_{LE_M} \|G\|_{LE_M^*} + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2} \|H\|_{L_t^1 L_x^2} \\
&\leq \epsilon \|u\|_{LE_M^1}^2 + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + C_\epsilon \left(\|G\|_{LE_M^*}^2 + \|H\|_{L_t^1 L_x^2}^2 \right),
\end{aligned} \tag{4.35}$$

where again the $\|G\|_{LE_M^*}^2 + \|H\|_{L_t^1 L_x^2}^2$ norms can be replaced with the $\|F\|_{LE_M^* + L_t^1 L_x^2}^2$ norm. Combining with the energy inequality (4.33), we obtain

$$\int_0^T \int_{\mathbb{R}^n} \frac{f}{r} u \square_m u dx dt \leq E[u](0) + \epsilon \|u\|_{LE_M^1}^2 + C_\epsilon \|F\|_{LE_M^* + L_t^1 L_x^2}. \tag{4.36}$$

We similarly bound

$$\int_0^T \int_{\mathbb{R}^n} f \partial_r u \square_m u dx dt \leq \int_0^T \int_{\mathbb{R}^n} |\nabla_{t,x} u| |F| dx dt \lesssim E[u](0) + \epsilon \|u\|_{LE_M^1}^2 + \|F\|_{LE_M^* + L_t^1 L_x^2}^2. \tag{4.37}$$

Combining the bulk term estimate (4.30) with the energy inequality (4.33) and inhomogeneous estimates (4.36), (4.37) with the expression (4.28) yields:

$$\begin{aligned}
\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \int_0^T \int_{\frac{R}{2} \leq |x| \leq R} |x|^{-1} |\nabla_{t,x} u|^2 + |x|^{-3} u^2 dx \\
\lesssim E[u](0) + \epsilon \|u\|_{LE_M^1}^2 + C_\epsilon \|\square_m u\|_{LE_M^* + L_t^1 L_x^2}^2
\end{aligned} \tag{4.38}$$

for any $\epsilon > 0$, independently of $R > 0$. In particular, taking the supremum over $R = 2^j$ for $j \in \mathbb{Z}$ and for $\epsilon \ll 1$ sufficiently small, we obtain

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \|u\|_{LE_M^1}^2 \lesssim E[u](0) + \|\square_m u\|_{LE_M^* + L_t^1 L_x^2}^2, \tag{4.39}$$

which concludes the proof of the theorem. \square

We remark that for dimensions $n \geq 4$, by combining our estimate with (4.1) the local energy norm LE_M can be strengthened to

$$\|u\|_{LE_M^1} := \|\nabla_{t,x} u\|_{LE_M} + \left\| |x|^{-\frac{3}{2}} u \right\|_{L_{t,x}^2}.$$

Let us also make a final remark on the dual norm $\|F\|_{LE_M^* + L_t^1 L_x^2}$. This is a natural norm in which to measure the inhomogeneity F , as we assume that the solution u is contained both in the finite energy class $L_t^\infty L_x^2$ and the finite local energy class LE_M^1 . As a result, the corresponding dual space takes the form $L_t^1 L_x^2 + LE_M^*$ which, in particular, allows one to control F in either of the norms.

1. Perturbations of the Minkowski Metric

The above proof can be adapted easily to variable coefficient perturbations of the Minkowski metric. To be precise, let us consider solutions u to the equation

$$\square_h u := \mathbf{D}_\alpha (m^{\alpha\beta} + h^{\alpha\beta}) \partial_\beta u = 0, \quad (4.40)$$

where m is the Minkowski metric as before, and $h^{\alpha\beta}$ is a symmetric 2-tensor field. This is a slight abuse of notation, as \square_h usually refers to the geometric operator $\mathbf{D}^\alpha \partial_\alpha$ with respect to the metric h . We instead use the Minkowski metric m to raise and lower indices, as well as to define the covariant derivative \mathbf{D} . We follow the approach of Metcalfe-Sogge in [40] and modify the proof of Theorem 4.4 to account for error terms. We require from the onset that $h^{\alpha\beta} \in C^1(\mathbb{R} \times \mathbb{R}^n)$, and we denote for simplicity,

$$|h| := \sum_{\alpha,\beta} |h^{\alpha\beta}|, \quad |\partial h| := \sum_{\alpha,\beta,\gamma} |\mathbf{D}_\gamma h^{\alpha\beta}|. \quad (4.41)$$

Within the framework of the local energy norm LE_M^1 , it is not quite enough to assume that h is bounded and sufficiently small in the sense that $|h|, |\partial h| \leq \epsilon \ll 1$, but additional flatness conditions are required. The main theorem is below and appears in a slightly different form than in [40].

THEOREM 4.5. *Let u solve the perturbed wave equation*

$$\square_h u = F, \quad (4.42)$$

where h is small and asymptotically flat in the sense that

$$\sum_{k \in \mathbb{Z}} \sup_{2^{k-1} \leq |x| \leq 2^k} (|h| + |x| |\partial h|) \leq \epsilon_h \quad (4.43)$$

for $\epsilon_h \ll 1$ sufficiently small. Then u satisfies the estimate

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \|u\|_{LE_M^1}^2 \lesssim E[u](0) + \|F\|_{LE_M^* + L_t^1 L_x^2}^2. \quad (4.44)$$

PROOF. Define the modified stress-energy tensor,

$$\begin{aligned} Q_{\alpha\beta}^h(u) &:= Q_{\alpha\beta}(u) + h_\alpha^\mu \partial_\mu u \partial_\beta u - \frac{1}{2} m_{\alpha\beta} (h^{\mu\nu} \partial_\mu u \partial_\nu u) \\ &= (\partial_\alpha u + h_\alpha^\mu \partial_\mu u) \partial_\beta u - \frac{1}{2} m_{\alpha\beta} (m^{\mu\nu} + h^{\mu\nu}) \partial_\mu u \partial_\nu u, \end{aligned} \quad (4.45)$$

and calculate its divergence,

$$\mathbf{D}^\alpha Q_{\alpha\beta}^h(u) = \partial_\beta u \square_h u - \frac{1}{2} (\mathbf{D}_\beta h^{\mu\nu}) \partial_\mu u \partial_\nu u. \quad (4.46)$$

As before, let us take $X = f(r) \partial_r$ and define the corresponding momentum density vector,

$$P_\alpha^{(X)}(u) = Q_{\alpha\beta}^h(u) X^\beta. \quad (4.47)$$

The modified stress-energy $Q_{\alpha\beta}^h$ is unfortunately not entirely symmetric, so we need to take care in taking the divergence of $P^{(X)}$. Let us calculate

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X)}(u) &= (Xu)(\square_h u) + (\mathbf{D}^\alpha X^\beta)(\partial_\alpha u + h_\alpha^\mu \partial_\mu u) \partial_\beta u - \frac{1}{2} (\mathbf{D}_X h^{\alpha\beta}) \partial_\alpha u \partial_\beta u \\ &\quad - \frac{1}{2} m_{\alpha\beta} (\mathbf{D}^\alpha X^\beta) (m^{\mu\nu} + h^{\mu\nu}) \partial_\mu u \partial_\nu u. \end{aligned} \quad (4.48)$$

Note that for $X = f(r) \partial_r$, we can use the exterior derivative on differential forms to write

$$\mathbf{D}_{(\alpha} X_{\beta)} = (d(f(r) dr))_{\alpha\beta} = 0,$$

so in fact we have

$$\mathbf{D}_\alpha X_\beta = {}^{(X)}\pi_{\alpha\beta}. \quad (4.49)$$

To eliminate the final ‘‘Lagrangian’’ term, let us modify the energy-momentum as before, defining

$$P_\alpha^{(X,q)}(u) = Q_{\alpha\beta}^h(u) X^\beta + qu (\partial_\alpha u + h_\alpha^\mu \partial_\mu u) - \frac{1}{2} (\partial_\alpha q) u^2, \quad (4.50)$$

and calculate its divergence as

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) &= \mathbf{D}^\alpha P_\alpha^{(X)}(u) + q(m^{\alpha\beta} + h^{\alpha\beta}) \partial_\alpha u \partial_\beta u + qu(\square_h u) \\ &\quad + u(\partial^\alpha q)(h_\alpha^\mu \partial_\mu u) - \frac{1}{2}(\square_m q)u^2 \end{aligned} \quad (4.51)$$

We again set $f = \frac{r}{r+R}$ for a fixed radius $R > 0$ and $q = (\frac{n-1}{2}) \frac{f}{r}$. With (4.49), we then have:

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X,q)}(u) &= (Xu + qu)(\square_h u) + {}^{(X)}\pi^{\alpha\beta} (\partial_\alpha u + h_\alpha^\mu \partial_\mu u) \partial_\beta u \\ &\quad - \frac{1}{2}(\mathbf{D}_X h^{\alpha\beta}) \partial_\alpha u \partial_\beta u - \frac{f'}{2}(m^{\alpha\beta} + h^{\alpha\beta}) \partial_\alpha u \partial_\beta u \\ &\quad - \left(\frac{n-1}{4}\right) \left(\square_m \frac{f}{r}\right) u^2 - \frac{n-1}{2} \frac{1}{(r+R)^2} u h_r^\alpha \partial_\alpha u \\ &= (Xu + qu)(\square_h u) + \frac{f'}{2}(\partial_t u)^2 + \frac{f'}{2}(\partial_r u)^2 + \left(\frac{f}{r} - \frac{f'}{2}\right) |\nabla u|^2 - \frac{n-1}{4} \left(\square_m \frac{f}{r}\right) u^2 \\ &\quad + f'(\partial_r u) h_r^\alpha \partial_\alpha u + \frac{f}{r} \sum_{j=1}^{n-1} (E_j u) h_{E_j}^\alpha \partial_\alpha u - \frac{1}{2}(\mathbf{D}_X h^{\alpha\beta}) \partial_\alpha u \partial_\beta u \\ &\quad - \frac{f'}{2} h^{\alpha\beta} \partial_\alpha u \partial_\beta u - \frac{n-1}{2} \frac{1}{(r+R)^2} u h_r^\alpha \partial_\alpha u \\ &=: (Xu + qu)(\square_h u) + Q_h^{(X,q)}(u) = (Xu + qu) + Q_m^{(X,q)}(u) + \text{error}, \end{aligned} \quad (4.52)$$

where we denote

$$\begin{aligned} Q_m^{(X,q)}(u) &= \frac{f'}{2}(\partial_t u)^2 + \frac{f'}{2}(\partial_r u)^2 + \left(\frac{f}{r} - \frac{f'}{2}\right) |\nabla u|^2 - \frac{n-1}{4} \left(\square_m \frac{f}{r}\right) u^2 \\ \text{error} &= f'(\partial_r u) h_r^\alpha \partial_\alpha u + \frac{f}{r} \sum_{j=1}^{n-1} (E_j u) h_{E_j}^\alpha \partial_\alpha u - \frac{1}{2}(\mathbf{D}_X h^{\alpha\beta}) \partial_\alpha u \partial_\beta u \\ &\quad - \frac{f'}{2} h^{\alpha\beta} \partial_\alpha u \partial_\beta u - \frac{n-1}{2} \frac{1}{(r+R)^2} u h_r^\alpha \partial_\alpha u. \end{aligned}$$

Integrating over a space-time slab $[0, T] \times \mathbb{R}^n$ and applying the divergence theorem, we have:

$$0 = \int_0^T \int_{\mathbb{R}^n} Q_m^{(X,q)}(u) dx dt + \int_0^T \int_{\mathbb{R}^n} \text{error} dx dt + \int_{\mathbb{R}^n} P_\alpha^{(X,q)}(u) T^\alpha dx \Big|_0^T. \quad (4.53)$$

From the proof of Theorem 4.4, we can bound:

$$\int_0^T \int_{\mathbb{R}^n} Q_m^{(X,q)}(u) dx dt \gtrsim \int_0^T \int_{\frac{R}{2} \leq |x| \leq R} |x|^{-1} |\nabla_{t,x} u|^2 + |x|^{-3} u^2 dx dt. \quad (4.54)$$

We will now show that after taking the supremum over $R = 2^j$ for $j \in \mathbb{Z}$, the error terms can be absorbed by $\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \|u\|_{L_{EM}^1}^2$ for $\epsilon_h \ll 1$ sufficiently small.

Term by term, we bound

$$\begin{aligned} |f' \partial_r u h_r^\alpha \partial_\alpha u| &\lesssim \frac{|h|}{r+R} |\nabla_{t,x} u|^2, \\ \left| \frac{f}{r} \sum (E_j u) h_{E_j}^\alpha \partial_\alpha u \right| &\lesssim \frac{|h|}{r+R} |\nabla_{t,x} u|^2, \\ \left| \frac{f'}{2} h^{\alpha\beta} \partial_\alpha u \partial_\beta u \right| &\lesssim \frac{|h|}{r+R} |\nabla_{t,x} u|^2, \\ \left| \frac{1}{2}(\mathbf{D}_X h^{\alpha\beta}) \partial_\alpha u \partial_\beta u \right| &\lesssim |\partial h| |\nabla_{t,x} u|^2, \\ \left| \frac{1}{(r+R)^2} u h_r^\alpha \partial_\alpha u \right| &\lesssim \frac{|h|}{r+R} |\nabla_{t,x} u|^2 + \frac{|h|}{(r+R)^3} u^2. \end{aligned} \quad (4.55)$$

To absorb each of these terms within the LE_M framework, we see that it is not quite enough to take $|h|, |\partial h| \ll 1$ sufficiently small. Instead, we split up the integral into a dyadic sum and use the LE_M and LE_M^* norms to bound appropriately:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \frac{|h|}{r+R} |\nabla_{t,x} u|^2 dx dt &= \sum_{k \in \mathbb{Z}} \int_0^T \int_{A_k} \frac{|h|}{r+R} |\nabla_{t,x} u|^2 dx dt \\ &\lesssim \sum_{k \in \mathbb{Z}} \sup_{2^{k-1} \leq |x| \leq 2^k} |h| \int_0^T \int_{A_k} |x|^{-1} |\nabla_{t,x} u|^2 dx dt \\ &\lesssim \left(\sum_{k \in \mathbb{Z}} \sup_{2^{k-1} \leq |x| \leq 2^k} |h| \right) \sup_{k \in \mathbb{Z}} \left(\int_0^T \int_{A_k} |x|^{-1} |\nabla_{t,x} u|^2 dx dt \right) \\ &\leq \epsilon_h \|\nabla_{t,x} u\|_{LE_M}^2. \end{aligned} \tag{4.56}$$

The same trick allows us to write

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} |\partial h| |\nabla_{t,x} u|^2 dx dt &= \int_0^T \int_{\mathbb{R}^n} (|x| |\partial h|) |x|^{-1} |\nabla_{t,x} u|^2 dx dt \\ &\lesssim \left(\sum_{k \in \mathbb{Z}} \sup_{2^{k-1} \leq |x| \leq 2^k} |x| |\partial h| \right) \sup_{k \in \mathbb{Z}} \left(\int_0^T \int_{A_k} |x|^{-1} |\nabla_{t,x} u|^2 dx dt \right) \\ &\leq \epsilon_h \|\nabla_{t,x} u\|_{LE_M}^2, \end{aligned} \tag{4.57}$$

as well as

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \frac{|h|}{(r+R)^3} u^2 dx dt &\lesssim \left(\sum_{k \in \mathbb{Z}} \sup_{2^{k-1} \leq |x| \leq 2^k} |h| \right) \sup_{k \in \mathbb{Z}} \left(\int_0^T \int_{A_k} |x|^{-3} u^2 dx dt \right) \\ &\leq \epsilon_h \||x|^{-1} u\|_{LE_M}^2. \end{aligned} \tag{4.58}$$

We next deal with the time boundary terms and prove a variant of the elementary energy inequality. Let us note that using $T = \partial_t$, we can write

$$\int_0^T \int_{\mathbb{R}^n} \partial_t u \square_h u - \frac{1}{2} (\mathbf{D}_t h^{\alpha\beta}) \partial_\alpha u \partial_\beta u dx dt + \int_{\mathbb{R}^n} Q_{\alpha\beta}^h T^\alpha T^\beta dx \Big|_0^T = 0. \tag{4.59}$$

Evaluating explicitly gives

$$Q_{\alpha\beta}^h T^\alpha T^\beta = \frac{1}{2} |\nabla_{t,x} u|^2 + \partial_t u h_t^\alpha \partial_\alpha u + \frac{1}{2} h^{\alpha\beta} \partial_\alpha u \partial_\beta u \approx |\nabla_{t,x} u|^2, \tag{4.60}$$

given that $|h| \ll 1$ is sufficiently small. Estimating in the spirit of (4.36), (4.37), this implies that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla_{t,x} u|^2(T, x) dx &\lesssim E[u](0) + \int_0^T \int_{\mathbb{R}^n} |\partial_t u| |\square_h u| dx dt + \int_0^T \int_{\mathbb{R}^n} |\partial h| |\nabla_{t,x} u|^2 dx dt \\ &\leq E[u](0) + \epsilon' \|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + (\epsilon + \epsilon_h) \|u\|_{LE_M^1}^2 + C_{\epsilon, \epsilon'} \|\square_h u\|_{LE_M^* + L_t^1 L_x^2}^2 \end{aligned} \tag{4.61}$$

for any $\epsilon' > 0$. Taking $\epsilon' \ll 1$ sufficiently small then gives another variant of the elementary energy inequality,

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 \lesssim E[u](0) + (\epsilon + \epsilon_h) \|u\|_{LE_M^1}^2 + C_\epsilon \|\square_h u\|_{LE_M^* + L_t^1 L_x^2}^2. \tag{4.62}$$

We can apply this to (4.50) using Cauchy's and Hardy's inequality and that $f \leq 1$:

$$\begin{aligned} \int_{\mathbb{R}^n} P_\alpha^{(X,q)}(u) T^\alpha dx &= \int_{\mathbb{R}^n} f \partial_r u \partial_t u + f \partial_r u h_t^\alpha \partial_\alpha u dx + \left(\frac{n-1}{2} \right) \frac{f}{r} u (\partial_t u + h_t^\alpha \partial_\alpha u) dx \\ &\lesssim \int_{\mathbb{R}^n} |\nabla_{t,x} u|^2(t, x) dx \lesssim E[u](0) + \|F\|_{LE_M^* + L_t^1 L_x^2}^2 + (\epsilon + \epsilon_h) \|\nabla_{t,x} u\|_{LE_M}^2. \end{aligned} \tag{4.63}$$

With the energy inequality (4.62), the inhomogeneous terms are easily controlled by

$$\int_0^T \int_{\mathbb{R}^n} f \partial_r u \square_h u dx dt \lesssim E[u](0) + (\epsilon + \epsilon_h) \|u\|_{LE_M^1}^2 + C_\epsilon \|\square_h u\|_{LE_M^* + L_t^1 L_x^2}^2, \tag{4.64}$$

$$\int_0^T \int_{\mathbb{R}^n} \frac{f}{r} u \square_h u dx dt \lesssim E[u](0) + (\epsilon + \epsilon_h) \|u\|_{LE_M^1}^2 + C_\epsilon \|\square_h u\|_{LE_M^* + L_t^1 L_x^2}^2. \quad (4.65)$$

Finally, we combine the bulk term estimate (4.54), the error estimates (4.56), (4.57), the energy inequality (4.62), the boundary term estimate (4.63), and the inhomogeneous term estimates (4.64), (4.65), with the space-time integral (4.53), we obtain

$$\begin{aligned} \|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \int_0^T \int_{\frac{R}{2} \leq |x| \leq R} |x|^{-1} |\nabla_{t,x} u|^2 + |x|^{-3} u^2 dx dt \\ \lesssim E[u](0) + (\epsilon + \epsilon_h) \|u\|_{LE_M^1}^2 + C_\epsilon \|\square_h u\|_{LE_M^* + L_t^1 L_x^2}^2. \end{aligned} \quad (4.66)$$

In particular, the bounding constant is independent of $R > 0$. We take the supremum of both sides of the inequality over $R = 2^j$ for $j \in \mathbb{Z}$; then for $\epsilon, \epsilon_h \ll 1$ sufficiently small, we have

$$\|\nabla_{t,x} u\|_{L_t^\infty L_x^2}^2 + \|u\|_{LE_M^1}^2 \lesssim E[u](0) + \|\square_h u\|_{LE_M^*}^2, \quad (4.67)$$

which concludes the proof of the theorem. \square

Localized Energy Estimates on Higher Dimensional Schwarzschild Space-Times

We now prove a localized energy estimate for solutions to the wave equation on higher dimensional Schwarzschild black holes. The Schwarzschild metric in (t, r, ω) coordinates takes the form

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) dt^2 + \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} dr^2 + r^2 d\Omega_{\mathbb{S}^{d+2}}^2. \quad (5.1)$$

We have denoted $d = n - 3$ so we may easily compare our calculations with the $(1 + 3)$ -dimensional case. Because the event horizon $r = r_s$ is a coordinate singularity, it is useful to work in coordinates which are non-degenerate in this region. A suitable alternative would be the ingoing Eddington-Finkelstein coordinates (v, r, ω) where $v = t + r_*$ and the metric takes the form:

$$ds^2 = - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) dv^2 + 2dvdr + r^2 d\Omega_{\mathbb{S}^{d+2}}^2. \quad (5.2)$$

However, the coordinate v is unfortunately degenerate in the sense that $v = \text{constant}$ slices are null hypersurfaces of the manifold, raising certain complications. To work around this, we introduce a new coordinate \tilde{t} as in [39] by setting

$$\tilde{t} = v - \mu_S(r), \quad (5.3)$$

where $\mu_S(r)$ is a smooth radial function and v is the ingoing Eddington-Finkelstein coordinate. In the (\tilde{t}, r, ω) coordinates, the metric takes the form

$$\begin{aligned} ds^2 = & - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) d\tilde{t}^2 + 2 \left(1 - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \mu'_S(r)\right) d\tilde{t}dr \\ & + \mu'_S(r) \left(2 - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \mu'_S(r)\right) dr^2 + r^2 d\Omega_{\mathbb{S}^{d+2}}^2. \end{aligned} \quad (5.4)$$

For the function μ_S , we require that

- (1) $\mu_S(r) \geq r_*$ for $r > r_s$ and $\mu_S(r) = r_*$ for $r > (r_s + r_{ps})/2$, so that the (\tilde{t}, r, ω) coincide with (t, r, ω) coordinates when $r > (r_s + r_{ps})/2$; and
- (2) $\mu'_S(r), 2 - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \mu'_S(r) > 0$, so that the $\tilde{t} = \text{constant}$ slices are spacelike.

We consider the Cauchy problem in the region $\mathcal{R} = \{\tilde{t} \geq 0, r \geq r_e\}$ for a radius $r_e < r_s$, where we assume without loss of generality that $r_s - r_e$ is sufficiently small relative to r_s . We prescribe initial data (u_0, u_1) along the slice $\Sigma_0 = \mathcal{R} \cap \{\tilde{t} = 0\}$. With the Killing vector field $T = \partial_{\tilde{t}}$ and a simple application of the divergence theorem, we derive a positive conserved energy for solutions to $\square_g u = 0$ along $\tilde{t} = \text{constant}$ slices. However, the conserved energy degenerates at the event horizon as T fails to be timelike there. For this reason, we work with a non-degenerate energy defined by:

$$E[u](\tilde{t}_0) := \int_{\Sigma_{\tilde{t}_0}} \left((\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 \right) r^{d+2} dr d\omega. \quad (5.5)$$

While the non-degenerate energy $E[u](\tilde{t})$ is not conserved along $\tilde{t} = \text{constant}$ slices, we showed in Chapter 3 that it remains bounded by a multiple of the initial energy $E[u](0)$ if $\square_g u = 0$ (see e.g. [22], [20]). We also set

$$E[u](\Sigma_{\mathcal{R}}^+) := \int_{\mathcal{R} \cap \{r=r_e\}} \left((\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 \right) r_e^{d+2} d\tilde{t} d\omega \quad (5.6)$$

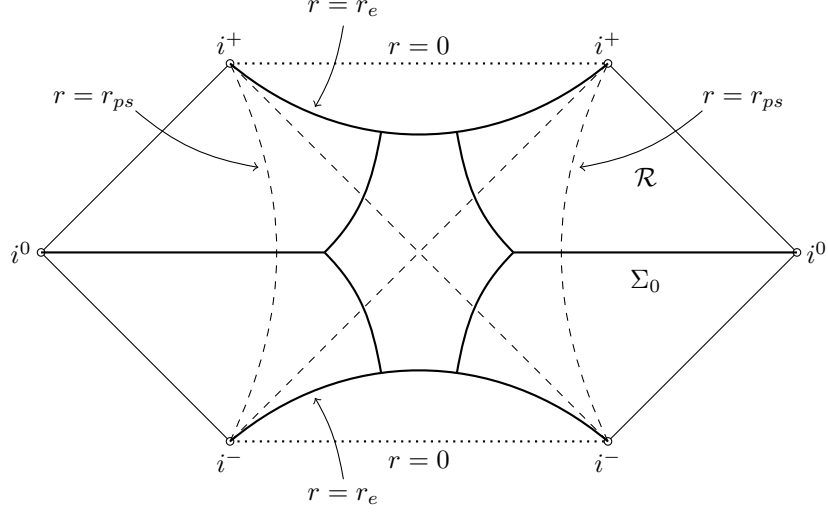


FIGURE 5.1. An illustration of the Schwarzschild black hole in the (\tilde{t}, r, ω) coordinates

to be the outgoing energy flux along the space-like hypersurface $\Sigma_{\mathcal{R}}^+ = \mathcal{R} \cap \{r = r_e\}$.

We modify the local energy norm LE_M used the previous chapter to capture the trapping phenomenon in the Schwarzschild space-time. We define a new local energy space by the norm,

$$\|u\|_{LE_S^1} := \left\| \left(1 - \frac{r_{ps}}{r}\right) \partial_{\tilde{t}} u \right\|_{LE_M} + \left\| \left(1 - \frac{r_{ps}}{r}\right) \nabla u \right\|_{LE_M} + \|\partial_r u\|_{LE_M} + \left\| r^{-\frac{3}{2}} u \right\|_{L_{\tilde{t},x}^2}, \quad (5.7)$$

as well as a dual norm LE_S^* to measure an inhomogeneous term:

$$\|F\|_{LE_S^*} := \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} F \right\|_{LE_M^*}. \quad (5.8)$$

The localized energy estimate was proved independently in [34] and [53] and is stated as follows:

THEOREM 5.1. *Let u be a solution to the wave equation $\square_g u = F$ on the Schwarzschild space-time. Then u satisfies:*

$$\sup_{\tilde{t}} E[u](\tilde{t}) + E[u](\Sigma_{\mathcal{R}}^+) + \|u\|_{LE_S^1}^2 \lesssim E[u](0) + \|\square_g u\|_{LE_S^*}^2. \quad (5.9)$$

To derive the localized energy estimate, we use a radial multiplier $X = f(r)\partial_{r^*}$. In particular, we calculate the divergence of the momentum density,

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X)}(u) &= (Xu)(\square_g u) + \frac{f(r)}{r} \left(1 - \frac{r_{ps}^{d+1}}{r^{d+1}}\right) |\nabla u|^2 + f'(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^2 (\partial_r u)^2 \\ &\quad - \left[\frac{1}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \partial_r (r^{d+2} f(r)) \right] \partial^\gamma u \partial_\gamma u. \end{aligned} \quad (5.10)$$

To eliminate the final Lagrangian term, we modify the momentum density as follows. For a smooth function q and 1-form m_α , we define:

$$P_\alpha^{(X,q,m)}(u) := P_\alpha^{(X)}(u) + qu\partial_\alpha u - \frac{1}{2}(\partial_\alpha q)u^2 + \frac{1}{2}m_\alpha u^2, \quad (5.11)$$

whose divergence then becomes:

$$\begin{aligned} \mathbf{D}^\alpha P_\alpha^{(X,q,m)}(u) &= \mathbf{D}^\alpha P_\alpha^{(X)}(u) + qu(\square_g u) + q\partial^\gamma u \partial_\gamma u - \frac{1}{2}(\square_g q - \mathbf{D}^\alpha m_\alpha) u^2 + (mu)u \\ &=: Q^{(X,q,m)}(u) + (Xu + qu)(\square_g u). \end{aligned} \quad (5.12)$$

We follow the exposition in [33] with slight modifications, and combining some ideas from [39]; in particular, we utilize the same multiplier $f(r)$.

PROOF. For a smooth, radial function $f(r)$, we set

$$X_0[f] = f(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \frac{\partial}{\partial r}, \quad q_0[f] = \frac{1}{2r^{d+2}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \partial_r (r^{d+2} f(r)). \quad (5.13)$$

In view of (5.10), we seek a function f such that

$$f(r_{ps}) = 0, \quad f'(r) > 0, \quad (5.14)$$

$$l(f) := -\frac{1}{4r^{d+2}} \partial_r \left[\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) r^{d+2} \partial_r \left(\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \frac{1}{r^{d+2}} \partial_r (r^{d+2} f(r)) \right) \right] \geq 0. \quad (5.15)$$

Unfortunately, it does not appear that a function satisfying all of these conditions exists, and we instead allow f to blow up along the event horizon. We set

$$g(r) = 1 - \frac{r_{ps}^{d+2}}{r^{d+2}}, \quad h(r) = \log \left(\frac{r^{d+1} - r_s^{d+1}}{\left(\frac{d+1}{2}\right) r_s^{d+1}} \right), \quad (5.16)$$

and then define f_0 by

$$f_0(r) = g(r) + A \frac{r_s^{d+1}}{r^{d+2}} a(h(r)) =: g(r) + h_\alpha(r). \quad (5.17)$$

The function $a \in C^2(\mathbb{R})$ is defined by

$$a(x) = \begin{cases} x, & x \leq 0, \\ x - \frac{2}{3\alpha^2} x^3 + \frac{1}{5\alpha^4} x^5, & 0 \leq x \leq \alpha, \\ \frac{8\alpha}{15}, & x \geq \alpha, \end{cases} \quad (5.18)$$

and serves to smooth out the logarithm near infinity. We will show that the constants A, α can be chosen so that the conditions (5.14), (5.15) are satisfied; specifically, we will take

$$A = \frac{d+2}{d+3} r_{ps}, \quad \alpha = 5 - \delta_\alpha, \quad (5.19)$$

for some $0 < \delta_\alpha \ll 1$ sufficiently small. The main difficulty with this construction is that $g' > 0$ for $r \geq r_s$ but that $h' < 0$ for r large, compounded with the fact that $l(g)$ is positive except for a small neighborhood about the event horizon, while $l(h)$ is negative for r large. Consequently, it seems that the only way to simultaneously control $f'_0, l(f_0)$ for all $r \geq r_s$ is to construct such a function a explicitly. In this case, we check easily that $a \in C^2$, though in the sequel we will need to replace f_0 by a similar multiplier which is smooth at the photon sphere. Our analysis splits into three regions:

- (1) near the event horizon, $r_s \leq r \leq r_{ps}$,
- (2) near the photon sphere, $r_{ps} \leq r \leq r_\alpha$, and
- (3) in the outgoing region, $r_\alpha \leq r$.

The radius r_α is implicitly defined by the value of r such that $\alpha = h(r_\alpha)$. In each of these regions, we show that $f'_0 > 0, l(f_0) > 0$.

Case 1: $r_s \leq r \leq r_{ps}$. In this region, we have

$$g'(r) = (d+2) \frac{r_{ps}^{d+2}}{r^{d+3}} > 0, \quad (5.20)$$

$$h'_\alpha(r) = -A(d+2) \frac{r_s^{d+1}}{r^{d+3}} \log \left(\frac{r^{d+1} - r_s^{d+1}}{\left(\frac{d+1}{2}\right) r_s^{d+1}} \right) + \frac{A(d+1)}{r^2} \frac{r_s^{d+1}}{r^{d+1} - r_s^{d+1}} > 0, \quad (5.21)$$

since the logarithm term has a good sign. Hence we have $f'_0(r) > 0$ trivially. We similarly calculate:

$$\begin{aligned} l(g) &= \left(\frac{d+2}{4}\right) \frac{1}{r^{2d+5}} \left[d(r^{d+1})^2 + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2(r_s^{d+1})^2 \right] \\ &= \left(\frac{d+2}{4}\right) \left[\frac{d}{r^3} + (d+3) \frac{r_s^{d+1}}{r^{d+4}} - (d+2)^2 \frac{(r_s^{d+1})^2}{r^{2d+5}} \right], \end{aligned} \quad (5.22)$$

$$\begin{aligned}
l(h_\alpha) &= -\frac{A(d+1)(d+3)r_s^{d+1}}{2r^{d+5}} \left(1 - \frac{r_{ps}^{d+1}}{r^{d+1}}\right) \\
&\geq A \frac{(d+1)(d+3)r_s^{d+1}}{r_{ps}} \left(-\frac{1}{r^{d+4}} + \left(\frac{d+3}{2}\right) \frac{r_s^{d+1}}{r^{2d+5}}\right).
\end{aligned} \tag{5.23}$$

We remark that $l(h_\alpha) > 0$ in $r_s \leq r \leq r_{ps}$ except at r_{ps} , where it vanishes. Moreover, $l(g)$ positive except for a small neighborhood of the photon sphere. Adding these two pieces together, we calculate:

$$\begin{aligned}
l(f_0) &\geq \frac{(d+2)}{4r^{2d+5}} \left[d(r^{d+1})^2 + r_s^{d+1} r^{d+1} \left((d+3) - \frac{2A(d+1)(d+3)}{r_{ps}(d+2)} \right) \right. \\
&\quad \left. + (r_s^{d+1})^2 \left(-(d+2)^2 + \frac{A(d+1)(d+3)^2}{r_{ps}(d+2)} \right) \right].
\end{aligned} \tag{5.24}$$

The term in brackets is a quadratic expression of r^{d+1} which reaches its minimum at

$$r^{d+1} = r_s^{d+1} \frac{(d+3)}{2d} \left((d+3) - \frac{2A(d+1)(d+3)}{r_{ps}(d+2)} \right) \leq r_s^{d+1}, \tag{5.25}$$

which lies inside the event horizon if

$$A \leq \frac{3}{2} \left(\frac{d+2}{d+3} \right) r_{ps}. \tag{5.26}$$

If this condition is satisfied, then the quadratic is minimized on the interval $r \in [r_s, r_{ps}]$ at $r = r_s$, so we can substitute this explicitly. Then $l(f_0) > 0$ if:

$$\begin{aligned}
0 &< d + (d+3) - (d+2)^2 - \frac{2A(d+1)(d+3)}{r_{ps}(d+2)} + \frac{A(d+1)(d+3)^2}{r_{ps}(d+2)} \\
&= -(d^2 + d + 1) + \frac{A}{r_{ps}} \left(\frac{d+3}{d+2} \right) (d+1)^2,
\end{aligned} \tag{5.27}$$

or simply that

$$A \geq r_{ps} \left(\frac{d+2}{d+3} \right) \left(1 - \frac{d}{(d+1)^2} \right). \tag{5.28}$$

Hence, the choice

$$A = \left(\frac{d+2}{d+3} \right) r_{ps} \tag{5.29}$$

is consistent with both conditions (5.26) and (5.28), and ensures that $l(f_0) > 0$.

Case 2: $r_{ps} \leq r \leq r_\alpha$. In this region, the smoothing function is $a(x) = x - \frac{2}{3\alpha^2}x^3 + \frac{1}{5\alpha^4}x^5$. For future reference, we calculate that:

$$a'(x) = \frac{(x^2 - \alpha^2)^2}{\alpha^4}, \quad a''(x) = \frac{4x(x^2 - \alpha^2)}{\alpha^4}, \quad a'''(x) = \frac{4}{\alpha^4}(3x^2 - \alpha^2). \tag{5.30}$$

On the interval $0 \leq x \leq \alpha$, we have the following bounds:

$$|a(x)| \leq \frac{8\alpha}{15}, \quad |a'(x)| \leq 1, \quad |a''(x)| \leq \frac{4x}{\alpha^2}, \quad -a'''(x) \geq -\frac{12x^2}{\alpha^4}. \tag{5.31}$$

We first show that $f'_0(r) > 0$:

$$\begin{aligned}
f'_0(r) &= \frac{(d+2)(d+3)}{2} \frac{r_{ps} r_s^{d+1}}{r^{d+3}} - \frac{(d+2)^2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+3}} a(h(r)) + \frac{d+2}{d+3} \frac{r_{ps} r_s^{d+1}}{r^{d+2}} a'(h(r)) h'(r) \\
&\geq \frac{(d+2)r_{ps} r_s^{d+1}}{r^{d+3}} \left(\frac{d+3}{2} - \frac{8\alpha}{15} \frac{d+2}{d+3} \right),
\end{aligned} \tag{5.32}$$

since $a'(h(r)), h'(r) \geq 0$. The first term in parentheses is positive if

$$\alpha < \frac{15}{16} \frac{(d+3)^2}{d+2},$$

where the right-hand-side of the inequality is minimized for $d = 1$. Consequently, $f'_0(r) > 0$ for $r_{ps} \leq r \leq r_\alpha$ if $\alpha < 5$, so we set $\alpha = 5 - \delta_\alpha$ for $0 < \delta_\alpha \ll 1$ sufficiently small.

We next consider the lower order term. Using the fact that $h'(r) = \frac{(d+1)r^d}{r^{d+1} - r_s^{d+1}}$, we record that

$$l\left(r^{-(d+2)}a(h(r))\right) = -\frac{(d+1)(d+3)}{2r^{2d+6}}(r^{d+1} - r_{ps}^{d+1})a'(h(r)) + \frac{(d+1)^2(d+5)}{4r^{d+5}}a''(h(r)) \\ - \frac{(d+1)^3}{4r^4}\left(\frac{1}{r^{d+1} - r_s^{d+1}}\right)a'''(h(r)). \quad (5.33)$$

We can then write:

$$l(f_0) = \frac{d+2}{4r^{2d+6}}(p(r) + n_1(r) + n_2(r) + n_3(r)), \quad (5.34)$$

where we denote

$$p(r) = r(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2r_s^{2d+2}), \\ n_1(r) = -r_{ps}r_s^{d+1}\frac{d+1}{2}(r^{d+1} - r_{ps}^{d+1})a'(h(r)), \\ n_2(r) = r_{ps}r_s^{d+1}\frac{(d+1)^2(d+5)}{d+3}r^{d+1}a''(h(r)), \\ n_3(r) = -r_{ps}r_s^{d+1}\frac{(d+1)^3}{d+3}\frac{r^{2d+2}}{r^{d+1} - r_s^{d+1}}a'''(h(r)).$$

The terms n_1, n_2 are negative and n_3 is unsigned; we will show that each of these contributions is absorbed by $p(r)$. In particular, we will prove that:

$$\frac{1}{3}p(r) + n_1(r) > 0, \quad (5.35)$$

$$\frac{1}{2}p(r) + n_2(r) > 0, \quad (5.36)$$

$$\frac{1}{6}p(r) + n_3(r) > 0. \quad (5.37)$$

PROOF OF (5.35). Using the bound $|a'(x)| \leq 1$ for $0 \leq x \leq \alpha$ and $r \geq r_{ps}$, we have:

$$\frac{1}{3}p(r) + n_1(r) \geq \frac{1}{3}r_{ps}(dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2r_s^{2d+2}) - r_{ps}r_s^{d+1}(d+1)(2r^{d+1} - (d+3)r_s^{d+1}) \\ = \frac{1}{3}r_{ps}(dr^{2d+2} - (5d+3)r_s^{d+1}r^{d+1} + (2d^2 + 8d + 5)r_s^{2d+2}) \\ = \frac{1}{3}r_{ps}\left(d\left(r^{d+1} - \frac{5d+3}{2d}r_s^{d+1}\right)^2 + \frac{1}{4d}(8d^3 + 7d^2 - 10d - 9)r_s^{2d+2}\right) \\ = \frac{1}{3}r_{ps}\left(d\left(r^{d+1} - \frac{5d+3}{2d}r_s^{d+1}\right)^2 + \frac{(d+1)^2(8d-9)}{4d}r_s^{2d+2}\right). \quad (5.38)$$

The right-hand-side of this equation is clearly positive for $d > 1$. In the case $d = 1$, we instead write the expression as a Taylor expansion about the photon sphere $r_{ps} = \sqrt{2}r_s$:

$$\frac{1}{3}p(r) + n_1(r) \geq \frac{1}{3}r^5 + \frac{4}{3}r_s^2r^3 - 4\sqrt{2}r_s^3r^2 - 3r_s^4r + 8\sqrt{2}r_s^5 \\ = \sqrt{2}r_s^5 - \frac{13}{3}r_s^4(r - r_{ps}) + \frac{20\sqrt{2}}{3}r_s^3(r - r_{ps})^2 + 8r_s^2(r - r_{ps})^3 \\ + \frac{5\sqrt{2}}{3}r_s(r - r_{ps})^4 + \frac{1}{3}(r - r_{ps})^5 \quad (5.39) \\ \geq \sqrt{2}r_s^5 - \frac{13}{3}r_s^4(r - r_{ps}) + \frac{20\sqrt{2}}{3}r_s^3(r - r_{ps})^2 \\ = \frac{\sqrt{2}}{3}r_s^3\left(3r_s^2 - \frac{13}{\sqrt{2}}r_s(r - r_{ps}) + 20(r - r_{ps})^2\right).$$

The right-hand-side of the inequality is a quadratic polynomial in $r - r_{ps}$ with no real roots, since its discriminant equals

$$\left(\frac{169}{2} - 240\right) r_s^2 < 0.$$

This proves (5.35). \square

PROOF OF (5.36). We will use the substitution $x = h(r) = \log\left(\frac{r^{d+1} - r_s^{d+1}}{\left(\frac{d+1}{2}\right)r_s^{d+1}}\right)$, so that $r^{d+1} - r_s^{d+1} = \left(\frac{d+1}{2}\right) r_s^{d+1} e^x$. In particular, we can rewrite:

$$\begin{aligned} \frac{1}{2}p(r) &\geq \frac{1}{2}r_{ps} (dr^{2d+2} + (d+3)r_s^{d+1}r^{d+1} - (d+2)^2r_s^{2d+2}) \\ &= \frac{1}{2}r_{ps} (d(r^{d+1} - r_s^{d+1})^2 + 3(d+1)r_s^{d+1}(r^{d+1} - r_s^{d+1}) - (d+1)^2r_s^{2d+2}) \\ &= \frac{(d+1)^2}{2}r_{ps}r_s^{2d+2} \left(\frac{d}{4}e^{2x} + \frac{3}{2}e^x - 1\right). \end{aligned}$$

Using the bound $|a''(x)| \leq \frac{4x}{\alpha^2} = \frac{4x}{25}$, we have:

$$\begin{aligned} n_2(r) &\geq -\frac{4x}{\alpha^2}r_{ps}r_s^{d+1} \frac{(d+1)^2(d+5)}{d+3} r^{d+1} \\ &= -\frac{4x}{\alpha^2}r_{ps}r_s^{d+1} \frac{(d+1)^2(d+5)}{d+3} (r^{d+1} - r_s^{d+1}) - \frac{4x}{\alpha^2}r_{ps}r_s^{2d+2} \frac{(d+1)^2(d+5)}{d+3} \\ &= -\frac{2}{\alpha^2}r_{ps}r_s^{2d+2} \frac{(d+1)^3(d+5)}{d+3} x e^x - \frac{4x}{\alpha^2}r_{ps}r_s^{2d+2} \frac{(d+1)^2(d+5)}{d+3}. \end{aligned}$$

Combining these two equations, we obtain:

$$\frac{1}{2}p(r) + n_2(r) \geq \frac{(d+1)^2}{2}r_{ps}r_s^{2d+2} \left(\frac{d}{4}e^{2x} + \frac{3}{2}e^x - 1 - \frac{4}{\alpha^2} \frac{(d+1)(d+5)}{d+3} x e^x - \frac{8}{\alpha^2} \frac{d+5}{d+3} x\right).$$

Denoting the quantity in parentheses by $q_2(x)$, it suffices to show that $q_2(x) \geq 0$ for $x \in [0, \alpha]$. We will prove that $q_2'(x) \geq 0$ for $x \in [0, \alpha]$ and $q_2(0) \geq 0$. We record that for $d \geq 1$,

$$q_2(0) = \frac{d}{4} + \frac{3}{2} - 1 = \frac{d}{4} + \frac{1}{2} > 0.$$

With $\alpha = 5$, and using the fact that $1 + x \leq e^x$ and $e^x \geq 1$ for $x \geq 0$, we calculate:

$$\begin{aligned} q_2'(x) &= \frac{d}{2}e^{2x} + \frac{3}{2}e^x - \frac{4}{\alpha^2} \frac{(d+1)(d+5)}{d+3} (1+x)e^x - \frac{8}{\alpha^2} \frac{d+5}{d+3} \\ &\geq \left(\frac{d}{2} - \frac{4}{25} \frac{(d+1)(d+5)}{d+3}\right) e^{2x} + \left(\frac{3}{2} - \frac{8}{25} \frac{d+5}{d+3}\right). \end{aligned} \tag{5.40}$$

Using the fact that $\frac{d+5}{d+3} \leq \frac{3}{2}$ for $d \geq 1$, each of these coefficients are seen to be positive. We have:

$$\frac{d}{2} - \frac{4}{25} \frac{(d+1)(d+5)}{d+3} \geq \frac{d}{2} - \frac{6}{25}(d+1) = \frac{1}{50}(13d - 12) > 0,$$

and

$$\frac{3}{2} - \frac{8}{25} \frac{d+5}{d+3} \geq \frac{3}{2} - \frac{12}{25} > 0.$$

Hence $q_2(x) > 0$ for $x \in [0, \alpha]$ with $\alpha = 5$. By continuity, the statement holds for $\alpha = 5 - \delta_\alpha$ with $0 < \delta_\alpha \ll 1$, which proves (5.36). \square

PROOF OF (5.37). We will again use the substitution $x = h(r) = \log\left(\frac{r^{d+1} - r_s^{d+1}}{\left(\frac{d+1}{2}\right)r_s^{d+1}}\right)$, so that $r^{d+1} - r_s^{d+1} = \left(\frac{d+1}{2}\right) r_s^{d+1} e^x$. We can rewrite:

$$\frac{1}{6}p(r) \geq \frac{1}{6}r_{ps}r_s^{2d+2}(d+1)^2 \left(\frac{d}{4}e^{2x} + \frac{3}{2}e^x - 1\right).$$

Since $\frac{1}{r^{d+1}-r_s^{d+1}}$ is maximized in $r \in [r_{ps}, r_\alpha]$ at $r = r_{ps}$, we have:

$$\begin{aligned} \frac{r^{2d+2}}{r^{d+1}-r_s^{d+1}} &= \frac{(r^{d+1}-r_s^{d+1})^2 + 2r_s^{d+1}(r^{d+1}-r_s^{d+1}) + r_s^{2d+2}}{r^{d+1}-r_s^{d+1}} \\ &= (r^{d+1}-r_s^{d+1}) + 2r_s^{d+1} + \frac{r_s^{2d+2}}{r^{d+1}-r_s^{d+1}} \\ &\leq r_s^{d+1} \left(\frac{d+1}{2} e^x + 2 \right) + \frac{2r_s^{d+1}}{d+1}. \end{aligned}$$

Combining this with the bound $-a'''(x) \geq -\frac{12}{\alpha^2}x^2$, we have:

$$\begin{aligned} n_3(r) &= r_{ps} r_s^{d+1} \frac{(d+1)^3}{d+3} \left(\frac{r^{2d+2}}{r^{d+1}-r_s^{d+1}} \right) \frac{4}{\alpha^2} \left(1 - \frac{3x^2}{\alpha^2} \right) \\ &= r_{ps} r_s^{d+1} \frac{(d+1)^3}{d+3} \left((r^{d+1}-r_s^{d+1}) + 2r_s^{d+1} + \frac{r_s^{2d+2}}{r^{d+1}-r_s^{d+1}} \right) \frac{4}{\alpha^2} \left(1 - \frac{3x^2}{\alpha^2} \right) \\ &\geq r_{ps} r_s^{2d+2} \frac{(d+1)^3}{d+3} \left[\frac{4}{\alpha^2} \left(1 - \frac{3x^2}{\alpha^2} \right) \left(\frac{d+1}{2} e^x + 2 \right) - \frac{24x^2}{\alpha^4} \frac{1}{d+1} \right]. \end{aligned}$$

Adding the two expressions, we can then bound:

$$\frac{1}{6}p(r) + n_3(r) \geq \frac{1}{6}r_{ps}r_s^{2d+2}(d+1)^2q_3(x), \quad (5.41)$$

where we denote

$$q_3(x) = \frac{d}{4}e^{2x} + \left(\frac{3}{2} + \frac{12}{\alpha^4} \frac{(d+1)^2}{d+3} \right) e^x - \frac{36}{\alpha^4} \frac{(d+1)^2}{d+3} x^2 e^x - \frac{144}{\alpha^4} \frac{d+2}{d+3} x^2.$$

As before, it suffices to show that $q_3(x) \geq 0$ for $x \in [0, \alpha]$; we will prove that $q_3'(x) \geq 0$ for $x \in [0, \alpha]$ and that $q_3(0) > 0$. We record that for $d \geq 1$, we have:

$$q_3(0) = \frac{d}{4} + \frac{3}{2} + \frac{12}{\alpha^4} \frac{(d+1)^2}{d+3} > 0.$$

Using the fact that $x \leq \alpha = 5$ and $x \leq e^x$ for $x \geq 0$, we calculate:

$$\begin{aligned} q_3'(x) &= \frac{d}{2}e^{2x} + \left(\frac{3}{2} + \frac{12}{\alpha^4} \frac{(d+1)^2}{d+3} \right) e^x - \frac{36}{\alpha^4} \frac{(d+1)^2}{d+3} x(x+2)e^x - \frac{288}{\alpha^4} \frac{d+2}{d+3} x \\ &\geq \frac{d}{2}e^{2x} + \left(\frac{3}{2} + \frac{12}{\alpha^4} \frac{(d+1)^2}{d+3} \right) e^x - \frac{252}{\alpha^4} \frac{(d+1)^2}{d+3} e^{2x} - \frac{388}{\alpha^4} \frac{d+2}{d+3} e^x \\ &= \left(\frac{d}{2} - \frac{36}{\alpha^4} \frac{(d+1)^2}{d+3} \right) e^{2x} + \left(\frac{3}{2} + \frac{12}{\alpha^2} \frac{(d+1)^2}{d+3} - \frac{288}{\alpha^4} \frac{d+2}{d+3} \right) e^x \\ &= \left(\frac{d}{2} - \frac{36}{\alpha^4} \frac{(d+1)^2}{d+3} \right) e^{2x} + \left(\frac{3}{2} + \frac{12}{\alpha^2} \frac{1}{d+3} (\alpha^2(d+1)^2 - 24(d+2)) \right) e^x \end{aligned} \quad (5.42)$$

With $\alpha = 5$ and $d \geq 1$, each of these coefficients are seen to be positive, since

$$\frac{d}{2} - \frac{36}{5^4} \frac{(d+1)^2}{d+3} \geq \frac{d}{2} - \frac{1}{5}(d+1) = \frac{1}{10}(3d-2) > 0$$

using the fact that $36 < 5^3$ and $\frac{d+1}{d+3} < 1$, and moreover

$$25(d+1)^2 - 24(d+2) = 25d^2 + 26d - 23 > 0.$$

Hence $q_3(x) > 0$ for $x \in [0, \alpha]$ with $\alpha = 5$. By continuity, the statement holds for $\alpha = 5 - \delta_\alpha$ with $0 < \delta_\alpha \ll 1$, which proves (5.37). \square

Case 3: $r \geq r_\alpha$. In this region, the smoothing function is constant, so for $A = \frac{d+2}{d+3}r_{ps}$ and $\alpha = 5$, we have

$$f_0(r) = 1 - \frac{r_{ps}^{d+2}}{r^{d+2}} + A \frac{8\alpha}{15} \frac{1}{r^{d+2}} = 1 - \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \left(\frac{d+3}{2} - \frac{8}{3} \frac{d+2}{d+3} \right), \quad (5.43)$$

and for the final coefficient we compute:

$$\frac{d+3}{2} - \frac{8}{3} \frac{d+2}{d+3} = \frac{3d^2 + 2d + 13}{6(d+3)} > 0, \quad (5.44)$$

so by continuity the statement holds for some $\alpha = 5 - \delta_\alpha$. This implies that f_0 is increasing and that

$$f_0'(r) \gtrsim \frac{1}{r^{d+3}}. \quad (5.45)$$

We also record that

$$l(f_0) \gtrsim \frac{1}{r^3}, \quad (5.46)$$

since $l(f_0) = l(g)$ is positive in this region. In particular, the $\frac{1}{r^3}$ weight is already stronger than the $(1+3)$ -dimensional case, where the lower order term containing u^2 must instead be bounded in an appropriate LE_M norm.

As the multiplier $f_0(r)$ is only C^2 , we would like to replace f_0 with a similar multiplier f_1 which is smooth at the photon sphere. To this end, we set

$$\begin{aligned} f_1(r) &= g(r) + \frac{d+2}{d+3} \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \left[(1 - \chi(r)) a(h(r)) + \chi(r) \left((\eta_j(r) * a)(h(r)) - Q_j(r) \right) \right] \\ &= f_0(r) + \frac{d+2}{d+3} \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \chi(r) \left((\eta_j * a)(h(r)) - Q_j(r) - a(h(r)) \right). \end{aligned} \quad (5.47)$$

Here η_j is a standard mollifier, which is an approximation of the identity for large j (see, e.g. [25]). In particular, if $\phi \in C^k$, then $\partial^\alpha(\eta_j * \phi) \rightarrow \partial^\alpha\phi$ uniformly on compact subsets of \mathbb{R} for $0 \leq \alpha \leq k$. The function $Q_j(r)$ is a second-degree polynomial (with small coefficients) chosen so that

$$\partial^\alpha \left((\eta_j * a)(h(r)) - Q_j(r) \right) \Big|_{r=r_{ps}} = \partial^\alpha a(h(r)) \Big|_{r=r_{ps}}, \quad |\alpha| \leq 2, \quad (5.48)$$

and $\chi \in C_c^\infty(\mathbb{R})$ is a smooth cutoff which is supported in a small neighborhood of r_{ps} and which equals the identity on a smaller neighborhood of r_{ps} . We note that $f_1(r) = f_0(r)$ except on the support of χ ; hence for any $\epsilon > 0$, we can choose j sufficiently large that

$$|\partial^\alpha(f_1(r) - f_0(r))| < \epsilon, \quad 0 \leq \alpha \leq 2. \quad (5.49)$$

We will show that f_1 preserves the desired properties of f_0 .

First, we note that $f_1(r_{ps}) = 0$ by construction. Moreover, we have $f_1' = f_0' > 0$ away from the support of χ . On the other hand, near $r = r_{ps}$ we already have $f_0' > 0$, so if j is chosen large enough, then $f_1' > 0$ as well. To show that $l(f_1) > 0$ on the support of χ , we rewrite:

$$\begin{aligned} l(f_1) &= l(f_0) + \left[l(f_1 - f_0) + \frac{1}{4} \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^2 (\eta_j * a)'''(h(r))(h'(r))^3 \right] \\ &\quad - \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^2 (\eta_j * a''')(h(r))(h'(r))^3. \end{aligned} \quad (5.50)$$

The term in brackets involves only second derivatives of $\eta_j * a$, and hence can be made uniformly small by taking j sufficiently large. On the other hand, $a''' \leq 0$ if $\text{supp}(\chi)$ is small enough, so that

$$(\eta_j * a''')(h(r)) \leq \epsilon \quad (5.51)$$

on $\text{supp}(\chi)$ if j is sufficiently large, and so the final term is of the form

$$- \frac{1}{4} \frac{d+2}{d+3} \frac{r_{ps}r_s^{d+1}}{r^{d+2}} \chi(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right)^2 (\eta_j * a''')(h(r))(h'(r))^3 = O(\epsilon)\chi(r). \quad (5.52)$$

Since $l(f_0) > 0$ in this region, we conclude that $l(f_1) > 0$ if $\epsilon \ll 1$ is sufficiently small.

We now smooth out the singularity at the event horizon by replacing $f_1(r)$ with

$$f(r) = \frac{1}{r^{d+2}} \chi_\epsilon(R), \quad (5.53)$$

where $R = r^{d+2} f_1(r)$ and $\chi \in C^\infty(\mathbb{R})$ is a smooth nondecreasing function such that

$$\chi(R) = \begin{cases} R, & -1 \leq R < \infty, \\ -2, & -\infty < R \leq -3. \end{cases} \quad (5.54)$$

We denote $\chi_\epsilon(R) = \epsilon^{-1} \chi(\epsilon R)$ for a small parameter ϵ to be specified. If $\epsilon R > -1$, then $f(r) = f_1(r)$ and we have $f'(r) \gtrsim \frac{1}{r^{d+3}}$ trivially. On the other hand, when $\epsilon R < -1$ we have $\chi(\epsilon R) \leq -1$, so we can bound

$$f'(r) = -\frac{d+2}{r^{d+3}} \chi_\epsilon(R) + \frac{1}{r^{d+2}} \chi'(\epsilon R) \partial_r(r^{d+2} f_1(r)) \geq \frac{d+2}{r^{d+3} \epsilon} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \quad (5.55)$$

using the fact that $\chi', \partial_r(r^{d+2} f_1(r))$ are nonnegative. In particular, $f'(r) \gtrsim \frac{1}{r^{d+3}}$ for $r > r_s$ independently of ϵ for $\epsilon \ll 1$ sufficiently small. In the region $\epsilon R \geq -1$, we have thus far controlled

$$Q^{(X_0, q_0)} \gtrsim \frac{1}{r^{d+3}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r u)^2 + \frac{1}{r} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \left(1 - \frac{r_{ps}}{r}\right)^2 |\nabla u|^2 + \frac{u^2}{r^3}. \quad (5.56)$$

To control $\partial_{\bar{t}} u$ in this region, we define

$$q = q_0 - \delta_1 q_1, \quad q_1 = \chi'(\epsilon R) \frac{1}{r^{d+3}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \left(1 - \frac{r_{ps}}{r}\right)^2, \quad (5.57)$$

where δ_1 is a small positive constant. This introduces a small Lagrangian factor, which yields a positive coefficient on $(\partial_{\bar{t}} u)^2$, while for $\delta_1 \ll 1$ sufficiently small, the remaining terms can be bootstrapped into (5.56). Indeed, for $\epsilon R \geq -1$, we have:

$$\begin{aligned} Q^{(X_0, q)}(u) &= Q^{(X_0, q_0)}(u) - \delta_1 q_1 \partial^\gamma u \partial_\gamma u + \frac{\delta_1}{2} (\square_g q_1) u^2 \\ &\gtrsim_{\delta_1} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \left(1 - \frac{r_{ps}}{r}\right)^2 \left(\frac{1}{r^{d+3}} (\partial_{\bar{t}} u)^2 + \frac{1}{r} |\nabla u|^2\right) \\ &\quad + \frac{1}{r^{d+3}} \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) (\partial_r u)^2 + \left(l(f) + \frac{\delta_1}{2} \square_g q_1\right) u^2. \end{aligned} \quad (5.58)$$

It remains to show $\delta_1 \ll 1$ sufficiently small implies that the lower order term containing u^2 has a good sign. We record that

$$l(f) + \frac{\delta_1}{2} \square_g q_1 = \chi'(\epsilon R) \left(l(f) + O\left(\frac{\delta_1}{r^{d+6}}\right)\right) + O(\epsilon) \chi''(\epsilon R) + O\left(\epsilon^2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}\right) \chi'''(\epsilon R), \quad (5.59)$$

since $\chi'', \chi''' \equiv 0$ for $\epsilon R > -1$ and $\chi', \chi'', \chi''' \equiv 0$ for $\epsilon R < -3$. Now for $\delta_1 \ll 1$ and $\epsilon R > -1$, it follows that

$$l(f) + \frac{\delta_1}{2} \square_g q_1 = l(f_1) + O\left(\frac{\delta_1}{r^{d+6}}\right) \gtrsim \frac{1}{r^3}. \quad (5.60)$$

Moreover, the first term in (5.59) is non-negative for $\epsilon R > -3$, and the choice of δ_1 does not depend on ϵ . When $\epsilon R < -3$, the whole quantity vanishes, so it remains only to analyze the region $-3 \leq \epsilon R \leq -1$.

Now R is a strictly increasing quantity and $R(r) \rightarrow -\infty$ as $r \searrow r_s$, so we assume without loss of generality that $\epsilon \ll 1$ is sufficiently small that $r_s < r < r_0^N$ when $\epsilon R < -1$, where r_0^N is the radius from the red-shift vector field N constructed in the previous chapter. If $\epsilon \ll 1$ is sufficiently small, then the red-shift effect easily controls the $O(\epsilon)$ term, while the final term in (5.59) is more dangerous. To control the factor

of $\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}$, we switch to the Regge-Wheeler coordinate r_* :

$$\begin{aligned}
& \int_0^{\tilde{t}_0} \int_{-3 \leq \epsilon R \leq -1} O\left(\epsilon^2 \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1}\right) \chi'''(\epsilon R) u^2 r^{d+2} dr d\omega d\tilde{t} \\
&= \int_0^{\tilde{t}_0} \int_{\mathbb{S}^{d+2}} \left(\int_{\{-3 \leq \epsilon R \leq -1\}} O(\epsilon^2) \chi'''(\epsilon R) u^2 r^{d+2} dr_* \right) d\omega d\tilde{t} \\
&\lesssim \int_0^{\tilde{t}_0} \int_{\mathbb{S}^{d+2}} \left(\int_{\{-3 \leq \epsilon R \leq -1\}} O(\epsilon^2) dr_* \right) \cdot \left(\sup_{-3 \leq \epsilon R \leq -1} u^2(\tilde{t}, r, \omega) \right) d\omega d\tilde{t} \\
&\lesssim \epsilon \int_0^{\tilde{t}_0} \int_{\mathbb{S}^{d+2}} \left(\sup_{-3 \leq \epsilon R \leq -1} u^2(\tilde{t}, r, \omega) \right) d\omega d\tilde{t} \leq \epsilon \int_0^{\tilde{t}_0} \int_{\mathbb{S}^{d+2}} \left(\sup_{r_s \leq r \leq r_0^N} u^2(\tilde{t}, r, \omega) \right) d\omega d\tilde{t} \\
&\lesssim_{r_0^N} \epsilon \int_0^{\tilde{t}_0} \int_{\mathbb{S}^{d+2}} \left(\int_{r_s}^{r_0^N} u^2 + (\partial_r u)^2 dr \right) d\omega d\tilde{t} \approx_{r_0^N} \epsilon \int_0^{\tilde{t}_0} \int_{\{r_s \leq r \leq r_0^N\}} (u^2 + (\partial_r u)^2) dV_g.
\end{aligned} \tag{5.61}$$

This follows from that fact that the interval of integration $\{-3 \leq \epsilon R \leq -1\}$ has size bounded by $\frac{C}{\epsilon}$ uniformly for small ϵ , since $R \approx r_*$ for $r \approx r_s$. The last line follows by applying a 1-dimensional Sobolev inequality in the bounded region $r_s \leq r \leq r_0^N$, so that the bound depends only on the radius r_0^N . We thus obtain:

$$\int_{\mathcal{R}} l(f) + \frac{\delta_1}{2} \square_g q_1 dV_g \gtrsim \int_{\mathcal{R} \cap \{r \geq r_0^N\}} \frac{u^2}{r^3} dV_g + O(\epsilon) \int_{\mathcal{R} \cap \{r_s < r < r_0^N\}} u^2 + (\partial_r u)^2 dV_g. \tag{5.62}$$

From the calculations in the previous chapter, and by relabelling r_0^N if necessary, we record that $-\mathbf{D}^\alpha N_\alpha = -\text{tr}^{(N)} \pi > 0$ for $r_s \leq r \leq r_0^N$, and that $\mathbf{D}^\alpha N_\alpha = 0$ for $r > r_1^N$. We now set

$$P_\alpha^{(N,0,-\delta_0(N-T))}(u) = P_\alpha^{(N)}(u) - \frac{\delta_0}{2} (N_\alpha - T_\alpha) u^2, \tag{5.63}$$

and calculate

$$Q^{(N,0,-\delta_0(N-T))}(u) = Q^{(N)}(u) - \frac{\delta_0}{2} (\mathbf{D}^\alpha N_\alpha) u^2 - \delta_0 u (Nu), \tag{5.64}$$

which satisfies

$$\begin{cases} Q^{(N,0,-\delta_0(N-T))}(u) \approx (\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 + u^2 & \text{for } r_s < r < r_0^N, \\ |Q^{(N,0,-\delta_0(N-T))}(u)| \lesssim (\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 + u^2 & \text{for } r_0^N < r < r_1^N, \\ Q^{(N,0,-\delta_0(N-T))}(u) = 0 & \text{for } r > r_1^N, \end{cases} \tag{5.65}$$

if $\delta_0 \ll 1$ is sufficiently small. By continuity, the first inequality holds for $r_e < r < r_0^N$ for $r_e < r_s$ sufficiently close to r_s . To control the gradient and lower order terms near the event horizon, we select X, q, m by

$$X = X_0[f] + \delta N, \quad q = q_0[f] - \delta_1 q_1, \quad m_\alpha = \delta \left[-\frac{\delta_0}{2} (N_\alpha - T_\alpha) \right], \tag{5.66}$$

where $\delta, \epsilon > 0$ are constants to be determined. If $\delta \ll 1$ sufficiently small, then for $r_0^N < r < r_1^N$, we have

$$\mathbf{D}^\alpha P_\alpha^{(X,q,m)}(u) \gtrsim (\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 + u^2, \tag{5.67}$$

which is possible since $f(r) = f_1(r)$ and $l(f) = l(f_1)$ in this range and because $f_1'(r) \gtrsim \frac{1}{r^{d+3}}$ and $l(f_1) \gtrsim \frac{1}{r^3}$. In particular, δ is chosen independently of ϵ . Finally, the unsigned $O(\epsilon)$ terms from (5.62) in the region $\{-3 \leq \epsilon R \leq -1\} \subset \{r_s < r < r_0^N\}$ can be absorbed by the red-shift effect for sufficiently small $\epsilon \ll \delta \ll 1$:

$$\int_{\mathcal{R}} Q^{(X,q,m)}(u) dV_g \gtrsim \int_{\mathcal{R}} \frac{1}{r^{d+3}} (\partial_r u)^2 + \left(1 - \frac{r_{ps}}{r}\right)^2 \left(\frac{1}{r^{d+3}} (\partial_{\tilde{t}} u)^2 + \frac{1}{r} |\nabla u|^2 \right) + \frac{u^2}{r^3} dV_g =: \|u\|_{LEW_S}^2. \tag{5.68}$$

We denote the right hand side of this expression by $\|u\|_{LEW_S}^2$, as we have yet to improve the weights at infinity to obtain the sharper LE_S norm.

Applying the divergence theorem in the region $\mathcal{R}(0, \tilde{t}_0) = \mathcal{R} \cap \{0 \leq \tilde{t} \leq \tilde{t}_0\}$ implies that:

$$0 = \int_{\mathcal{R}(0, \tilde{t}_0)} \mathbf{D}^\alpha P_\alpha^{(X, q, m)}(u) dV_g + \int_{\Sigma_{\tilde{t}}} P_\alpha^{(X, q, m)} n_{\Sigma_{\tilde{t}}}^\alpha dS \Big|_{\tilde{t}=0}^{\tilde{t}=\tilde{t}_0} + \int_{\Sigma_{\mathcal{R}}^+} P_\alpha^{(X, q, m)} n_{r_e}^\alpha dS. \quad (5.69)$$

Here $n_{\Sigma_{\tilde{t}}}$ and n_{r_e} are the future-pointing unit normals to $\Sigma_{\tilde{t}}$ and $\Sigma_{\mathcal{R}}^+$, respectively. In particular, $\Sigma_{\tilde{t}_0}$ and $\Sigma_{\mathcal{R}}^+$ are outgoing boundary components in $\mathcal{R}(0, \tilde{t}_0)$ while Σ_0 is an ingoing boundary component, which explains the signs of the terms in (5.69). It remains to show that the boundary terms have the correct sign. Since the vector field $T = \partial_{\tilde{t}}$ is Killing, we can add a large multiple CT to our multiplier X without affecting $Q^{(X, q, m)}$. We can rewrite:

$$\begin{aligned} \int_{\Sigma_{\tilde{t}_0}} P_\alpha^{(X+CT, q, m)}(u) n_{\Sigma_{\tilde{t}_0}}^\alpha dS &= \int_{\Sigma_{\tilde{t}_0}} \langle P^{(X+CT, q, m)}, -d\tilde{t} \rangle r^{d+2} dr d\omega, \\ \int_{\Sigma_{\mathcal{R}}^+} P_\alpha^{(X+CT, q, m)} n_{r_e}^\alpha dS &= \int_{\Sigma_{\mathcal{R}}^+} \langle P^{(X+CT, q, m)}, dr \rangle r_e^{d+2} d\omega d\tilde{t}. \end{aligned}$$

We remark that since $T = \partial_{\tilde{t}}$ is timelike away from the event horizon, and since the red-shift vector field N and unit normal $n_{\Sigma_{\tilde{t}_0}}$ to $\Sigma_{\tilde{t}_0}$ are everywhere timelike, we must have

$$\langle P^{(T)}, -d\tilde{t} \rangle \gtrsim \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2) \quad (5.70)$$

$$\langle P^{(N)}, -d\tilde{t} \rangle \gtrsim \delta ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2). \quad (5.71)$$

This allows us to bound:

$$\begin{aligned} \langle P^{(X+CT)}, -d\tilde{t} \rangle &= (X_0 u) \langle du, -d\tilde{t} \rangle + \delta \langle P^{(N)}(u), -d\tilde{t} \rangle + C \langle P^{(T)}, -d\tilde{t} \rangle \\ &= O\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2) + \delta \langle P^{(N)}(u), -d\tilde{t} \rangle + C \langle P^{(T)}, -d\tilde{t} \rangle \\ &\gtrsim \left(\delta + C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)\right) ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2), \end{aligned} \quad (5.72)$$

given that the constant C is sufficiently large. Now since q, q' are supported in the region $\epsilon R \geq -3$ and $\epsilon \ll 1$ is previously fixed constant, we can write

$$|q| \lesssim \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \frac{1}{r}, \quad |q'| \lesssim \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) \frac{1}{r^{d+3}}, \quad (5.73)$$

though we need to take extra care that none of the bounds that follow depend on our choice of ϵ . Combining this with the fact that m is compactly supported allows us to similarly bound:

$$\begin{aligned} \langle P^{(0, q, m)}, -d\tilde{t} \rangle &= (qu) \langle du, -d\tilde{t} \rangle - \frac{1}{2} (\langle dq, -d\tilde{t} \rangle - \langle m, -d\tilde{t} \rangle) u^2 \\ &= O\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2) + O\left(\frac{1}{r^2}\right) u^2. \end{aligned} \quad (5.74)$$

The first of these terms is easily absorbed for a choice of C large. The lower order term can be controlled by an appropriate Hardy inequality, of the form:

$$\int_{r_e}^\infty \frac{u^2}{r^2} r^{d+2} dr \lesssim C^{-\frac{1}{2}} \int_{r_e}^\infty \left[C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1 \right] (\partial_r u)^2 r^{d+2} dr, \quad (5.75)$$

where C, r_e are such that $0 < r_s - r_e < C^{-1}$. To prove the inequality, we take a smooth bounded function γ with $\gamma(r_e) = 0$, and use the fundamental theorem of calculus to write:

$$\begin{aligned} 0 &= \int_{r_e}^{\infty} \partial_r (r^{d+2} \gamma u^2) dr = \int_{r_e}^{\infty} \partial_r (\gamma r^{d+2}) u^2 dr + \int_{r_e}^{\infty} 2\gamma(r) u \partial_r u r^{d+2} dr \\ &\geq \int_{r_e}^{\infty} \left(r^{-(d+2)} \partial_r (\gamma r^{d+2}) - \frac{C^{\frac{1}{2}} \gamma^2(r)}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} \right) u^2 r^{d+2} dr \\ &\quad - C^{-\frac{1}{2}} \int_{r_e}^{\infty} \left[C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1 \right] (\partial_r u)^2 r^{d+2} dr. \end{aligned} \quad (5.76)$$

With the choice

$$\gamma(r) = \epsilon \frac{r - r_e}{r^2}, \quad (5.77)$$

we calculate the first term in parentheses to be:

$$\epsilon \frac{1}{r^2} + \epsilon \frac{r - r_e}{r^3} - \epsilon^2 C^{\frac{1}{2}} \frac{1}{r^4} \frac{(r - r_e)^2}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} \gtrsim \frac{1}{r^2} \quad (5.78)$$

for $\epsilon \ll 1$ sufficiently small, where in particular ϵ and the bounding constant do not depend on C . Rearranging the terms in (5.76) then yields (5.75). The final term in (5.78) can be rewritten as:

$$C^{\frac{1}{2}} \frac{1}{r^4} \frac{(r - r_e)^2}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} = C^{\frac{1}{2}} \frac{r - r_e}{r^3} \frac{\left(1 - \frac{r_s}{r}\right)}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} + C^{\frac{1}{2}} \frac{r - r_e}{r^4} \frac{r_s - r_e}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1}. \quad (5.79)$$

Using the assumption that $0 < r_s - r_e < C^{-1}$, this expression can be estimated by

$$\left| \frac{\left(1 - \frac{r_s}{r}\right)}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} \right| \lesssim C^{-1}, \quad \frac{r_s - r_e}{C \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + 1} \lesssim C^{-1}, \quad (5.80)$$

which proves (5.78).

Applying the inequality (5.75) to the time boundary terms for C sufficiently large then yields:

$$\int_{\Sigma_{\tilde{t}_0}} \langle P^{(X+CT, q, m)}, -d\tilde{t} \rangle r^{d+2} dr d\omega \approx \int_{\Sigma_{\tilde{t}_0}} ((\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2) r^{d+2} dr d\omega = E[u](\tilde{t}_0). \quad (5.81)$$

For the outgoing energy flux along $\Sigma_{\mathcal{R}}^+$, we remark that for $r = r_e$,

$$\langle T, T \rangle = O\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right), \quad \langle X_0, X_0 \rangle = O\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right), \quad \langle N, N \rangle < 0,$$

uniformly for r_e sufficiently close to r_s . This implies that

$$\begin{aligned} \langle X + CT, X + CT \rangle &= \langle X_0, X_0 + 2\delta N + CT \rangle + \delta^2 \langle N, N \rangle + 2\delta C \langle T, N \rangle \\ &= O\left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right) + \delta^2 \langle N, N \rangle + 2\delta C \langle T, Y \rangle < 0, \end{aligned} \quad (5.82)$$

if r_e is sufficiently close to r_s . Moreover, we have $\langle dr, dr \rangle < 0$ for $r_e < r_s$, which implies that

$$\left\langle P^{(X+CT)}, dr \right\rangle \Big|_{r=r_e} \gtrsim (\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2,$$

which follows directly from the properties of the stress-energy tensor (in particular, the dominant energy condition; see e.g. [27]). The remaining terms are:

$$\left\langle P^{(0,0,m)}(u), dr \right\rangle = \frac{1}{2} \langle m, dr \rangle u^2, \quad (5.83)$$

because q is supported away from (and exterior to) the event horizon. On the other hand, we explicitly have:

$$m = -\delta\delta_0(N - T) = -\delta\delta_0 Y = -\delta\delta_0 \hat{Y} \quad \text{on } \mathcal{H}^+. \quad (5.84)$$

The definition of \hat{Y} in (3.13) we can write

$$-\frac{1}{2}\hat{Y} = \frac{\partial}{\partial r} - \left(1 - \frac{r_s^{d+1}}{r^{d+1}}\right)^{-1} \frac{\partial}{\partial t} \quad (5.85)$$

in the (t, r, ω) coordinates, which implies that $\langle m, dr \rangle > 0$ on $r = r_s$. In particular, if r_e is sufficiently close to r_s , then by continuity we have $\langle m, dr \rangle > 0$ on $r = r_e$, which implies

$$\left\langle P^{(X, q, m)}, dr \right\rangle \Big|_{r=r_e} \gtrsim (\partial_{\tilde{t}} u)^2 + (\partial_r u)^2 + |\nabla u|^2 + u^2, \quad (5.86)$$

and hence

$$\int_{\mathcal{R} \cap \{r=r_e\}} \left\langle P^{(X, q, m)}, dr \right\rangle r_e^{d+2} d\omega d\tilde{t} \gtrsim E[u](\Sigma_{\mathcal{R}}^+). \quad (5.87)$$

Combining (5.68), (5.81), (5.87) then implies:

$$\sup_{\tilde{t} > 0} E[u](\tilde{t}) + E[u](\Sigma_{\mathcal{R}}^+) + \|u\|_{LEW_S}^2 \lesssim E[u](0) + \left| \int_{\mathcal{R}} (Xu + qu)(\square_g u) dV_g \right|. \quad (5.88)$$

For simplicity, we denote $\mathcal{R}_j = \mathcal{R} \cap \{2^{j-1} \leq r \leq 2^j\}$ for $j \in \mathbb{Z}$. We then bound the final term in two pieces, as follows:

$$\begin{aligned} \left| \int_{\mathcal{R}} (Xu)(\square_g u) dV_g \right| &\lesssim \sum_j 2^{-\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right) \nabla u \right\|_{L^2(\mathcal{R}_j)} \cdot 2^{\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{L^2(\mathcal{R}_j)} \\ &\lesssim \left(\sup_j 2^{-\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right) \nabla u \right\|_{L^2(\mathcal{R}_j)} \right) \cdot \left(\sum_k 2^{\frac{k}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{L^2(\mathcal{R}_j)} \right) \\ &= \left\| \left(1 - \frac{r_{ps}}{r}\right) \nabla u \right\|_{LE_M} \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{LE_M^*} \lesssim \|u\|_{LE_S} \|\square_g u\|_{LE_S^*} \\ &\leq \epsilon \|u\|_{LE_S}^2 + C_\epsilon \|\square_g u\|_{LE_S^*}^2, \end{aligned} \quad (5.89)$$

for any $\epsilon > 0$. The term involving u can be estimated similarly by:

$$\begin{aligned} \left| \int_{\mathcal{R}} (qu)(\square_g u) dV_g \right| &\lesssim \sum_j 2^{-\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right) r^{-1} u \right\|_{L^2(\mathcal{R}_j)} \cdot 2^{\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{L^2(\mathcal{R}_j)} \\ &\lesssim \left(\sup_j 2^{-\frac{j}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right) r^{-1} u \right\|_{L^2(\mathcal{R}_j)} \right) \cdot \left(\sum_k 2^{\frac{k}{2}} \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{L^2(\mathcal{R}_j)} \right) \\ &\approx \left(\sup_j \left\| \left(1 - \frac{r_{ps}}{r}\right) r^{-\frac{3}{2}} u \right\|_{L^2(\mathcal{R}_j)} \right) \cdot \left\| \left(1 - \frac{r_{ps}}{r}\right)^{-1} \square_g u \right\|_{LE_M^*} \\ &\lesssim \left\| r^{-\frac{3}{2}} u \right\|_{L_{t,x}^2} \|\square_g u\|_{LE_S^*} \leq \|u\|_{LE_S^1} \|\square_g u\|_{LE_S^*} \\ &\leq \epsilon \|u\|_{LE_S^1}^2 + C_\epsilon \|\square_g u\|_{LE_S^*}^2. \end{aligned} \quad (5.90)$$

Plugging these bounds into (5.88), we obtain:

$$\sup_{\tilde{t} > 0} E[u](\tilde{t}) + E[u](\Sigma_{\mathcal{R}}^+) + \|u\|_{LEW_S}^2 \lesssim E[u](0) + \epsilon \|u\|_{LE_S^1}^2 + C_\epsilon \|\square_g u\|_{LE_S^*}^2. \quad (5.91)$$

It remains still to improve the optimize the radial weights at infinity, and correspondingly absorb the final LE_S norm in the right hand side of (5.91) for $\epsilon > 0$ sufficiently small. We rely on the methods of [40], [43] for small perturbations of Minkowski space, which we proved in Theorem 4.5. Let $\beta \in C^\infty(\mathbb{R})$ be a smooth cutoff which equals

$$\begin{cases} \beta(r) = 1, & r \geq 2R, \\ \beta(r) = 0, & r \leq R, \end{cases} \quad (5.92)$$

where $R \gg r_s$ is a sufficiently large radius that the Schwarzschild metric is small and asymptotically flat in the sense of (4.43). The function βu solves the wave equation

$$\square_g(\beta u) = \beta(\square_g u) + [\square_g, \beta]u, \quad (5.93)$$

and satisfies a localized energy estimate of the form:

$$\begin{aligned} \|\nabla_{t,x}(\beta u)\|_{L_t^\infty L_x^2}^2 + \|\beta u\|_{LE_M^1}^2 &\lesssim \|\nabla_{t,x}(\beta u)(0, \cdot)\|_{L_x^2}^2 + \|\square_g(\beta u)\|_{LE_M^*}^2 \\ &\leq \|\nabla_{t,x}(\beta u)(0, \cdot)\|_{L_x^2}^2 + \|\beta \square_g u\|_{LE_M^*}^2 + \|[\square_g, \beta]u\|_{LE_M^*}^2 \\ &\lesssim \|\nabla_{t,x}(\beta u)(0, \cdot)\|_{L_x^2}^2 + \|\square_g u\|_{LE_S^*}^2 + \|u\|_{LEW_S}^2 \\ &\lesssim E[u](0) + \epsilon \|u\|_{LE_S^1}^2 + C_\epsilon \|\square_g u\|_{LE_S^*}^2 \end{aligned} \quad (5.94)$$

This follows from the fact that $\|\nabla_{t,x}(\beta u)(0, \cdot)\|_{L_x^2}^2 \lesssim E[u](0)$, and because β is supported away from $r = r_{ps}$. Moreover, $[\square_g, \beta]$ is compactly supported, and can thus be estimated by the LEW_S norm of u , which is controlled by (5.91). To strengthen the LEW_S norm in (5.91) to the LE_S^1 norm, we will use the fact that

$$\|u\|_{LEW_S}^2 + \|\beta u\|_{LE_M}^2 \gtrsim \|u\|_{LE_S^1}^2. \quad (5.95)$$

We now add a small multiple of (5.94) to (5.91), and correspondingly take $\epsilon \ll 1$ sufficiently small that the LE_S^1 terms on the right hand side can be absorbed. This yields the desired estimate and completes the proof of the theorem. \square

Localized Energy Estimates on (1+4)-Dimensional Myers-Perry Space-Times

In this chapter, we prove localized energy estimates for solutions to the wave equation on the (1 + 4)-dimensional Myers-Perry space-time as in [35]. Henceforth, we use the subscripts mp and S to refer to the Myers-Perry and Schwarzschild metrics, respectively. Since the $(t, r, \theta, \phi, \psi)$ coordinates degenerate along the event horizon $r = r_+$, we will define a new set of coordinates which are tailored to behave well in this region. We begin with the ingoing coordinates $(v_+, r, \theta, \psi_+, \phi_+)$ defined in Chapter 2, in which the Myers-Perry metric takes the form:

$$ds^2 = -dv_+^2 + 2dv_+dr + \frac{r_s^2}{\rho^2} (dv_+ + a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+)^2 + \rho^2 d\theta^2 + 2(a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+) dr + (r^2 + a^2) \sin^2 \theta d\phi_+^2 + (r^2 + b^2) \cos^2 \theta d\psi_+^2. \quad (6.1)$$

The ingoing coordinates are non-degenerate near the event horizon, but as in the Schwarzschild case, the coordinate v_+ is unfortunately null in the sense that $v_+ = \text{constant}$ slices are null hypersurfaces of the manifold. To work around this, we set

$$\tilde{t} = v_+ - \mu_{mp}(r), \quad (6.2)$$

where μ_{mp} is a smooth radial function. In the $(\tilde{t}, r, \theta, \phi_+, \psi_+)$ coordinates, the metric becomes:

$$ds^2 = -d\tilde{t}^2 + 2 \left(1 - \left(1 - \frac{r_s^2}{\rho^2} \right) \mu'_{mp}(r) \right) d\tilde{t}dr + \mu'_{mp}(r) \left(2 - \left(1 - \frac{r_s^2}{r^2} \right) \mu'_{mp}(r) \right) dr^2 + 2 \left(1 + \frac{r_s^2}{\rho^2} \mu'_{mp}(r) \right) (a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+) dr + \frac{r_s^2}{\rho^2} (d\tilde{t} + a \sin^2 \theta d\phi_+ + b \cos^2 \theta d\psi_+)^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi_+^2 + (r^2 + b^2) \cos^2 \theta d\psi_+^2. \quad (6.3)$$

For the function μ_{mp} , we require that

- (1) $\mu_{mp}(r) \geq r_*$ for $r \geq r_+$ and $\mu_{mp}(r) = r_*$ for $r > (r_+ + r_{ps})/2$ so that $\tilde{t} = t$ for $r > (r_+ + r_{ps})/2$;
- (2) $\mu'_{mp}(r), 2 - \left(1 - \frac{r_s^2}{\rho^2} \right) \mu'_{mp}(r) > 0$, so that the $\tilde{t} = \text{constant}$ slices space-like.

Here r_* is given by the relation $dr_* = \frac{(r^2 + a^2)(r^2 + b^2)}{\Delta} dr$ and is analogous to the Regge-Wheeler coordinate in the Schwarzschild case. In particular, we select μ_{mp} such that $|\mu_{mp} - \mu_S| = O(\epsilon_0)$.

We consider the Cauchy problem in $\mathcal{R} = \{\tilde{t} \geq 0, r \geq r_e\}$, for a radius $r_- < r_e < r_+$, where we assume without loss of generality that $r_+ - r_e \ll r_s$. We analogously prescribe initial data (u_0, u_1) along the slice $\mathcal{R} \cap \{\tilde{t} = 0\}$.

As in the Schwarzschild case, we work with a (non-conserved) energy on $\tilde{t} = \text{constant}$ slices of the form:

$$E[u](\tilde{t}_0) = \int_{\Sigma_{\tilde{t}_0}} ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2) r^3 dr d\omega. \quad (6.4)$$

We define

$$E[u](\Sigma_{\mathcal{R}}^+) = \int_{\mathcal{R} \cap \{r=r_e\}} ((\partial_{\tilde{t}}u)^2 + (\partial_r u)^2 + |\nabla u|^2) r_e^3 dr d\omega \quad (6.5)$$

to be the outgoing energy flux on the space-like hypersurface $\Sigma_{\mathcal{R}}^+ = \mathcal{R} \cap \{r = r_e\}$.

Unlike the Schwarzschild space-time, the trapping phenomenon can no longer be characterized in terms of position alone and must be described in phase space. For this reason, we define new localized energy norms to characterize the more complicated nature of trapping on this background.

As in the Schwarzschild case, we have $\tilde{t} = t$ near the trapped set, so it suffices for now to work in t, r, ϕ, ψ, θ coordinates. Let $\tau, \xi, \Phi, \Psi, \Theta$ denote Fourier variables for the coordinates t, r, ϕ, ψ, θ . We are slightly abusing notation, since we used some of these variables earlier to denote constants of motion. From this point on, however, this notation is reserved solely for the Fourier variables. We denote

$$p(r, \theta, \tau, \xi, \Phi, \Psi, \Theta) = g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + 2g^{t\psi}\tau\Psi + g^{rr}\xi^2 + g^{\theta\theta}\Theta^2 + g^{\phi\phi}\Phi^2 + g^{\psi\psi}\Psi^2 \quad (6.6)$$

to be the Hamiltonian for the geodesic flow. The Hamilton flow equations dictate that

$$\dot{r} = \frac{\partial p}{\partial \xi} = 2g^{rr}\xi = \frac{2\Delta}{r^2\rho^2}\xi, \quad \dot{\xi} = -\frac{\partial p}{\partial r}. \quad (6.7)$$

Explicitly, we can express:

$$\begin{aligned} \rho^2\dot{\xi} &= -\frac{\partial}{\partial r}(\rho^2 p) + \frac{\partial}{\partial r}(\rho^2)p \\ &= 2rR_{a,b}(r, \tau, \Phi, \Psi)\Delta^{-2} + \frac{\partial}{\partial r}(\rho^2)p - 2\frac{\partial}{\partial r}(\rho^2 g^{rr})\xi^2, \end{aligned} \quad (6.8)$$

where we we denote:

$$\begin{aligned} R_{a,b} &= \left(r^4(r^2 + b^2)(r^2 - 2r_s^2 + b^2) + a^4(r^2 + b^2)^2 + a^2(b^2(4r^4 - 2r_s^2r^2 + r_s^4) + 2r^4(r^2 - r_s^2) + 2b^4r^2) \right) \tau^2 \\ &\quad + \left(2ar_s^2(r^4 + b^2(2r^2 - r_s^2) + b^4)\Phi + 2br_s^2(r^4 + a^2(2r^2 - r_s^2) + a^4)\Psi \right) \tau \\ &\quad + \left(b^2(r^4 + 2(r_s^2 + b^2)r^2 + (r^4 - b^4)) - a^2(r^4 - 2b^2(r^2 - r_s^2) + b^4) \right) \Phi^2 \\ &\quad + \left(a^6 - a^4(b^2 + 2r_s^2 - 2r^2) - b^2r^4 + a^2(r_s^4 - 2r_s^2r^2 + r^2(r^2 - 2b^2)) \right) \Psi^2 \\ &\quad - 2abr_s^2(2r^2 + a^2 + b^2 - r_s^2)\Phi\Psi. \end{aligned} \quad (6.9)$$

For future reference, we note the identity

$$\frac{\partial}{\partial r}(\rho^2 p) = -2rR_{a,b}\Delta^{-2} + \frac{\partial}{\partial r}\left(\frac{\Delta}{r^2}\right)\xi^2. \quad (6.10)$$

We remark that $p = 0$ along every null geodesic; moreover, we showed in Chapter 2 that every trapped null geodesic has constant r , so (6.7) implies that $\xi \equiv 0$ along any trapped geodesic. Then $\dot{\xi} = 0$, and so (6.8) reads

$$\rho^2\dot{\xi} = 2rR_{a,b}(r, \tau, \Phi, \Psi)\Delta^{-2} = 0. \quad (6.11)$$

Thus, corresponding to any trapped null geodesic there exists a root $r_{a,b}(\tau, \Phi, \Psi)$ solving

$$R_{a,b}(r_{a,b}, \tau, \Phi, \Psi) = 0. \quad (6.12)$$

We henceforth make the ansatz that trapped null geodesics occur only in a sufficiently small, say $O(|a| + |b|) = O(\epsilon_0)$, neighborhood of the photon sphere $r = r_{ps}$. This is intuitively true, since the Myers-Perry metric is a small perturbation of the Schwarzschild geometry. It is not clear how to prove this rigorously due to the highly technical nature of our calculations. Nevertheless, the assumption is sufficient for our work as our localized energy estimate degenerates only in an $O(\epsilon_0)$ neighborhood of $r = r_{ps}$; if there were any trapped null geodesics outside the neighborhood, it would not be possible for us to prove the estimate without further loss.

We note that $\tau^{-2}R_{a,b}$ is a polynomial of degree 8 in r and that

$$\tau^{-2}R_{0,0} = r^6(r^2 - 2r_s^2) = r^6(r^2 - r_{ps}^2). \quad (6.13)$$

We remark that because $p = 0$ along any null geodesic, we must have

$$g^{tt}\tau^2 + 2g^{t\phi}\tau\Phi + 2g^{t\psi}\tau\Psi + g^{\phi\phi}\Phi^2 + g^{\psi\psi}\Psi^2 + 2g^{\phi\psi}\Phi\Psi = -g^{rr}\xi^2 - g^{\theta\theta}\Theta^2 \leq 0, \quad (6.14)$$

which we can rewrite as

$$-g^{tt} \geq 2g^{t\phi} \left(\frac{\Phi}{\tau} \right) + 2g^{t\psi} \left(\frac{\Psi}{\tau} \right) + g^{\phi\phi} \left(\frac{\Phi}{\tau} \right)^2 + g^{\psi\psi} \left(\frac{\Psi}{\tau} \right)^2 + 2g^{\phi\psi} \left(\frac{\Phi}{\tau} \right) \left(\frac{\Psi}{\tau} \right). \quad (6.15)$$

Since $-g^{tt} > 0$ at $r = r_{ps}$, we can bound

$$|\Phi|, |\Psi| \leq C_{mp}|\tau| \quad (6.16)$$

for any null geodesic near $r \approx r_{ps}$. Fixing a sufficiently small neighborhood V about $r = r_{ps}$, it follows that in V , the polynomial $\tau^{-2}R_{a,b}$ is a small perturbation of $\tau^{-2}R_{0,0}$ for $|a|, |b| \leq \epsilon_0 \ll r_s$ sufficiently small, and hence has a simple root $r_{a,b}(\tau, \Phi, \Psi)$ in V . By the implicit function theorem, we have

$$r_{a,b}(\tau, \Phi, \Psi) \in C^\infty(\{|\Phi|, |\Psi| \leq C_{mp}|\tau|\}), \quad (6.17)$$

depending smoothly on $a, b \in [-\epsilon_0, \epsilon_0]$.

We are now ready to define the localized energy spaces as in [62]. Ideally, we would replace the factor of $r - r_{ps}$ in (5.7) with a quantization of $r - r_{a,b}(\tau, \Phi, \Psi)$. However, any such quantization depends nontrivially on τ , which makes uniform energy bounds on \hat{t} slices difficult to prove. Instead, near $r = r_{ps}$ we factor

$$p = g^{tt}(\tau - \tau_1(r, \theta, \xi, \Phi, \Psi, \Theta))(\tau - \tau_2(r, \theta, \xi, \Phi, \Psi, \Theta)), \quad (6.18)$$

for some real smooth 1-homogeneous symbols τ_i . We remark that the τ_i are well defined since $g^{tt} < 0$ and $g^{rr}\xi^2 + g^{\theta\theta}\Theta^2 + g^{\phi\phi}\Phi^2 + g^{\psi\psi}\Psi^2 \geq 0$ near $r = r_{ps}$; hence the polynomial p always has real roots τ_i , which are smooth and are 1-homogeneous in ξ, Φ, Ψ, Θ . We then define

$$c_i(r, \theta, \xi, \Phi, \Psi, \Theta) = \chi_{\geq 1}(r - r_{a,b}(\tau_i, \Phi, \Psi)), \quad (6.19)$$

where $\chi_{\geq 1}$ is a smooth symbol which equals 1 for frequencies $\gg 1$ and equals 0 for frequencies $\lesssim 1$. The cutoff serves to transform the homogeneous symbol into a classical one. As the τ_i by construction satisfy $|\Phi|, |\Psi| \leq C_{mp}|\tau_i|$, the symbols $c_i = c_i(r, \theta, \xi, \Phi, \Psi, \Theta)$ are smooth and well-defined for all $r, \theta, \xi, \Phi, \Psi, \Theta$. We note that the cutoff $\chi_{\geq 1}$ does not alter our estimates, as low frequencies are already controlled without degeneracy near the trapped set. Indeed, let $V \times \mathbb{S}^3$ be a neighborhood of the photon sphere $\{r = r_{ps}\} \times \mathbb{S}^3$; let $\chi \in C_c^\infty(V \times \mathbb{S}^3)$ with $\chi \equiv 1$ near $r = r_{ps}$, and let η be a smooth symbol supported in a neighborhood of $\{r = r_{ps}\} \times \mathbb{S}^3$, which equals 1 for frequencies $\lesssim 1$, and equals 0 for frequencies $\gg 1$. Then we have the trivial estimate:

$$\|\chi\eta(x, D)\chi u\|_{L^2} \lesssim \|\chi u\|_{L^2} \lesssim \left\| r^{-\frac{3}{2}}u \right\|_{L^2}. \quad (6.20)$$

We use the symbols c_i to define microlocally weighted function spaces in $V \times \mathbb{S}^3$. We set

$$\|u\|_{L_{c_i}^2}^2 = \|c(x, D)u\|_{L^2}^2 + \|u\|_{H^{-1}}^2. \quad (6.21)$$

To measure an inhomogeneous term, we define the dual norm

$$\|g\|_{c_i L^2}^2 = \inf_{g=c_i(x, D)g_1+g_2} \left(\|g_1\|_{L^2}^2 + \|g_2\|_{H^1} \right). \quad (6.22)$$

We set our local energy norm to be:

$$\begin{aligned} \|u\|_{LE_{mp}^1} &= \|\chi(D_t - \tau_2(x, D))\chi u\|_{L_{c_1}^2} + \|\chi(D_t - \tau_1(x, D))\chi u\|_{L_{c_2}^2} + \|(1 - \chi^2)\partial_t u\|_{LE_M} \\ &\quad + \|(1 - \chi^2)\nabla\psi\|_{LE_M} + \|\partial_r u\|_{LE_M} + \left\| r^{-\frac{3}{2}}u \right\|_{L^2}. \end{aligned} \quad (6.23)$$

To see that this norm captures the trapping on the Myers-Perry space-time, note that every trapped null geodesic corresponds to a root $r_{a,b}(\tau, \Phi, \Psi)$ and satisfies $p = g^{tt}(\tau - \tau_1)(\tau - \tau_2) = 0$, where τ_1, τ_2 are distinct. In particular, we must have either $\tau = \tau_1$ or $\tau = \tau_2$, and so the factor $D_t - \tau_2$ restricts to the cone $\tau = \tau_1$, while the factor $D_t - \tau_1$ restricts to the cone $\tau = \tau_2$. To measure an inhomogeneous term, we define a dual norm by

$$\|F\|_{LE_{mp}^*} = \|(1 - \chi)F\|_{LE_M^*} + \|\chi F\|_{c_1 L^2 + c_2 L^2}. \quad (6.24)$$

Our main result is the following estimate.

THEOREM 6.1. *Let $|a|, |b| \leq \epsilon_0 \ll r_s$ be sufficiently small and $r_- < r_e < r_+$, and let u solve $\square_{mp}u = F$ in \mathcal{R} . Then u satisfies*

$$\|u\|_{LE_{mp}^1}^2 + \sup_{\tilde{t} > 0} E[u](\tilde{t}) + E[u](\Sigma_{\mathcal{R}}^+) \lesssim E[u](0) + \|\square_{mp}u\|_{LE_{mp}^*}^2. \quad (6.25)$$

The proof of this theorem is from [35], though we include additional details as necessary. In particular, we closely follow the exposition from [63].

PROOF. The main idea is to use the multiplier method. Unfortunately, due to the complicated nature of the trapping, no differential operator provides us with a positive local energy norm. Instead, we will use the smoothed out vector field X (from the previous chapter) to control the LE_{mp} norm away from the trapped set and a pseudodifferential correction near the trapped set. Since we would like to establish uniform energy bounds, we will define this correction to be a first order differential operator in t , which will allow us to integrate by parts with respect to time.

Let X, q, m be as in Chapter 5. In particular, for the Schwarzschild metric, we know that

$$Q_S^{(X,q,m)}(u) \gtrsim \frac{1}{r^2}(\partial_r u)^2 + \left(1 - \frac{r_{ps}}{r}\right)^2 \left(\frac{1}{r^2}(\partial_{\tilde{t}} u)^2 + \frac{1}{r}|\nabla u|^2\right) + \frac{u^2}{r^3}.$$

We can perturb off this statement for the Myers-Perry metric. Precisely, let $\partial = (\partial_t, \partial_x)$, where $x = r\omega$ are Euclidean-like coordinates. Then we have:

$$|\partial^\alpha[(g_{mp})_{ij} - (g_S)_{ij}]| \lesssim_\alpha \epsilon_0 r^{-2-|\alpha|}, \quad |\partial^\alpha[(g_{mp})^{ij} - (g_S)^{ij}]| \lesssim_\alpha \epsilon_0 r^{-2-|\alpha|}.$$

Then we easily have

$$|Q_S^{(X,q,m)}(u) - Q_{mp}^{(X,q,m)}(u)| \lesssim \epsilon_0 \left(\frac{1}{r^2}|\partial u|^2 + \frac{1}{r^4}u^2\right),$$

but this only implies that

$$Q_{mp}^{(X,q,m)} \gtrsim \frac{1}{r^2}(\partial_r u)^2 + \left[\left(1 - \frac{r_{ps}}{r}\right)^2 - C\epsilon_0\right] \left(\frac{1}{r^2}(\partial_{\tilde{t}} u)^2 + \frac{1}{r}|\nabla u|^2\right) + \frac{u^2}{r^3}, \quad (6.26)$$

which fails to be positive definite near $r = r_{ps}$.

In order to work around this, we will add a pseudodifferential correction to X and q . To quantize the symbols, we will use a Weyl calculus which is adapted to the Myers-Perry volume element $dV_{mp} = r\rho^2 dr d\omega$. We slightly abuse notation and re-define the Weyl quantization as

$$s^w := \frac{r}{\rho} s^w \frac{\rho}{r}, \quad (6.27)$$

so that real symbols get quantized to self-adjoint operators with respect to $L^2(dV_{mp})$. We define skew-adjoint and self-adjoint pseudodifferential operators S, E , respectively, by

$$S = i s_1^w + s_0^w \partial_t, \quad E = e_0^w + \frac{1}{i} e_{-1}^w \partial_t, \quad (6.28)$$

where $s_1 \in S^1$, $s_0, e_0 \in S^0$, and $e_{-1} \in S^{-1}$ are real symbols with kernels supported close to $r = r_{ps}$. Then we can compute:

$$\Re \int_{\mathcal{R}} (-\square_{mp}u) \cdot (S + E)u dV_{mp} = \int_{\mathcal{R}} Qu \cdot u dV_{mp} + BDR^{mp}(u), \quad (6.29)$$

where

$$Q = \frac{1}{2} \left([-\square_{mp}, S] + (-\square_{mp})E + E(-\square_{mp}) \right) = q_2^w + 2q_1^w D_t + q_0^w D_t^2 + q_{-1}^w D_t^3 \quad (6.30)$$

for symbols $q_j \in S^j$.

Since the Weyl quantization is only with respect to the spatial variables, we define the matrix-valued pseudodifferential operator

$$\tilde{Q} = \begin{pmatrix} q_2^w & q_1^w \\ q_1^w & q_0^w \end{pmatrix}, \quad (6.31)$$

which acts on $\vec{u} = (u, D_t u)$. Let

$$IQ^{mp}(u) = \int_{\mathcal{R}} \tilde{Q}\vec{u} \cdot \vec{u} dV_{mp} = \int_{\mathcal{R}} q_2^w u \cdot \bar{u} + 2\Re q_1^w u \cdot D_t u + q_0^w D_t u \cdot \overline{D_t u} dV_{mp}. \quad (6.32)$$

We will choose S, E so that $q_{-1} = 0$; in this case, after integrating by parts in time, (6.29) becomes

$$\Re \int_{\mathcal{R}} (-\square_{mp} u) \cdot (S + E)u dV_{mp} = IQ^{mp}(u) + BDR^{mp}(u). \quad (6.33)$$

The exact form of the boundary terms does not matter, since we will pick the symbols $s_i \in \epsilon_0 S^i$ to be small when $\epsilon_0 \ll r_s$. In particular, since there are no boundary terms at $r = r_e$, this means that

$$|BDR^{mp}(u)| \lesssim \epsilon_0 (E[u](0) + E[u](\tilde{t}_0)). \quad (6.34)$$

The localized energy estimates now follow if we can choose S and E so that

$$\int_{\mathcal{R}} Q^{(X,q,m)}(u) dV_g + IQ^{mp}(u) \gtrsim \|u\|_{LE_{mp}^1}^2. \quad (6.35)$$

Due to the use of the Weyl calculus, the principal symbol of the quadratic form on the left in (6.35) is

$$\frac{1}{2i} \{p, X + s\} + p(q + e), \quad (6.36)$$

where for convenience we denote

$$s = i(s_1 + \tau s_0), \quad e = e_0 + \tau e_{-1}. \quad (6.37)$$

In order for (6.35) to hold, the above symbol must dominate the principal part of the LE_{mp} norm, i.e.

$$\frac{1}{2i} \{p, X + s\} + p(q + e) \gtrsim c_2^2(\tau - \tau_1)^2 + c_1^2(\tau - \tau_1)^2 + \xi^2. \quad (6.38)$$

Unfortunately, this is not enough since \tilde{Q} is a matrix valued operator of second order, so the Fefferman-Phong inequality does not hold. What we do instead is write the left hand side as a sum of squares of smooth symbols dominating the right hand side. We have:

LEMMA 6.2. *Let $|a|, |b| \leq \epsilon_0 \ll r_s$ be sufficiently small. Then there exist smooth homogeneous symbols $s \in \epsilon_0(S_{hom}^1 + \tau S_{hom}^0)$ and $e \in \epsilon_0(S_{hom}^0 + S_{hom}^{-1})$ so that for $r \approx r_{ps}$, we have*

$$\rho^2 \left(\frac{1}{2i} \{p, X + s\} + p(q + e) \right) = \sum_{j=1}^{11} \mu_j^2 \gtrsim c_2^2(\tau - \tau_1)^2 + c_1^2(\tau - \tau_2)^2 + \xi^2, \quad (6.39)$$

for smooth homogeneous symbols $\mu_j \in S_{hom}^1 + \tau S_{hom}^0$ depending smoothly on a, b , and in addition satisfy:

- (1) the μ_j are differential operators for $a = b = 0$, $j = 1, \dots, 9$,
- (2) μ_{10}, μ_{11} are small in the sense that

$$\mu_{10}, \mu_{11} \in \epsilon_0^{\frac{1}{2}}(S_{hom}^1 + S_{hom}^0).$$

PROOF OF LEMMA 6.2. We will start by reformulating the results of Chapter 5 in the setting of pseudodifferential operators. In the Schwarzschild space-time, the vector field X takes the form

$$X = f_S(r) \frac{\partial}{\partial r_*} = f_S(r) \left(1 - \frac{r_s^{d+1}}{r^{d+1}} \right) \frac{\partial}{\partial r}.$$

Near $r = r_{ps}$, we can write express symbol of this operator by

$$if_S(r) \left(1 - \frac{r_s^2}{r^2} \right) \xi = if(r)(r - r_{ps})\xi,$$

for a smooth function $f > 0$. Then near $r = r_{ps}$, we can write

$$Q_S^{(X,q,m)}(u) = q_S^{\alpha\beta} \partial_\alpha u \partial_\beta u + q_S^0 u^2, \quad (6.40)$$

where the principal symbols are

$$q_S = q_S^{\alpha\beta} \eta_\alpha \eta_\beta = \frac{1}{2i} \{p_S, X\} + qp_S, \quad q_S^0 = -\frac{1}{2} \square_S q. \quad (6.41)$$

The symbol of $-\square_S$ is

$$p_S(r, \omega, \tau, \xi, \lambda) = - \left(1 - \frac{r_s^2}{r^2}\right)^{-1} \tau^2 + \left(1 - \frac{r_s^2}{r^2}\right) \xi^2 + \frac{1}{r^2} \lambda^2, \quad (6.42)$$

where λ stands for the spherical Fourier variable.

We now compute

$$r^2\{p_S, X\} = \{r^2 p_S, X\} - \{r^2, X\} p_S = 2irf(r)(r - r_{ps})p_S.$$

We then explicitly compute the Poisson bracket

$$\begin{aligned} \frac{1}{2i}\{r^2 p_S, X\} &= \frac{1}{2} \frac{\partial}{\partial \xi} (r^2 p_S) \frac{\partial}{\partial r} (f(r)(r - r_{ps})\xi) - \frac{f(r)(r - r_{ps})}{2} \frac{\partial}{\partial r} (r^2 p_S) \\ &= \frac{f(r)(r - r_{ps})}{2} \left[2r \left(1 - \frac{r_s^2}{r^2}\right)^{-1} - r^2 \left(1 - \frac{r_s^2}{r^2}\right)^{-2} \left(\frac{2r_s^2}{r^3}\right) \right] \tau^2 \\ &\quad + (r^2 - r_s^2) (f'(r - r_{ps}) + f) \xi^2 - rf(r)(r - r_{ps})\xi^2 \\ &= \left(\frac{r^3 f(r)(r + r_{ps})(r - r_{ps})^2}{(r^2 - r_s^2)^2} \right) \tau^2 + \left((r^2 - r_s^2)f + (r - r_{ps})(f'(r^2 - r_s^2) - rf) \right) \xi^2. \end{aligned}$$

Using the fact that $f > 0$, we can then express the coefficients as squares,

$$\begin{aligned} \alpha_S^2(r) &= \frac{r^3 f(r)(r + r_{ps})(r - r_{ps})^2}{(r^2 - r_s^2)^2}, \\ \beta_S^2(r) &= (r^2 - r_s^2)f + (r - r_{ps})(f'(r^2 - r_s^2) - rf), \end{aligned}$$

near $r = r_{ps}$ for smooth functions α_S, β_S . We also denote

$$\tilde{q}(r) = q - r^{-1}f(r)(r - r_{ps}).$$

With these substitutions, we can now express

$$\begin{aligned} r^2 q_S &= \frac{1}{2i}\{r^2 p_S, X\} + (1 - r^{-1}f(r)(r - r_{ps})) (r^2 p_S) \\ &= \alpha_S^2(r)\tau^2 + \beta_S^2(r)\xi^2 + \tilde{q}(r)(r^2 p_S). \end{aligned} \quad (6.43)$$

Due to the results in Chapter 5, we know that

$$q_S \gtrsim \xi^2 + (r - r_{ps})^2(\tau^2 + \lambda^2), \quad q_S^0 > 0 \quad (6.44)$$

near $r = r_{ps}$, which implies that \tilde{q} is a multiple of $(r - r_{ps})^2$. We can thus rewrite

$$-g^{tt}r^2\tilde{q}(r) = \left(1 - \frac{r_s^2}{r^2}\right)^{-1} r^2\tilde{q}(r) = \nu(r)\alpha_S^2(r),$$

for a smooth function $0 < \nu(r) < 1$. The symbol λ^2 of the spherical Laplacian can also be written as a sum of squares of differential symbols,

$$\lambda^2 = \sum_{i=1}^6 \lambda_i^2$$

where in Euclidean coordinates we can write

$$\{\lambda_i\} = \{x_j \eta_k - x_k \eta_j : 1 \leq j < k \leq 4\}.$$

This leads to an expression of $r^2 q_S$ as a sum of squares

$$r^2 q_S = (1 - \nu(r))\alpha_S^2(r)\tau^2 + \beta_S^2(r)\xi^2 + \nu(r)\alpha_S^2(r)r^{-2} \left(1 - \frac{r_s^2}{r^2}\right) \left((r^2 - r_s^2)\xi^2 + \sum_{i=1}^6 \lambda_i^2 \right). \quad (6.45)$$

For the Myers-Perry metric, it will be convenient to work with $\rho^2 p$ rather than p . For this reason, we note the identity

$$\rho^2 \left(\frac{1}{2i}\{p, X + s\} + p(q + e) \right) = \frac{1}{2i}\{\rho^2 p, X + s\} + (\rho^2 p) \left((q - \rho^{-2}\{\rho^2, X\}) + (e - \rho^{-2}\{\rho^2, s\}) \right) \quad (6.46)$$

for any X, s, q, e , and correspondingly define

$$\tilde{q} = q - 2\{\log \rho, X\}, \quad \tilde{e} = e - 2\{\log \rho, s\}.$$

We will use (6.46) to switch between these two notations.

The natural counterpart for the Myers-Perry space-time is the symbol

$$\tilde{s} = if(r)(r - r_{a,b}(\tau, \Phi, \Psi))\xi, \quad (6.47)$$

which is a homogeneous symbol that coincides with X in the Schwarzschild case $a = b = 0$, and it is well-defined for r near r_{ps} and $|\Phi|, |\Psi| < C_{mp}|\tau|$. In particular, it is well-defined in a neighborhood of the characteristic set $p = 0$, which is all we need. Using the identity (6.10), we can now compute the Poisson bracket on the characteristic set $\{p = 0\}$:

$$\begin{aligned} \frac{1}{2i}\{\rho^2 p, \tilde{s}\} &= -\frac{1}{2}\frac{\partial}{\partial r}(\rho^2 p)f(r)(r - r_{a,b}(\tau, \Phi, \Psi)) + g^{rr}\frac{\partial}{\partial r}(f(r)(r - r_{a,b}(\tau, \Phi, \Psi)))\xi^2 \\ &= \left(\frac{r}{\Delta^2}f(r)\right)R_{a,b}(r - r_{a,b}(\tau, \Psi, \Phi)) \\ &\quad + \left[\frac{\Delta}{r^2}f(r) + (r - r_{a,b}(\tau, \Phi, \Psi))\left(\frac{\Delta}{r^2}f'(r) - \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\Delta}{r^2}\right)f(r)\right)\right]\xi^2 \end{aligned}$$

Since $r_{a,b}(\tau, \Phi, \Psi)$ is the unique (simple) root of $\tau^{-2}R_{a,b}$ near $r = r_{ps}$ and is close to r_{ps} , we can express

$$\frac{1}{2i}\{\rho^2 p, \tilde{s}\} = \alpha^2(r, \tau, \Phi, \Psi)\tau^2(r - r_{a,b}(\tau, \Phi, \Psi))^2 + \beta^2(r, \tau, \Phi, \Psi)\xi^2 \quad (6.48)$$

on $\{p = 0\}$ and near $r = r_{ps}$, where $\alpha, \beta \in S_{hom}^0$ are positive symbols.

Unfortunately \tilde{s} is not a polynomial in τ , which is problematic as we want s^w to be a first order differential operator in t . To modify the symbol \tilde{s} , we refer to the Malgrange Preparation Theorem (see Theorem 7.5.6 of [29] for a proof), whose full statement is below.

THEOREM 6.3 (Malgrange Preparation Theorem). *Let $f(t, x)$ be a C^∞ function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ near $(0, 0)$ such that*

$$f(0, 0) = \frac{\partial f}{\partial t}(0, 0) = \dots = \frac{\partial^{k-1} f}{\partial t^{k-1}}(0, 0) = 0, \quad \frac{\partial^k f}{\partial t^k}(0, 0) \neq 0$$

for some integer $k \geq 0$, and let $g(t, x)$ be a C^∞ function in a neighborhood of $(0, 0)$. Then

$$g(t, x) = q(t, x)f(t, x) + \sum_{j=0}^{k-1} t^j r_j(x)$$

where $q(t, x)$ and $r_j(x)$ are C^∞ functions in a neighborhood of $(0, 0)$ and 0, respectively.

We note that since $|r_{ps} - r_{a,b}(\tau, \Phi, \Psi)| \lesssim \epsilon_0$, we have

$$\tilde{s} - if(r)(r - r_{ps})\xi \in \epsilon_0 S_{hom}^1.$$

Applying the theorem to the homogeneous symbol p , we can express

$$\frac{1}{i}\tilde{s} = f(r)(r - r_{ps})\xi + s_0(r, \theta, \xi, \Theta, \Phi, \Psi) + s_1(r, \theta, \xi, \Theta, \Phi, \Psi)\tau + \gamma(r, \theta, \tau, \xi, \Theta, \Phi, \Psi)p, \quad (6.49)$$

for homogeneous symbols $s_1 \in \epsilon_0 S_{hom}^1$, $s_0 \in \epsilon_0 S_{hom}^0$, and $\gamma \in \epsilon_0 S_{hom}^{-1}$. We define the desired symbol s by

$$s = i(s_1 + s_0\tau).$$

However, the Poisson bracket $\frac{1}{2i}\{\rho^2 p, s\}$ is generally a third degree polynomial of τ . In the Schwarzschild case (6.45), this expression is a quadratic polynomial in τ with leading order coefficient

$$e_S = (1 - \nu(r))\alpha_S^2(r).$$

Hence, dividing the expression by $(\tau - \tau_1)(\tau - \tau_2)$ and accounting for remainder terms, we can express

$$\frac{1}{2i}\{\rho^2 p, X + s\} + \tilde{q}(\rho^2 p) = \gamma_2 + \gamma_1\tau + (e_S + (e_0 + e_{-1}\tau))(\tau - \tau_1)(\tau - \tau_2), \quad (6.50)$$

for symbols $\gamma_j \in S_{hom}^j$ and $e_j \in \epsilon_0 S_{hom}^j$. We will show that this expression can be written as a sum of squares, modulo a remainder term of the form $\epsilon_0(S_{hom}^0 + \tau S_{hom}^{-1})$. Since the e_j are already $O(\epsilon_0)$, they will be contained in this error.

Note that since $\{\rho^2 p, X + s\} = \{\rho^2 p, \tilde{s}\}$ on the characteristic set $\{p = 0\}$, we can in fact calculate γ_1, γ_2 explicitly by evaluating each of these quantities on $\tau = \tau_i$. Indeed, we define

$$\alpha_i = \frac{2|\tau_i|}{\tau_1 - \tau_2} \alpha(r, \tau_i, \Phi, \Psi)(r - r_{a,b}(\tau_i, \Phi, \Psi)) \in S_{hom}^0, \quad \beta_i = \beta(r, \tau_i, \Phi, \Psi). \quad (6.51)$$

Substituting $\tau = \tau_i$ into (6.48), we calculate

$$\frac{1}{2i} \{\rho^2 p, \tilde{s}\}(\tau_i) = \frac{1}{4} \alpha_i^2 (\tau_1 - \tau_2)^2 + \beta_i^2 \xi^2.$$

Making the same substitution into (6.50), this also equals

$$\frac{1}{2i} \{\rho^2 p, \tilde{s}\}(\tau_i) = \gamma_2 + \gamma_1 \tau_i.$$

Solving the system of equations, we then obtain:

$$\begin{aligned} \gamma_2 &= \frac{1}{4} (\tau_1 - \tau_2) (\tau_1 \alpha_2^2 - \tau_2 \alpha_1^2) + \left(\frac{\tau_1 \beta_2^2 - \tau_2 \alpha_2^2}{\tau_1 - \tau_2} \right) \xi^2, \\ \gamma_1 &= \frac{1}{4} (\tau_1 - \tau_2) (\alpha_1^2 - \alpha_2^2) + \left(\frac{\beta_1^2 - \beta_2^2}{\tau_1 - \tau_2} \right) \xi^2. \end{aligned}$$

To obtain the desired sum of squares expression, we combine these two expressions:

$$\begin{aligned} \gamma_2 + \tau \gamma_1 &= \frac{1}{4} \left((\tau_1 - \tau_2) (\tau_1 \alpha_2^2 - \tau_2 \alpha_1^2) + \tau (\tau_1 - \tau_2) (\alpha_1^2 - \alpha_2^2) \right) \\ &\quad + \left(\frac{\tau_1 \beta_2^2 - \tau_2 \alpha_2^2}{\tau_1 - \tau_2} + \tau \frac{\beta_1^2 - \beta_2^2}{\tau_1 - \tau_2} \right) \xi^2. \end{aligned} \quad (6.52)$$

The first term can be rewritten as:

$$\begin{aligned} (\tau_1 - \tau_2) (\tau_1 \alpha_2^2 - \tau_2 \alpha_1^2) + \tau (\tau_1 - \tau_2) (\alpha_1^2 - \alpha_2^2) &= ((\tau - \tau_2) - (\tau - \tau_1)) \left((\tau - \tau_2) \alpha_1^2 - (\tau - \tau_1) \alpha_2^2 \right) \\ &= (\tau - \tau_2)^2 \alpha_1^2 + (\tau - \tau_1)^2 \alpha_2^2 - (\alpha_1^2 + \alpha_2^2) (\tau - \tau_1) (\tau - \tau_2) \\ &= \nu(r) (\alpha_1 (\tau - \tau_2) - \alpha_2 (\tau - \tau_1))^2 + (1 - \nu(r)) (\alpha_1 (\tau - \tau_2) + \alpha_2 (\tau - \tau_1))^2 \\ &\quad - ((\alpha_1 - \alpha_2)^2 + 4(1 - \nu(r)) \alpha_1 \alpha_2) (\tau - \tau_1) (\tau - \tau_2). \end{aligned} \quad (6.53)$$

We thus label

$$e_{mp} = \frac{1}{4} (\alpha_1 - \alpha_2)^2 + (1 - \nu(r)) \alpha_1 \alpha_2 \in S_{hom}^0.$$

Note that in the Schwarzschild case $a = b = 0$, we simply have

$$\tau_2 = -\tau_1, \quad \alpha_1 = \alpha_2 = \alpha_S, \quad \beta_1 = \beta_2 = \beta_S.$$

Since the α_i depend smoothly on a, b , we have $e_{mp} - e_S \in \epsilon_0(S_{hom}^0)$, so in fact e_{mp} cancels out the e_S term in (6.50) up to a small error.

For the second term in (6.52), we have

$$\frac{\tau_1 \beta_2^2 - \tau_2 \alpha_2^2}{\tau_1 - \tau_2} + \tau \frac{\beta_1^2 - \beta_2^2}{\tau_1 - \tau_2} = \frac{\tau - \tau_2}{\tau_1 - \tau_2} \beta_1^2 - \frac{\tau - \tau_1}{\tau_1 - \tau_2} \beta_2^2.$$

Considering each of these terms separately, we can rewrite

$$\begin{aligned} \frac{\tau - \tau_2}{\tau_1 - \tau_2} &= \frac{(\tau_1 - \tau_2) + (\tau - \tau_1)}{\tau_1 - \tau_2} = 1 + \frac{(\tau - \tau_1)(\tau_1 - \tau_2)}{\tau_1 - \tau_2} \\ &= 1 + \frac{(\tau - \tau_1) ((\tau - \tau_2) - (\tau - \tau_1))}{\tau_1 - \tau_2} \\ &= 1 - \frac{(\tau - \tau_1)^2}{(\tau_1 - \tau_2)^2} + \frac{(\tau - \tau_1)(\tau - \tau_2)}{(\tau_1 - \tau_2)^2}. \end{aligned}$$

However, this also equals

$$\begin{aligned} \frac{\tau - \tau_2}{\tau_1 - \tau_2} &= \frac{(\tau - \tau_2)((\tau - \tau_2) - (\tau - \tau_1))}{\tau_1 - \tau_2} \\ &= \frac{(\tau - \tau_2)^2}{(\tau_1 - \tau_2)^2} - \frac{(\tau - \tau_1)(\tau - \tau_2)}{(\tau_1 - \tau_2)^2}. \end{aligned}$$

Combining these two expressions, and carrying out the same exercise for $\frac{\tau - \tau_1}{\tau_1 - \tau_2}$, we then obtain:

$$\begin{aligned} \frac{\tau - \tau_2}{\tau_1 - \tau_2} &= \frac{1}{2} + \frac{(\tau - \tau_2)^2 - (\tau - \tau_1)^2}{(\tau_1 - \tau_2)^2}, \\ -\frac{\tau - \tau_1}{\tau_1 - \tau_2} &= \frac{1}{2} + \frac{(\tau - \tau_1)^2 - (\tau - \tau_2)^2}{(\tau_1 - \tau_2)^2}. \end{aligned}$$

We also calculate:

$$\frac{(\tau - \tau_1)^2 + (\tau - \tau_2)^2}{(\tau_1 - \tau_2)^2} - 1 = 2 \frac{(\tau - \tau_1)(\tau - \tau_2)}{(\tau_1 - \tau_2)^2}.$$

Combining all of these expressions and letting C be a large positive constant, we express the second term in (6.52) as

$$\frac{1}{2}(\beta_1^2 + \beta_2^2 - C\epsilon_0) + \frac{(C\epsilon_0 - \beta_2^2 + \beta_1^2)(\tau - \tau_2)^2}{2(\tau_1 - \tau_2)^2} + \frac{(C\epsilon_0 - \beta_1^2 + \beta_2^2)(\tau - \tau_1)^2}{2(\tau_1 - \tau_2)^2} + O(\epsilon_0)(\tau - \tau_1)(\tau - \tau_2). \quad (6.54)$$

We note that $\beta_1 - \beta_2 \in \epsilon_0 S^0$, so that the second and third terms are positive for C sufficiently large, which can be chosen independently of ϵ_0 . Then $\epsilon_0 \ll 1$ sufficiently small ensures that the first term is also positive.

Finally, we can write

$$\begin{aligned} \frac{1}{2i}\{\rho^2 p, X + s\} + \tilde{q}(\rho^2 p) &= \frac{\nu(r)}{4}(\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2 + \frac{1 - \nu(r)}{4}(\alpha_1(\tau - \tau_2)^2 + \alpha_2(\tau - \tau_1))^2 \\ &\quad + \frac{(C\epsilon_0 - \beta_2^2 + \beta_1^2)(\tau - \tau_2)^2}{2(\tau_1 - \tau_2)^2} + \frac{(C\epsilon_0 - \beta_1^2 + \beta_2^2)(\tau - \tau_1)^2}{2(\tau_1 - \tau_2)^2} \\ &\quad + \frac{1}{2}(\beta_1^2 + \beta_2^2 - C\epsilon_0) - \tilde{e}(\rho^2 p) \end{aligned} \quad (6.55)$$

for a symbol $\tilde{e} \in \epsilon_0(S_{hom}^0 - S_{hom}^{-1})$. We then define $e = \tilde{e} + 2\{\log \rho, s\} \in \epsilon_0(S_{hom}^0 + S_{hom}^{-1})$ to cancel the final term. Then the expression is clearly a sum of squares, and with (6.46), we obtain the desired bound

$$\rho^2 \left(\frac{1}{2i}\{p, X + s\} + (q + e)p \right) \gtrsim c_1^2(\tau - \tau_2)^2 + c_2^2(\tau - \tau_1)^2 + \xi^2. \quad (6.56)$$

In particular, we rely on the fact that $\alpha_i^2 \gtrsim c_i^2$.

We next identify the individual summands so as to satisfy the statement of Lemma 6.2. We define:

$$\begin{aligned} \mu_i^2 &= \frac{\lambda_i^2}{\lambda^2 + (r^2 - r_s^2)\xi^2} \frac{\nu(r)}{4} (\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2, \quad i = 1, \dots, 6, \\ \mu_7^2 &= \frac{(r^2 - r_s^2)\xi^2}{\lambda^2 + (r^2 - r_s^2)\xi^2} \frac{\nu(r)}{4} (\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2, \\ \mu_8^2 &= \frac{1 - \nu(r)}{4} (\alpha_1(\tau - \tau_1) + \alpha_2(\tau - \tau_2))^2, \quad \mu_9^2 = \frac{1}{2} (\beta_1^2 + \beta_2^2 - C\epsilon_0) \xi^2, \\ \mu_{10}^2 &= \frac{(C\epsilon_0 - \beta_2^2 + \beta_1^2)(\tau - \tau_2)^2}{2(\tau_1 - \tau_2)^2} \xi^2, \quad \mu_{11}^2 = \frac{(C\epsilon_0 - \beta_1^2 + \beta_2^2)(\tau - \tau_1)^2}{2(\tau_1 - \tau_2)^2} \xi^2. \end{aligned}$$

In particular,

$$\sum_{i=1}^7 \mu_i^2 = \frac{\nu(r)}{4} (\alpha_1(\tau - \tau_2) - \alpha_2(\tau - \tau_1))^2,$$

which accounts for the first term in (6.55).

For the Schwarzschild case $a = b = 0$, we compare with (6.45). We note that $\tau_1 = -\tau_2$ and that

$$\tau_1^2 = \tau_2^2 = \left(1 - \frac{r_s^2}{r^2}\right) \left(\frac{1}{r^2}\lambda^2 + \left(1 - \frac{r_s^2}{r^2}\right)\xi^2\right),$$

so in fact

$$\begin{aligned} \mu_i^2 &= \frac{\nu(r)}{r^2} \left(1 - \frac{r_s^2}{r^2}\right) \alpha_s^2(r) \lambda_i^2, \quad i = 1, \dots, 6, \\ \mu_7^2 &= \nu(r) \alpha_S^2(r) \left(1 - \frac{r_s^2}{r^2}\right)^2 \xi^2, \quad \mu_8^2 = (1 - \nu(r)) \alpha_S^2(r) \tau^2, \quad \mu_9^2 = \beta_S^2(r) \xi^2, \end{aligned}$$

and moreover $\mu_{10}^2 = \mu_{11}^2 = 0$ since $\beta_1 = \beta_2 = \beta_S$. This completes the proof of the lemma. \square

We now use the lemma to complete the proof of the theorem. First, we truncate away the low frequencies, so we re-define

$$s := \chi_{>1}s, \quad e := \chi_{>1}e,$$

where χ is a smooth symbol which equals 1 for frequencies $\gg 1$ and 0 for frequencies $\lesssim 1$. This serves to transform the homogeneous symbols into smooth symbols. Let $\chi \in C_c^\infty$ be a smooth cutoff function which is supported near $r = r_{ps}$ and which equals 1 on a neighborhood of $r = r_{ps}$. This gives a smooth partition of unity in r ,

$$1 = \chi^2(r) + \chi_o^2(r), \quad \chi_o^2(r) = 1 - \chi^2(r).$$

We now define the operators

$$S = \chi s^w \chi, \quad \tilde{E} = \chi e^w \chi,$$

for which we then have the expression

$$Q[S, \tilde{E}] = \frac{1}{2} \left([-\square_{mp}, S] + (-\square_m p) \tilde{E} + \tilde{E} (-\square_{mp}) \right) = q_2^w + q_1^w D_t + q_0^w D_t^2 + q_{-1}^w D_t^3, \quad (6.57)$$

for smooth symbols $q_j \in S^j$. Note that each of these operators have integral kernels $K(x, y)$ that are supported in a small neighborhood of (r_{ps}, r_{ps}) due to the order in which we applied cutoffs. The principal symbol of (6.57) is

$$q_p[S, \tilde{E}] = \frac{1}{2i} \{p, \chi s \chi\} + p(\chi e \chi) = \chi^2 \left(\frac{1}{2i} \{p, s\} + p e \right) + \frac{1}{i} \chi s \{p, \chi\}. \quad (6.58)$$

By construction, $q_p[S, \tilde{E}]$ is at most a second order polynomial in τ . We can then write:

$$Q[S, \tilde{E}] - \left(q_p[S, \tilde{E}] \right)^w \in \Psi^0 + \Psi^{-1} D_t + \Psi^{-2} D_t^2 + \Psi^{-3} D_t^3,$$

which follows from the relevant properties of the Weyl calculus, namely, that

$$\begin{aligned} [a^w, b^w] - \{a, b\}^w &\in \Psi^{m_1+m_2-3}, \\ a^w b^w + b^w a^w - 2(ab)^w &\in \Psi^{m_1+m_2-2}, \end{aligned} \quad (6.59)$$

for any symbols $a \in S^{m_1}$ and $b \in S^{m_2}$ (see [30]). In particular, we have that $q_{-1}^w \in \Psi^{-3}$. We then re-define

$$E := \chi e^w \chi - e_{aux}^w D_t,$$

where the operator e_{aux}^w is chosen so that the D_t^3 term in (6.57) vanishes. In particular, we require that

$$g^{tt} e_{aux}^w + e_{aux}^w g^{tt} = q_{-1}^w.$$

This can be done by defining the kernel $K[e_{aux}^w]$ to be

$$K[e_{aux}^w](x, y) = \frac{K[q_{-1}^w](x, y)}{g^{tt}(x) + g^{tt}(y)},$$

which is well-defined in a neighborhood of $r = r_{ps}$ since $g^{tt} < 0$, and moreover is supported near (r_{ps}, r_{ps}) since $K[q_{-1}^w]$ is supported near (r_{ps}, r_{ps}) . By construction we also have $e_{aux}^w \in \Psi^{-3}$. In particular, we have

$$Q[S, E] = q_2^w + q_1^w D_t + q_0^w D_t^2 \quad \text{mod } \Psi^{-1} D_t + \Psi^{-2} D_t^2,$$

where the remainder terms all have kernels that are supported on a neighborhood of (r_{ps}, r_{ps}) .

We now split the space-time integral $IQ^{(S,E)}(u)$ into two parts:

$$IQ^{(S,E)}(u) = IQ_p(u) + IQ_{aux}(u), \quad (6.60)$$

where $IQ_p(u)$ comprises the bulk terms given by

$$Q_p = \chi \left(\frac{1}{2i} \{p, s\} + pe \right)^w \chi = q_{2,p}^w + q_{1,p}^w D_t + q_{0,p}^w D_t^2$$

for smooth symbols $q_{j,p} \in S^j$, so that

$$IQ_p(u) = \int_{\mathcal{R}(0, \bar{t}_0)} q_{2,p}^w u \cdot \bar{u} + 2\Re q_{1,p}^w u \cdot \overline{D_t u} + q_{0,p}^w D_t u \cdot \overline{D_t u} dV_{mp}. \quad (6.61)$$

We also calculate:

$$[-\square_{mp}, \chi s^w \chi] = \chi [-\square_{mp}, s^w] \chi + \chi s^w [-\square_{mp}, \chi] + [-\square_{mp}, \chi] s^w \chi, \quad (6.62)$$

and similarly

$$(-\square_{mp}) \chi e^w \chi + \chi e^w \chi (-\square_{mp}) = \chi \left((-\square_{mp}) e^w + e^w (-\square_{mp}) \right) \chi + [-\square_{mp}, \chi] e^w \chi - \chi e^w [-\square_{mp}, \chi]. \quad (6.63)$$

In particular, we can write

$$\frac{1}{2} \chi \left([-\square_{mp}, s^w] + e^w (-\square_{mp}) + (-\square_{mp}) e^w \right) \chi - Q_p \in \chi \left(\Psi^0 + \Psi^{-1} D_t + \Psi^{-2} D_t^2 \right) \chi. \quad (6.64)$$

The remainder terms $IQ_{aux}(u) = IQ^{(S,E)}(u) - IQ_p(u)$ are then given by

$$Q_{aux} = \chi (s^w - e^w) [-\square_{mp}, \chi] + [-\square_{mp}, \chi] (s^w + e^w) \chi + \chi (r_{2,aux}^w + r_{1,aux}^w D_t + r_{0,aux}^w D_t^2) \chi$$

for smooth symbols $r_{j,aux} \in S^{j-2}$. Then $IQ_{aux}(u)$ reads

$$\begin{aligned} IQ_{aux}(u) &= \int_{\mathcal{R}(0, \bar{t}_0)} (\chi (s^w - e^w) [-\square_{mp}, \chi] + [-\square_{mp}, \chi] (s^w + e^w) \chi) u \cdot \bar{u} dV_{mp} \\ &\quad + \int_{\mathcal{R}(0, \bar{t}_0)} (\chi r_{2,aux}^w \chi) u \cdot \bar{u} + 2\Re (\chi r_{1,aux}^w \chi) u \cdot \overline{D_t u} + (\chi r_{0,aux}^w \chi) D_t u \cdot \overline{D_t u} dV_{mp}. \end{aligned} \quad (6.65)$$

Note that since $\chi \equiv 1$ on a neighborhood of the trapped set, the commutator $[-\square_{mp}, \chi]$ is a first order differential operator is zero there. We can then control each of these terms using the LE_{mp}^1 norm, which we remark is non-degenerate away from the trapped set. Indeed, with respect to the L^2 inner product, we have:

$$|\langle \chi s^w [-\square_{mp}, \chi] u, u \rangle| = |\langle s^w [-\square_{mp}, \chi] u, \chi u \rangle| \lesssim \|[-\square_{mp}, \chi] u\|_{L_t^2 H_x^1} \|\chi u\|_{L^2} \lesssim \|u\|_{LE_{mp}^1}^2,$$

$$|\langle [-\square_{mp}, \chi] s^w \chi u, u \rangle| = |\langle \chi u, s^w [-\square_{mp}, \chi] u \rangle| \lesssim \|\chi u\|_{L^2} \|[-\square_{mp}, \chi] u\|_{L_t^2 H_x^1} \lesssim \|u\|_{LE_{mp}^1}^2,$$

and similarly for s^w replaced with e^w . This follows from the fact that $s, e \in S^1$. To estimate the remainder terms corresponding to $r_{j,aux}$, we have:

$$|\langle \chi r_{2,aux} \chi u, u \rangle| = |\langle r_{0,aux} \chi u, \chi u \rangle| \lesssim \|\chi u\|_{L^2}^2 \lesssim \|u\|_{LE_{mp}^1}^2,$$

$$\begin{aligned} |\langle \chi r_{1,aux} \chi u, D_t u \rangle| &= |\langle r_{1,aux} \chi u, \chi D_t u \rangle| \leq \|r_{1,aux} \chi u\|_{L_t^2 H_x^1} \|\chi D_t u\|_{H_x^{-1}} \\ &\lesssim \|\chi u\|_{L^2} \|\chi D_t u\|_{L_t^2 H_x^{-1}} \lesssim \|u\|_{LE_{mp}^1}^2 + \|\chi D_t u\|_{L_t^2 H_x^{-1}}^2, \end{aligned}$$

$$|\langle \chi r_{0,aux} \chi D_t u, D_t u \rangle| = |\langle r_{0,aux} \chi D_t u, \chi D_t u \rangle| \lesssim \|r_{0,aux} \chi D_t u\|_{L_t^2 H_x^1} \|\chi D_t u\|_{L_t^2 H_x^{-1}} \lesssim \|\chi D_t u\|_{L_t^2 H_x^{-1}}^2$$

Combining these bounds yields:

$$|IQ_{aux}(u)| \lesssim \|u\|_{LE_{mp}^1}^2 + \|\chi D_t u\|_{L_t^2 H_x^{-1}}^2. \quad (6.66)$$

To prove (6.35), we will show that

$$\int_{\mathcal{R}(0, \bar{t}_0)} Q_{mp}^{(X,q,m)}(u) dV_{mp} + IQ_p(u) \gtrsim \|u\|_{LE_{mp}^1}^2 - \epsilon_0 \|\chi D_t u\|_{L_t^2 H_x^{-1}}^2. \quad (6.67)$$

We split up this expression into terms corresponding to trapping and non-trapping regions, respectively. In particular, we have

$$\int_{\mathcal{R}(0, \bar{t}_0)} Q_{mp}^{(X, q, m)}(u) dV_{mp} + IQ_p(u) = IQ_{ps}(u) + IQ_o(u), \quad (6.68)$$

where we denote

$$IQ_o(u) = \int_{\mathcal{R}(0, \bar{t}_0)} \chi_o^2 Q_{mp}^{(X, q, m)}(u) dV_{mp}, \quad (6.69)$$

to correspond to the non-trapping region, and

$$IQ_{ps}(u) = IQ_p(u) + \int_{\mathcal{R}(0, \bar{t}_0)} \chi^2 Q_{mp}^{(X, q, m)}(u) dV_{mp} \quad (6.70)$$

to correspond to the trapping region.

Away from the trapped set, the calculations from (6.26) easily imply that

$$IQ_o(u) \gtrsim \int_{\mathcal{R}(0, \bar{t}_0)} \chi_o^2 \left(\frac{1}{r^2} (\partial_{\bar{t}} u)^2 + \frac{1}{r^2} (\partial_r u)^2 + \frac{1}{r} |\nabla u|^2 + \frac{u^2}{r^3} \right) dV_{mp} \quad (6.71)$$

for $\epsilon_0 \ll 1$ sufficiently small.

Near the trapped set, we can express

$$\begin{aligned} IQ_{ps}(u) &= \int_{\mathcal{R}(0, \bar{t}_0)} (q_{2,p}^w u \cdot \bar{u} + 2\Re q_{1,p}^w u \cdot \overline{D_t u} + q_{1,p}^w D_t u \cdot \overline{D_t u}) dV_{mp} \\ &\quad + \int_{\mathcal{R}(0, \bar{t}_0)} \chi^2 (q_{mp}^{\alpha\beta} \partial_\alpha u \partial_\beta u + q_{mp}^0 u^2) dV_{mp}. \end{aligned} \quad (6.72)$$

We note that the coefficients $q_{mp}^{\alpha\beta}, q_{mp}^0$ are given by

$$q_{mp}^{\alpha\beta} \eta_\alpha \eta_\beta = \frac{1}{2i} \{p, X\} + qp, \quad q_{mp}^0 = -\frac{1}{2} \square_{mp} q > 0,$$

given that $\epsilon_0 \ll r_s$ is sufficiently small. Moreover, the symbols $q_{j,p}$ are defined by

$$q_{2,p}^w + 2q_{1,p}^w D_t + q_{0,p}^w D_t^2 = \chi \left(\frac{1}{2i} \{p, s\} + pe \right)^w \chi.$$

Unfortunately, Lemma 6.2 does not directly yield a sum-of-squares representation for this expression, as the cutoff χ appears in different places in IQ_{ps} . If we were to commute one of the terms in (6.72) with χ , we would destroy either the positivity of q_{mp}^0 or the compact support of the kernels of $q_{j,p}^w$.

Instead, we rewrite the symbols μ_j as

$$\mu_j = \mu_{j,0} + (\mu_j - \mu_{j,0}),$$

where $\mu_{j,0}$ is the symbol corresponding to the Schwarzschild case $a = b = 0$. In particular, Lemma 6.2 ensures that $\mu_{j,0}$ are differential operators, as well as that $|\mu_j - \mu_{j,0}| = O(\epsilon_0)$. We define the operators

$$M_j = \chi \mu_{j,0}(x, D) + (\mu_j - \mu_{j,0})^w \chi,$$

where the first term corresponds to the standard quantization, as opposed to the Weyl calculus. We can then express IQ_{ps} as

$$IQ_{ps}(u) = \int_{\mathcal{R}(0, \bar{t}_0)} \sum_j |M_j u|^2 + \chi^2 q_{mp}^0 u^2 dV_{mp} + \int_{\mathcal{R}(0, \bar{t}_0)} r_2^w u \cdot u + 2\Re r_1^w u \cdot \overline{D_t u} + r_0^w D_t u \cdot \overline{D_t u} dV_{mp} \quad (6.73)$$

for symbols $r_j \in \epsilon_0 S^{j-2}$, by the Weyl calculus.

Combining (6.71) and (6.73) and estimating the remainder terms in the spirit of (6.66), we obtain

$$\begin{aligned} \int_{\mathcal{R}(0, \bar{t}_0)} \left(\chi_o^2 \frac{1}{r^2} |\nabla_{t,x} u|^2 + \frac{u^2}{r^3} + \sum_j |M_j u|^2 \right) dV_{mp} &\lesssim \int_{\mathcal{R}(0, \bar{t}_0)} Q_{mp}^{(X, q, m)}(u) dV_{mp} + IQ_{ps}(u) \\ &\quad + \epsilon_0 (\|\chi u\|_{L^2}^2 + \|\chi D_t u\|_{L_t^2 H_x^{-1}}^2). \end{aligned} \quad (6.74)$$

We now show that the left-hand-side of this equation dominates the local energy norm $\|u\|_{LE_{m_p}^1}^2$. Indeed, this is straightforward away from the trapped region due to the first two terms. Near the trapped set, we instead note that the principal symbol of M_k is $\chi\mu_k$, for which we have

$$\chi^2 \sum_j \mu_j^2 \gtrsim \chi (c_1^2(\tau - \tau_2)^2 + c_2^2(\tau - \tau_1)^2 + \xi^2) \chi.$$

To translate this into an operator bound, we rely on the following variant of the Sharp Garding inequality from [60]:

THEOREM 6.4. *Let $a_j, b \in C^{1,1}S^1$ be real symbols with $|b| \lesssim \sum_{j=1}^N |a_j|$. Then*

$$\|b(x, D)\|_{L^2} \lesssim \sum_{j=1}^N \|a_j(x, D)u\|_{L^2} + \|u\|_{L^2}.$$

With the cutoffs now in place, we obtain

$$\|u\|_{LE_{m_p}^1}^2 \lesssim IQ_{ps}(u) + \int_{\mathcal{R}(0, \tilde{t}_0)} Q_{mp}^{(X, q, m)}(u) dV_{mp} + \epsilon_0 \|\chi D_t u\|_{L_t^2 H_x^{-1}},$$

and it remains only to absorb the final term. To do this, let $Q = q(x, D) \in \Psi^{-1}$ be a self-adjoint pseudodifferential operator supported near the photon sphere. We can then use $(g^{tt})^{-1}Q^2$ to integrate by parts:

$$\begin{aligned} \Re \int_{\mathcal{R}(0, \tilde{t}_0)} \chi((g^{tt})^{-1}\square_{mp})u \cdot \overline{Q^2\chi}udV_{mp} &\gtrsim \|Q\chi D_t u\|_{L^2}^2 - \|QD_t u\|_{L^2} \|Q\chi\nabla_x u\|_{L^2} \\ &\quad - \|Q\chi\nabla_x u\|_{L^2}^2 - E[u](\tilde{t}_0) - E[u](0) \\ &\gtrsim (1 - \epsilon) \|Q\chi D_t u\|_{L^2}^2 - C_\epsilon \|Q\chi\nabla_x u\|_{L^2}^2 \\ &\quad - E[u](\tilde{t}_0) - E[u](0) \\ &\gtrsim \|Q\chi D_t u\|_{L^2}^2 - \|\chi u\|_{L^2} - E[u](\tilde{t}_0) - E[u](0), \end{aligned}$$

if $\epsilon \ll 1$ is sufficiently small. In particular, the factor of $(g^{tt})^{-1}$ ensures that no additional terms involving $D_t u$ show up. We can also bound the left-hand-side of this inequality by

$$\left| \Re \int_{\mathcal{R}(0, \tilde{t}_0)} \chi((g^{tt})^{-1}\square_{mp})u \cdot \overline{Q^2\chi}udV_{mp} \right| \lesssim \|\chi\square_{mp}u\|_{L_t^2 H_x^{-1}} \|Q^2\chi u\|_{L^2} \lesssim \|\chi\square_{mp}u\|_{L_t^2 H_x^{-1}}^2 + \|\chi u\|_{L^2}^2.$$

Hence, we obtain

$$\begin{aligned} \|Q\chi D_t u\|_{L^2}^2 &\lesssim \|\chi u\|_{L^2}^2 + \|\chi\square_{mp}u\|_{L_t^2 H_x^{-1}}^2 + E[u](0) + E[u](\tilde{t}_0) \\ &\lesssim \|u\|_{LE_{m_p}^1}^2 + \|\square_{mp}u\|_{LE_{m_p}^*}^2 + E[u](0) + E[u](\tilde{t}_0). \end{aligned} \tag{6.75}$$

Since Q is arbitrary, this implies that

$$\|\chi D_t u\|_{L_t^2 H_x^{-1}} \lesssim \|u\|_{LE_{m_p}^1}^2 + \|\square_{mp}u\|^2 + LE_{m_p}^* + E[u](0) + E[u](\tilde{t}_0). \tag{6.76}$$

Finally, for $\epsilon_0 \ll 1$ sufficiently, we obtain (6.35), which completes the proof of the theorem. \square

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