Capacity Investment Strategies under Operational Flexibility

by
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ABSTRACT

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Capacity Investment Strategies under Operational Flexibility.
(Under the direction of Professor Tekin.)

Operational flexibility has been attractive in many industries to hedge against demand uncertainty and to promote profits by decreasing lost sales, saving on investments and providing higher quality service. Hence, it is extremely important to develop quantitative models that will provide insights on how to manage systems with some form of flexibility in their operations. In this research, we propose to study optimal capacity investment, resource allocation and pricing decisions of a central decision maker that manages multiple resources which can be utilized flexibly to satisfy demands from multiple market segments. The main objectives of the proposed research are 1) to develop quantitative models in order to determine the optimal capacity investment decisions for multiple resources that can be used flexibly to satisfy stochastic demands from multiple customer segments, 2) to develop easy-to-implement computational algorithms for computing optimal or near-optimal solutions, 3) to quantify the benefits of managing multiple resources that can be used flexibly.

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CONTENTS

List of Figures

LIST OF TABLES

Chapter 1

Introduction

Global competition has forced most manufacturing and service industries to persistently strive to decrease their operating costs, and still improve efficiency and quality of their services. Responding economically to the specific needs of customers has become a major challenge. Firms are continuously reorganizing themselves and making use of various strategies to match their resources to varying demands from different market segments.

Operational flexibility has been one strategy that companies use to hedge against demand uncertainty, and to promote profits by decreasing lost sales, saving on investments and providing higher quality service. Operational flexibility broadly implies various tactics such as use of flexible resources (i.e., plants, machines, workers), product and volume flexibility, delayed product differentiation, and flexible sourcing, pricing and distribution of goods and services. In this research, we focus on two aspects of operational flexibility: Resource flexibility and ex-post pricing (i.e., the ability to set prices after observing the demand patterns (see, e.g., Van Mieghem and Dada (1999)).

In particular, we study optimal capacity investment, resource allocation/reallocation and pricing decisions of a firm that manages multiple resources, which can be utilized flexibly to satisfy demands from multiple market segments. We use the term resource in the broad sense to mean manufacturing capacity or inventory. We consider there are multiple customer classes whose demands are characterized by random market sizes and selling prices. Furthermore, the firm has the ability to use a particular resource capacity to satisfy demands from different market segments, other than its own, at the expense of a reallocation cost to hedge against demand uncertainty. In general, due to the long production lead times and contractual agreements, the capacity investment decisions for resources must be made long before the market sizes are known with accuracy. On the other hand, reallocation and pricing decisions can be postponed until more information about the actual market conditions is obtained.

We focus on the following two-stage problem: In the first stage, the firm makes its capacity investment decision for multiple resources in the face of uncertain demand so as to maximize the total expected profit. In the second stage, after the market sizes are realized, the firm jointly determines its prices and capacity reallocations to maximize the total profit based on the capacity investment decisions made in the first stage.

We study models that address the strategic capacity investment decisions faced by a number of industries, such as manufacturing companies that operate reconfigurable plants, retailers with multiple sales outlets in different geographical locations, etc. For example, consider a car manufacturing company that sells its vehicles through its dealers which are geographically distributed within a region. The major source of demand for each dealer is its local community. The company has to decide how many vehicles from each model to put in the inventory at each dealer in the beginning of a selling season under demand uncertainty. The time between two replenishments is usually long (e.g., six months), and hence, this decision can be treated as a single-period problem. After the company allocates the vehicles to dealers, the demand becomes observable as the sales are made. It is highly likely that the actual demand does not match the supply at each dealer. An effective way to balance demand and supply is to adjust the selling price based on the realized market potential and reallocate the vehicles among the dealers, if needed. Such an operational flexibility allows the company to generate more profits by matching supply with demand.

Another example can be a manufacturing company that operates multiple plants which are reconfigurable to produce a variety of products. The manufacturing capacity that will be allocated to each product is determined when demand is highly uncertain. This especially applies to seasonal products. As the beginning of the season approaches, more information about the demand is collected, and the company makes its pricing decisions and reallocates manufacturing capacities to different products at a cost, if necessary.

The main objectives of this research are as follows:

- to develop quantitative models in order to determine the optimal capacity investment decisions for multiple resources that can be used flexibly to satisfy stochastic demands from multiple customer segments;
- to develop easy-to-implement computational algorithms for computing optimal or near-optimal solutions;
- to quantify the benefits of managing multiple resources that can be used flexibly.

In Chapter 2, we present the literature review. In Chapter 3, we focus on capacity investment, resource allocation and pricing decisions faced by a central decision maker that manages two resources which can be used flexibly to satisfy demands from two market segments. We consider situations where the capacity investment decisions have to be made in the face of uncertain demand. On the other hand, resource allocation and pricing decisions are made after the uncertainty about demand is resolved. Accordingly, the decision making process is formulated as a two-stage stochastic programming problem. The second stage problem determines the optimal resource allocations and selling prices given the realized demands and resource capacities, and the first stage problem seeks for the optimal capacities given the random demands. We explicitly solve the second stage problem for the two-resource system, and further investigate the properties of the first stage problem. We find that depending on the magnitudes of the unit costs of the resources, the optimal capacity investment strategy takes one of the following three forms: (1) Do not invest in any of the resources; (2) Invest in one of the resources; (3) Invest in both resources.

In Chapter 4, we allow for multiple (more than two) resources in the system. We investigate the structural properties of the optimal solution of the second stage problem, and then provide heuristic methods to solve the second stage problem efficiently. In the numerical experiments, we first investigate the performance of the heuristics, and then study the impact of the system parameters on the optimal investment strategies. The first stage problem is solved by Monte Carlo simulation, which requires an efficient, yet accurate, solution of the stage two problem.

In Chapter 5, we consider two extensions of the multi-resource model. First, we relax the assumption that each facility has its own market. Next, we consider that resources are utilized through multiple periods instead of a single period. This multi-period model is more realistic, however, it has a more complicated structure. Therefore, we focus on studying the properties and structure of the optimal solution, and searching for efficient computational algorithms to solve the problem.

Chapter 2

Literature Review

The problem addressed in this dissertation is closely related to two streams of literature. The first stream of literature focuses on strategic models that address the issue of how to make investment decisions in environments with a mix of dedicated and flexible resources. The second stream of literature develops models for inventory procurement decisions in multi-product inventory systems where one product can be substituted for another. Both streams of literature investigate the effects of future demand uncertainty on investment/procurement decisions and quantify the benefits of operational flexibility gained through flexible production capacity or inventory substitution.

Operational flexibility has been of interest to many researchers for a long time. De Groote (1994) proposed a general framework of flexibility which is based on three elements: the set of technologies whose flexibility is going to be compared, the set of environments in which those technologies might be operated, and a performance criterion for the evaluation of different technologies in different environments. Flexibility is considered as a property of technologies whereas diversity is a property of environments. Flexibility is a hedge against the diversity of the environment. The formal definition of flexibility is given in De Groote (1994) as follows: "A particular technology is said to be more flexible than another if an increase in the diversity of the environment yields a more desirable change in performance with this particular technology than the change that would be obtained with the other technology under the same conditions". This characterization of flexibility yields three attractive strategic properties, each related to a different optimization problem. First, while allocating two different environments to two different technologies, the overall performance of the system is improved if the more diverse environment is allocated to the more flexible technology. Second, while selecting or designing the best technology for a given environment, an increase in the diversity of the environment makes it more desirable to select a more flexible technology. Finally, an increase in the flexibility of the technology makes it more attractive to operate in a more diverse environment. De Toni and Tonchia (1998) conducted a thorough literature review on manufacturing flexibility and contributes to the conceptual systemization of the flexibility. In the paper, the authors discussed six aspects of the flexibility: definition, factors which determine the request for flexibility, classification, measurement, choices for flexibility and interpretation. Robert and Joseph(1984) studied the flexibility as an economic concept. The paper formalized the notion of flexibility in a sequential decision making context, and quantified its value based on the amount of the available information. Overall, these studies provided general ideas on the concept of flexibility for a wide range of systems.

The literature on strategic models for systems with a mix of dedicated and flexible resource is summarized below: Fine and Freund (1990) considered an investment decision model for a flexible manufacturing system. They analyzed a multi-product system where each product can be produced by a dedicated resource and also by a flexible resource shared by all product types. They developed a discrete stochastic programming model to investigate the necessary and sufficient conditions to invest in flexible capacity, and discussed the properties of the optimal profit function and optimal capacity levels. They also presented numerical results to analyze the sensitivity of solutions to the correlation and variability in demand. For the two-resource case, their numerical results show that when demands for resources are perfectly positively correlated, it is not optimal to invest in flexible capacity. Van Mieghem (1998) considered a two-resource firm with the option of investing in dedicated and/or flexible resources. Using a multi-dimensional newsvendor model developed by Harrison and Van Mieghem (1999), the paper presents the conditions for investing in dedicated and/or flexible capacities. Unlike Fine and Freund (1990), Van Mieghem showed that under certain conditions, it may be optimal to invest in flexible capacity even when the demands for resources are perfectly positively correlated. Gupta et. al. (1992) considered a firm that faces uncertain demands for several product groups and needs to decide how much of dedicated capacity and how much of flexible capacity to acquire. They particularly focused on the dependence of investment policy on initial capacities, as most firms facing the problem are not likely to be entirely new. They showed that if initial capacities are lower than the levels that would be optimal in absence of initial capacities, the investment decision is a simple "acquire-up-to" (optimal levels) for each capacity type. On the other hand, if some initial capacity is "too high", the optimal additions to others depend on its value in a non-linear fashion. Netessine et. al. (2002) considered limited flexibility in service systems that provide multiple services. Each lower level service can be fulfilled by higher level of services. An example application is given from car rental industry where, say, demand for an economy car may be ungraded to a luxury car if there are no economy cars available. Netessine et. al. (2002) discussed the properties of such systems under the assumption that service may be upgraded by only one class. They presented an algorithm to compute the optimal investment levels and discussed the impact of demand correlation to the optimal capacities. For two customer classes, they showed that as correlation in between two demand types increases, the flexible capacity will shift to the dedicated capacity. When there are more than two customer classes, the change of correlation in between two demand types, besides affecting the investment level of two corresponding capacities (say, A and B) for these two demand types, also affects the investment level of other capacities indirectly. As the correlation increases, these changes follow an alternating pattern. It means that if the investment level of a type of capacity, say C, (neither A nor B) changes, the optimal investment level of the capacity which can be used to satisfy the demand for capacity C and the optimal investment level of the capacity which can be substituted by capacity C change in the opposite direction. All of these papers assume that the prices are fixed, and focus on determining optimal resource allocations and optimal capacity investment decisions.

There are some recent papers that consider allocation, pricing and capacity investment decision, simultaneously. Bish and Wang (2004) investigated a model similar to the one in Van Mieghem (1998). In their model, the resource investment decision is made first under demand uncertainty, and the pricing and capacity allocation decisions are made later when demands are realized. Their results confirm most of Van Mieghem's (1998) conclusions. Chod and Rudi (2005) investigated a simpler model, where a single flexible resource satisfies two distinct demand classes without dedicated resources. They also considered a different pricing model such that the demand for a resource does not only depend on its own price but also depends on the price of its alternative. The two key drivers of flexibility such as demand variability and demand correlation are characterized in the paper. Assuming normal distribution for demands, when correlation increases, the optimal flexible capacity increases and the optimal profit decreases. Positive demand correlation remains undesirable. The benefit of flexibility is most significant when the demand levels are highly variable and negatively correlated. When demand variability increases, both the optimal flexible capacity level and the optimal profit increase.

There are also quite a few closely related literature focusing on multi-product inventory systems with substitutions. The earliest work is due to Ignall and Veinott (1969) who considered the multi-product inventory problem with one-way substitution and zero setup costs. Bassok et. al. (1999) considered a single period multi-product inventory problem with full downward substitution, i.e., demand for some product can only be substituted by the products with higher quality. They formulated the problem as a two-stage profit maximization model. At the first stage, given the initial inventories, the problem is to decide optimal inventory procurement amount before the demand is known. At the second stage, given the realized demand and the inventory levels, the optimal demand substitution quantities are determined in order to maximize the expected profit. The downward substitution assumption yields to a simple optimal substitution strategy for the second stage problem, and the objective function of the first stage problem is concave and submodular. As a result, they proved that the optimal quantities of resources to purchase at the first stage are non-increasing in the initial inventory levels, which means that the higher the initial inventory levels are the lower quantities of resources should be purchased. Rao et. al. (2002) considered a similar single period multi-product inventory problem which is also formulated as a two stage problem. Unlike Bassok et. al. (1999), in the first stage, besides determining the optimal quantity to produce (purchase), decision on which products to produce needs to be made. In the second stage, they consider a similar substitution structure but relax the assumption that the unit substitution costs are identical. Moreover, they include the setup cost of production in their model. They use a network flow approach and use dynamic programming and simulation based optimization to develop effective heuristics. The paper provides some insights on issues such as the effect of demand variance and cost parameters on the optimal number of product types to produce, the amount produced or inventoried, and the benefits of substitution. Karaesmen and Ryzin (2004) considered an overbooking problem with multiple reservation and inventory classes, in which the multiple inventory classes may be used as substitutes to satisfy the demand of a given reservation class. The problem is similar as the one discussed in Bassok et. al. (1999), but arises in a variety of revenue management contexts. They modeled this problem as a two-period optimization problem, and showed that the expected revenue function is submodular in the overbooking levels. They also proposed a stochastic gradient algorithm to find the joint optimal overbooking levels. Eppen (1979) developed a single-period, single-product inventory model with several individual sources of demand. It is a multi-location problem with an opportunity for centralization. The paper shows that under reasonable assumptions, the expected holding and penalty costs in a decentralized system exceed those in a centralized system.

Analysis of single period two product substitution problems have been extensively studied. McGillivary and Silver (1978) considered a case where products have identical costs and there is a fixed probability that a customer demand for a stocked out product can be substituted by another available product. They showed that when the substitution probability is close to 1 or the stock level of the substitutable products is high, substantial cost savings can be obtained. Pasternack and Drezner (1991) considered a similar system where the substitution probability is one. They compared the optimal stocking levels to the corresponding inventory levels without substitution. Gerchak et. al. (1996) considered two single-period production processes which both involve the production of the products of two grades, higher and lower. Demand for lower-grade products can be met by high-grade units. They showed that both expected profit functions are concave and derive the optimality conditions. Deterministic versions of the substitution problem are studied by Tripathy et. al. (1999) and Li and Tirupati (1994). Tripathy et. al. (1999) addressed the discrete multidimensional assortment problem, which seeks the optimal sizes of a product to stock from among a discrete set of possible ones and determines the optimal stock level. They modeled the problem as a facility location problem and propose a heuristic procedure to approximate the optimal solution. Li and Tirupati (1994) considered a multi-period multi-product dynamic investment model. Assuming that the demands for different products at different periods are known, they formulated the problem as a two-stage deterministic programming problem, and provided a heuristic method, which gives acceptable solutions efficiently.

Similar as Fine and Freund (1990), Van Mieghem (1998) and Bish and Wang (2004), we investigate the optimal capacity investment strategies for multiple resources. Instead of studying the problem for small systems (i.e., at most one flexible resource) as in these papers, we study a general model, which allows any number of resources and a more flexible reallocation/substitution strategy. Comparing to Bassok et. al. (1999) and Rao et. al. (2002), which addressed inventory management, under substitution models, our work considers an arbitrary substitution structure and pricing power, which can better reflect the reality of the demand-supply markets. Furthermore, Bassok et. al. (1999) and Rao et. al. (2002) studied the problem from the inventory control perspective, and we focus on capacity investment strategies and the effect of the variation of the environment on the optimal investment policies. Since our model addresses more general problems, the results can be applied to more realistic problems although the analysis is more complicated. In this dissertation, we formulate our models as two-stage stochastic programming problems. With the insights that we obtain from the analytic results, we develop heuristic methods based on "Marginal Reallocation Profit". The insights from this research enhance the understanding in investment decisions under operational flexibility (i.e., pricing and reallocation/substitution), and provide practical tools to solve realistic size problems.

Chapter 3

Capacity Investment Strategies for Systems with Two Resources

3.1 Introduction

We study optimal capacity investment, resource allocation and pricing decisions of a firm that manages multiple resources which can be utilized flexibly to satisfy demands from multiple market segments. We use the term resource in the broad sense to mean manufacturing capacity or inventory. We consider that each resource has its own primary market characterized by a random market size and a selling price for the products. We assume that the firm has the monopoly power to set the prices in each market. Furthermore, the firm has the ability to use a particular resource capacity to satisfy demands from a different market segment, other than its own, at the expense of a reallocation cost to hedge against demand uncertainty.

In general, due to the long lead times and contractual agreements, the capacity investment decisions for resources must be made long before the market sizes are known with accuracy. On the other hand, reallocation and pricing decisions can be postponed until more information about the actual market conditions is obtained. The ability to set prices after observing the demand patterns is termed as ex-post (postponed) pricing in the literature (see, for example, Van Mieghem and Dada (1999)).

In this research, we investigate models that apply reallocation and ex-post pricing strategies simultaneously in order to find the optimal capacity investment decisions for multiple resources. In particular, we focus on the following two-stage problem: In the first stage, the firm makes its capacity investment decision for multiple resources in the face of uncertain demand. In the second stage, after the market sizes are realized, the firm jointly determines its prices and capacity reallocations to maximize the total profit based on the capacity investment decisions made in the first stage.

This model addresses the strategic capacity investment decisions faced by a number of industries, such as manufacturing companies that operate reconfigurable plants, retailers with multiple sales outlets in different geographical locations, etc. For example, consider a car manufacturing company that sells its vehicles through its dealers which are geographically distributed within a region. The major source of demand for each dealer is its local community. The company has to decide how many vehicles from each model to put in the inventory at each dealer in the beginning of a selling season under demand uncertainty. The time between two replenishments is usually long (i.e., one year), and hence, this decision can be treated as a single-period problem. After the company allocates the vehicles to dealers, the demand becomes observable as the sales are made. It is highly likely that the actual demand does not match the supply at each dealer. An effective way to balance demand and supply is to adjust the selling price based on the realized market potential and reallocate the vehicles among the dealers if needed. Such an operational flexibility allows the company to generate more profits by matching supply with demand.

Another example can be a manufacturing company that operates multiple plants which are reconfigurable to produce a variety of products. As a result of long production lead times, the manufacturing capacity that will be allocated to each product is determined when demand is highly uncertain. This especially applies to seasonal products. As the beginning of the season approaches, more information about the demand is collected, and the company makes its pricing decisions and reallocates manufacturing capacities to different products at a cost, if necessary.

Due to the simplicity the two-resource system, we can obtain some nice results from analysis which enable us to better understand the model. Therefore, in the rest of this chapter we focus on the two-resource system. A multi-resource version of the problem will be addressed in the next chapter.

The remainder of this chapter is organized as follows: In section 3.2, we introduce a two-stage optimization model to address the two-resource capacity investment problem. In section 3.3, we solve the stage 2 model for the two-resource system. In section 3.4, we solve the stage 1 model. Finally, in section 3.5, we study the impact of demand correlation on the optimal capacity investment decision via a numerical study.

3.2 Model Formulation

We consider a firm that serves two markets where demand for market i is controlled by the unit selling price p_i according to the following linear, downward sloping function

$$D_i = \Gamma_i - \alpha_i p_i$$

where $\alpha_i > 0$ is the slope, and Γ_i is the intercept that denotes the random market size of demand $i,\ i = 1,2$. We assume that Γ_i is a nonnegative continuous random variable, $i = 1,2$. Market i is primarily served by resource i, but it can also be served by resource j $(i \neq j)$ at a nonnegative reallocation (i.e., substitution) cost $k_{ji},\ i, j = 1,2$. We assume that $k_{12} + k_{21} > 0$ to avoid a trivial case that resource 1 and resource 2 can replace each other with no cost so that they can aggregated into a single resource. Let c_j denote the unit cost of investing in resource $j,\ j = 1,2$. The company commits to resource capacities $\overrightarrow{x} = (x_1, x_2)$ before the market sizes of demands $\overrightarrow{\Gamma} = (\Gamma_1, \Gamma_2)$ are realized, in order to maximize the expected total profit. Let x_j denote the capacity acquired for resource $j,\ j = 1,2$. We denote a realization of $\overrightarrow{\Gamma} = (\Gamma_1, \Gamma_2)$ by $\overrightarrow{\gamma} = (\gamma_1, \gamma_2)$. Once the realization $\overrightarrow{\gamma}$ of $\overrightarrow{\Gamma}$ is observed, the company makes its pricing and resource allocation decisions so as to maximize its total expected profit under the resource investment decisions made earlier. Let z_{ji} $(i, j = 1,2.\ \ i \neq j)$ denote the amount of demand i satisfied by resource j once demand is observed. Then, the model can be formulated as a two-stage optimization problem. Stage 1 problem P_1 makes the investment decisions as follows:

Stage 1 (P_1):

$$\max_{\overrightarrow{x}} \Pi(\overrightarrow{x}) = E[\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})] - \sum_{i=1}^{2} c_i x_i$$

subject to:

$$x_1,\, x_2 \geq 0$$

$E[\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})]$ is the expected revenue when the resource capacity vector is $\overrightarrow{x}$, where $\Phi^*(\overrightarrow{x}, \overrightarrow{\gamma})$ is the optimal objective function value of the stage 2 problem P_2, which decides the optimal prices and allocates the resources optimally to fulfill the demand based on an observed demand, $d_i = \overrightarrow{\gamma_i} - \alpha_i p_i,\ i = 1,2.$

Stage 2 P_2:

$$\Phi^*(\overrightarrow{x}, \overrightarrow{\gamma}) = \max_{z_{12},\, z_{21},\, p_1,\, p_2} p_1(\gamma_1 - \alpha_1 p_1) + p_2(\gamma_2 - \alpha_2 p_2) - k_{12} z_{12} - k_{21} z_{21} \quad (3.1)$$

$$s.t.\quad \gamma_1 - \alpha_1 p_1 \leq x_1 - z_{12} + z_{21} \qquad (\lambda_1) \quad (3.2)$$

$$\gamma_2 - \alpha_2 p_2 \leq x_2 + z_{12} - z_{21} \qquad (\lambda_2) \quad (3.3)$$

$$z_{ij} \geq 0 \ \ i, j = 1,2 \text{ and } i \neq j \qquad (u_{ij}) \quad (3.4)$$

$$\gamma_i - \alpha_i p_i \geq 0 \quad i = 1,2 \qquad (\beta_i) \quad (3.5)$$

$$p_i \geq 0 \quad i = 1,2 \quad (3.6)$$

The stage 2 model P_2 maximizes the profit given the resource investment vector $\overrightarrow{x}$ and the realized market size vector $\overrightarrow{\gamma}$. In P_2, inequality (3.2) and inequality (3.3) ensure that the demands from market 1 and market 2 do not exceed the total available capacities for market 1 and market 2, respectively. Inequalities (3.4), (3.5) and (3.6) are the nonnegativity constraints on the resource allocations, demands and prices, respectively. The objective function of P_2 is concave with respect to p_i, and it is linear in $z_{ij},\ i, j = 1,2$ and $i \neq j$. Moreover, all the constraints are linear. Therefore, P_2 is a concave problem.

We would like to note that when $k_{12} = 0$ and $k_{21} \to \infty$, the above model can be considered a generalization of the model investigated in Netessine et. al. (2002) with two types of resources and pricing power. (i.e., the prices are also decision variables.)

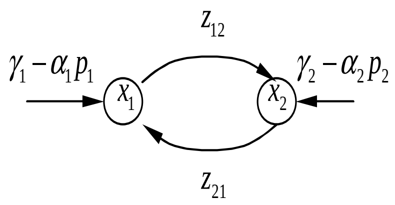

Figure 3.1: An illustration of the 2-resource system

3.3 Optimal Solution of the Stage 2 Model

In this section, we focus on solving P_2 for the two-resource system. The purpose of studying the two-resource model is two-fold:

1. To derive the explicit optimal solution of the two-resource model;

2. To obtain insights on the optimal solution of P_2 for two resources, and use these insights to develop an efficient numerical algorithm for the general model (i.e., model with more than two resources).

Let λ_1, λ_2, u_{12}, u_{21}, β_1, β_2 be the Lagrange multipliers corresponding to the constraints of P_2 given in $(3.2)-(3.5)$, respectively. Note that we omit the last constraint in P_2 because the optimal solution cannot result in $p_i < 0$ since the demand for each resource i, $i = 1,2$, is nonnegative. Without loss of generality, we assume $k_{21} \geq k_{12}$. Furthermore, in order to avoid the discussion of trivial cases, we assume $x_1 > 0$ and $x_2 > 0$.

Since the objective function is concave, the Karush-Kuhn-Tucker (K-K-T) conditions are necessary and sufficient for the optimal solution. The K-K-T conditions of P_2 are given as follows:

$$p_i^* = \frac{\gamma_i + \alpha_i(\lambda_i^* - \beta_i^*)}{2\alpha_i} \qquad \forall i = 1,2 \tag{3.7}$$

$$\lambda_j^* - \lambda_i^* = k_{ij} - u_{ij}^* \quad i,j = 1,2 \;\; i \neq j \tag{3.8}$$

$$\lambda_i^*(-\gamma_i + \alpha_i p_i^* + x_i - z_{ij}^* + z_{ji}^*) = 0 \quad i,j = 1,2 \;\; i \neq j \tag{3.9}$$

$$\beta_i^*(\gamma_i - \alpha_i p_i^*) = 0 \quad i = 1,2 \tag{3.10}$$

$$u_{ij}^* z_{ij}^* = 0 \quad i,j = 1,2 \text{ and } i \neq j \tag{3.11}$$

From (3.8), we have that $\lambda_2^* - \lambda_1^* = k_{12} - u_{12}^*$ and $\lambda_1^* - \lambda_2^* = k_{21} - u_{21}^*$. Summing these two equalities, we obtain $u_{12}^* + u_{21}^* = k_{12} + k_{21} > 0$. Therefore, at least one of u_{12}^* and u_{21}^* should be positive. From (3.11), we conclude that at least one of z_{12}^* and z_{21}^* should be equal to zero in the optimal solution. In addition, we show the following two results regarding the optimal values of the Lagrange multipliers:

Proposition 1. If $\lambda_i^* = 0$, then $\beta_i^* = 0,\ i = 1,2$.

Proof: If $\lambda_i^* = 0$, from (3.7) we have that $p_i^* = \frac{\gamma_i - \alpha_i \beta_i^*}{2\alpha_i}$. Then,

$$\gamma_i - \alpha_i p_i^* = \gamma_i - \alpha_i \left(\frac{\gamma_i - \alpha_i \beta_i^*}{2\alpha_i}\right) = \frac{\gamma_i + \alpha_i \beta_i^*}{2} \geq 0$$

since $\gamma_i \geq 0$ and $\beta_i^* \geq 0$. If $\gamma_i - \alpha_i p_i^* = 0$, we have $\gamma_i = 0$ and $\beta_i^* = 0$. If $\gamma_i - \alpha_i p_i^* > 0$, from (3.10), it follows that $\beta_i^* = 0$. Therefore, $\beta_i^* = 0$. ∎

Proposition 2. If $u_{ij}^* > 0$, then $\beta_i^* = 0$ for $i, j = 1,2$ and $i \neq j$.

Proof: If $u_{ij}^* > 0$, then $z_{ij}^* = 0$ from (3.11). First, assume that $\lambda_i^* = 0$, then $\beta_i^* = 0$ from Proposition 1. Second, assume that $\lambda_i^* > 0$. Then, from (3.9) and using $z_{ij}^* = 0$

$$-\gamma_i + \alpha_i p_i^* + x_i - z_{ij}^* + z_{ji}^* = 0 \Rightarrow p_i^* = \frac{\gamma_i - x_i - z_{ji}^*}{\alpha_i}$$

Then

$$\gamma_i - \alpha_i p_i^* = \gamma_i - \alpha_i \left(\frac{\gamma_i - x_i - z_{ji}^*}{\alpha_i}\right) = x_i + z_{ji}^* > 0$$

since $x_i > 0$ and $z_{ji}^* \geq 0$. From (3.10), it follows that $\beta_i^* = 0$. ∎

In order to describe the optimal solution, let us divide the demand space into 10 disjoint sets as follows:

$\Omega_0 = \{\gamma_1 \leq 2x_1,\ \gamma_2 \leq 2x_2\}$

$\Omega_1 = \{\gamma_1 \leq 2x_1, 2x_2 < \gamma_2 \leq 2x_2 + \alpha_2 k_{12}\}$

$\Omega_2 = \{\gamma_2 \leq 2x_2, 2x_1 < \gamma_1 \leq 2x_1 + \alpha_1 k_{21}\}$

$\Omega_3 = \{\gamma_1 > 2x_1, \gamma_2 > 2x_2, -k_{21} \leq \frac{\gamma_2 - 2x_2}{\alpha_2} - \frac{\gamma_1 - 2x_1}{\alpha_1} \leq k_{12}\}$

$\Omega_4 = \{\gamma_2 > 2x_2 + \alpha_2 k_{12}, \gamma_1 + \gamma_2 \leq 2x_1 + 2x_2 + \alpha_2 k_{12}\}$

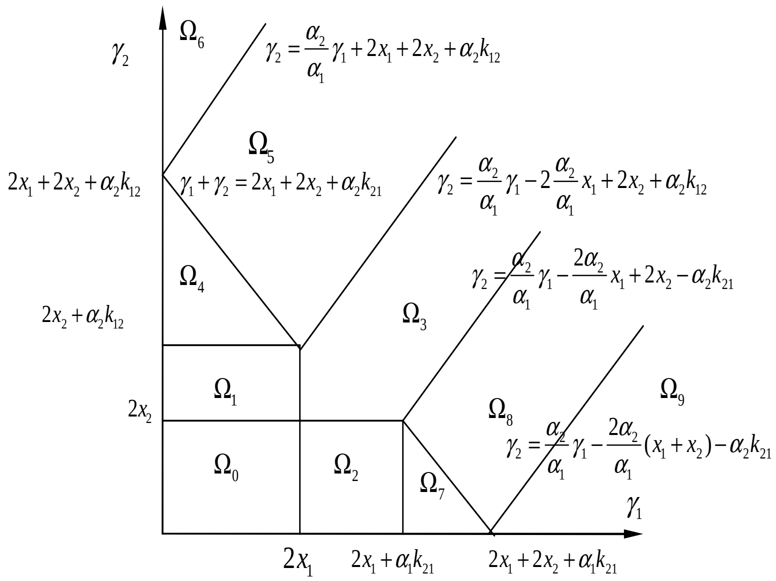

Figure 3.2: Optimal solution of the stage 2 problem with two types of resources

$\Omega_5 = \{\gamma_1 + \gamma_2 > 2x_1 + 2x_2 + \alpha_2 k_{12}, \frac{\gamma_2 - 2x_2}{\alpha_2} - \frac{\gamma_1 - 2x_1}{\alpha_1} > k_{12}, \frac{\gamma_1}{\alpha_1} \geq \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2}\}$

$\Omega_6 = \{\frac{\gamma_1}{\alpha_1} < \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2}\}$

$\Omega_7 = \{\gamma_1 > 2x_1 + \alpha_2 k_{21}, \gamma_1 + \gamma_2 \leq 2x_1 + 2x_2 + \alpha_1 k_{21}\}$

$\Omega_8 = \{\gamma_1 + \gamma_2 > 2x_1 + 2x_2 + \alpha_1 k_{21}, \frac{\gamma_1 - 2x_1}{\alpha_1} - \frac{\gamma_2 - 2x_2}{\alpha_2} > k_{21}, \frac{\gamma_2}{\alpha_2} \geq \frac{\gamma_1 - 2x_1 - 2x_2 - \alpha_1 k_{21}}{\alpha_1}\}$

$\Omega_9 = \{\frac{\gamma_2}{\alpha_2} < \frac{\gamma_1 - 2x_1 - 2x_1 - \alpha_1 k_{21}}{\alpha_1}\}$

An illustration of the demand space with these 10 disjoint sets is given in Figure 3.2. Using these sets, we present the following result that characterizes the optimal solution to the stage 2 model.

Proposition 3. Given realizations γ_1, γ_2 of random variables Γ_1, Γ_2 and a resource investment vector $\overrightarrow{x}$, the optimal solution of the stage 2 model can be expressed as

If $\gamma \in \Omega_0, p_1^* = \frac{\gamma_1}{2\alpha_1}$, $p_2^* = \frac{\gamma_2}{2\alpha_2}$, $z_{12}^* = z_{21}^* = 0$.

If $\gamma \in \Omega_1$, $p_1^* = \frac{\gamma_1}{2\alpha_1}$, $p_2^* = \frac{\gamma_2 - x_2}{\alpha_2}$, $z_{12}^* = z_{21}^* = 0$.

If $\gamma \in \Omega_2$, $p_1^* = \frac{\gamma_1 - x_1}{\alpha_1}$, $p_2^* = \frac{\gamma_2}{2\alpha_2}$, $z_{12}^* = z_{21}^* = 0$.

If $\gamma \in \Omega_3$, $p_1^* = \frac{\gamma_1 - x_1}{\alpha_1}$, $p_2^* = \frac{\gamma_2 - x_2}{\alpha_2}$, $z_{12}^* = z_{21}^* = 0$.

If $\gamma \in \Omega_4$, $p_1^* = \frac{\gamma_1}{2\alpha_1}$, $p_2^* = \frac{\gamma_2 + \alpha_2 k_{12}}{2\alpha_2}$, $z_{12}^* = \frac{\gamma_2 - 2x_2 - \alpha_2 k_{12}}{2}$, $z_{21}^* = 0$.

If $\gamma \in \Omega_5$, $p_1^* = \frac{\gamma_1}{2\alpha_1} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{2(\alpha_1 + \alpha_2)}$, $p_2^* = \frac{\gamma_2}{2\alpha_2} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_1 k_{12}}{2(\alpha_1 + \alpha_2)}$,

$z_{12}^* = \frac{\alpha_1 \alpha_2}{2(\alpha_1 + \alpha_2)}(\frac{\gamma_2 - 2x_2}{\alpha_2} - \frac{\gamma_1 - 2x_1}{\alpha_1} - k_{12})$, $z_{21}^* = 0$.

If $\gamma \in \Omega_6$, $p_1^* = \frac{\gamma_1}{\alpha_1}$, $p_2^* = \frac{\gamma_2 - x_1 - x_2}{\alpha_2}$, $z_{12}^* = x_1$, $z_{21}^* = 0$.

If $\gamma \in \Omega_7$, $p_1^* = \frac{\gamma_1 + \alpha_1 k_{21}}{2\alpha_1}$, $p_2^* = \frac{\gamma_2}{2\alpha_2}$, $z_{21}^* = \frac{\gamma_1 - 2x_1 - \alpha_2 k_{21}}{2}$, $z_{12}^* = 0$.

If $\gamma \in \Omega_8$, $p_1^* = \frac{\gamma_1}{2\alpha_1} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_2 k_{21}}{2(\alpha_1 + \alpha_2)}$, $p_2^* = \frac{\gamma_2}{2\alpha_2} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_1 k_{21}}{2(\alpha_1 + \alpha_2)}$,

$z_{12}^* = 0$, $z_{21}^* = \frac{\alpha_1 \alpha_2}{2(\alpha_1 + \alpha_2)} (\frac{\gamma_1 - 2x_1}{\alpha_1} - \frac{\gamma_2 - 2x_2}{\alpha_2} - k_{21})$.

If $\gamma \in \Omega_9$, $p_1^* = \frac{\gamma_1 - x_1 - x_2}{\alpha_1}$, $p_2^* = \frac{\gamma_2}{\alpha_2}$, $z_{12}^* = 0$, $z_{21}^* = x_2$.

Proof: Based on the K-K-T conditions given in (3.7) – (3.11), in the optimal solution, we can have one of the following two cases:

Case 1. $u_{12}^* > 0$ and $u_{21}^* > 0$.

Case 2. $u_{12}^* > 0$ and $u_{21}^* = 0$ or $u_{12}^* = 0$ and $u_{21}^* > 0$.

Below we will investigate these two cases, respectively.

Case 1. $u_{12}^* > 0$ and $u_{21}^* > 0$.

From Proposition 2, it follows that $\beta_1^* = \beta_2^* = 0$. From (3.11), it follows that $z_{12}^* = z_{21}^* = 0$. Below we will investigate four subcases for a) $\lambda_1^* = \lambda_2^* = 0$, b) $\lambda_1^* = 0$ and $\lambda_2^* > 0$, c) $\lambda_1^* > 0$ and $\lambda_2^* >= 0$, and d) $\lambda_1^* > 0$ and $\lambda_2^* > 0$, respectively.

a) When $\lambda_1^* = 0$ and $\lambda_2^* = 0$, from (3.7),

$$p_i^* = \frac{\gamma_i}{2\alpha_i}, \quad i = 1,2 \tag{3.12}$$

From (3.8), $u_{12}^* = k_{12}$ and $u_{21}^* = k_{21}$.

From (3.9), using $z_{12}^* = z_{21}^* = 0$ and $\lambda_1^* = \lambda_2^* = 0$,

$$p_i^* \geq \frac{\gamma_i - x_i}{\alpha_i} \tag{3.13}$$

From (3.12) and (3.13),

$$p_i^* = \frac{\gamma_i}{2\alpha_i} \geq \frac{\gamma_i - x_i}{\alpha_i} \quad i = 1,2$$

If $\gamma_i \leq 2x_i$ for $i = 1,2$, K-K-T condition (3.10) is satisfied with $p_i^* = \frac{\gamma_i}{2\alpha_i}$ and $\beta_i^* = 0$, $i = 1,2$.

Hence, if $\gamma_i \leq 2x_i$ for $i = 1,2$, and $k_{12},\ k_{21} \geq 0$ then

$$p_i^* = \frac{\gamma_i}{2\alpha_i} \ \ i = 1,2 \text{ and } z_{12}^* = z_{21}^* = 0.$$

b) When $\lambda_1^* = 0$ and $\lambda_2^* > 0$, from (3.7), $p_1^* = \frac{\gamma_1}{2\alpha_1}$ and $p_2^* = \frac{\gamma_2+\alpha_2\lambda_2^*}{2\alpha_2}$.

From (3.9), $p_1^* \geq \frac{\gamma_1 - x_1}{\alpha_1}$ and $p_2^* = \frac{\gamma_2 - x_2}{\alpha_2}$.

Using $p_2^* = \frac{\gamma_2+\alpha_2\lambda_2^*}{2\alpha_2} = \frac{\gamma_2 - x_2}{\alpha_2}$, $\lambda_2^* = \frac{\gamma_2 - 2x_2}{\alpha_2}$.

From (3.8), $u_{12}^* = k_{12} - \lambda_2^* = k_{12} - \frac{\gamma_2 - 2x_2}{\alpha_2}$, and $u_{21}^* = k_{21} + \lambda_2^* = k_{21} + \frac{\gamma_2 - 2x_2}{\alpha_2}$.

Hence, if $\gamma_1 \leq 2x_1,\ \gamma_2 \geq 2x_2$ and $k_{12} > \frac{\gamma_2 - 2x_2}{\alpha_2}$, K-K-T condition (3.10) is also satisfied and

$$p_1^* = \frac{\gamma_1}{2\alpha_1},\ p_2^* = \frac{\gamma_2 - x_2}{\alpha_2},\ z_{12}^* = z_{21}^* = 0$$

with $\lambda_2^* = \frac{\gamma_2 - 2x_2}{\alpha_2}$.

c) The case with $\lambda_1^* > 0$ and $\lambda_2^* = 0$ is symmetric with case b. By following the same steps, it is straightforward to see that if $\gamma_1 > 2x_1,\ \gamma_2 \leq 2x_2$ and $k_{21} > \frac{\gamma_1 - 2x_1}{\alpha_1}$, K-K-T conditions are satisfied with

$$p_1^* = \frac{\gamma_1}{2\alpha_1},\ p_2^* = \frac{\gamma_2 - x_2}{\alpha_2},\ z_{12}^* = z_{21}^* = 0$$

and $\lambda_1^* = \frac{\gamma_1 - 2x_1}{\alpha_1}$.

d) When $\lambda_1^* > 0$, and $\lambda_2^* > 0$, from (3.9), $p_i^* = \frac{\gamma_i - x_i}{\alpha_i},\ i = 1,2$.

Using (3.7) and (3.9), $p_i^* = \frac{\gamma_i - x_i}{\alpha_i} = \frac{\gamma_i + \alpha_i\lambda_i^*}{2\alpha_i} \Rightarrow \lambda_i^* = \frac{\gamma_i - 2x_i}{\alpha_i}\ i = 1,2$.

From (3.8), $u_{12}^* = k_{12} - \lambda_2^* + \lambda_1^* = k_{12} - \frac{\gamma_2 - 2x_2}{\alpha_2} + \frac{\gamma_1 - 2x_1}{\alpha_1}$, and $u_{21}^* = k_{21} - \lambda_1^* + \lambda_2^* = k_{21} - \frac{\gamma_1 - 2x_1}{\alpha_1} + \frac{\gamma_2 - 2x_2}{\alpha_2}$.

Since $\lambda_i^* > 0, i = 1,2$, and $u_{12}^*,\ u_{21}^* > 0$, if $\gamma_i > 2x_i,\ i = 1,2$, and $k_{ij} - \frac{\gamma_j - 2x_j}{\alpha_j} + \frac{\gamma_i - 2x_i}{\alpha_i} > 0,\ i,j = 1,2,\ i \neq j$, then $p_i^* = \frac{\gamma_i - x_i}{\alpha_i}\ i = 1,2$ and $z_{12}^* = z_{21}^* = 0$ with

$$p_i^* = \frac{\gamma_i - 2x_i}{\alpha_i},\ i = 1,2 \text{ and } u_{ij}^* = k_{ij} - \frac{\gamma_j - 2x_j}{\alpha_j} + \frac{\gamma_i - 2x_i}{\alpha_i}.$$

Case 2. For this case, we will only consider $u_{12}^* = 0$ and $u_{21}^* > 0$. The case with $u_{12}^* > 0$ and $u_{21}^* = 0$ is symmetric, and its proof follows in the same lines.

When $u_{12}^* = 0$ and $u_{21}^* > 0$, from (3.11), $z_{12}^* \geq 0$ and $z_{21}^* = 0$. From Proposition 2, $\beta_2^* = 0$. From (3.8), $\lambda_2^* - \lambda_1^* = k_{12} - u_{12}^*$, and with $u_{12}^* = 0$, we have $\lambda_2^* = \lambda_1^* + k_{12}$ and $u_{21}^* = k_{21} - \lambda_1^* + \lambda_2^*$. Consequently, we consider two subcases with $\lambda_1^* = 0$ and $\lambda_2^* = k_{12}$, and $\lambda_1^* > 0$ and $\lambda_2^* = \lambda_1^* + k_{12} > 0$, respectively.

a) When $\lambda_1^* = 0$ and $\lambda_2^* = k_{12}$, then from (3.7), $p_1^* = \frac{\gamma_1 - \alpha_1\beta_1^*}{2\alpha_1}$ and $p_2^* = \frac{\gamma_2 + k_{12}\alpha_2}{2\alpha_2}$. Then, $\gamma_1 - \alpha_1 p_1^* = \gamma_1 - \alpha_1(\frac{\gamma_1 - \alpha_1\beta_1^*}{2\alpha_1}) = \frac{\gamma_1 + \alpha_1\beta_1^*}{2} > 0$. Since $\gamma_1 > 0$ and $\beta_1^* \geq 0$. From (3.10), we conclude that $\beta_1^* = 0$.

   i) If $k_{12} = 0$, then $\lambda_i^* = \beta_i^* = 0$, $p_i^* = \frac{\gamma_i}{2\alpha_i}$, $i = 1,2$ and $u_{21}^* = k_{21}$. From (3.9),

$$-\gamma_1 + \alpha_1 p_1^* + x_1 - z_{12}^* \geq 0 \Rightarrow z_{12}^* \leq -\frac{\gamma_1}{2} + x_1, \text{ and}$$

$$-\gamma_2 + \alpha_2 p_2^* + x_2 + z_{12}^* \geq 0 \Rightarrow z_{12}^* \geq \frac{\gamma_2}{2} - x_2.$$

   Hence, $\frac{\gamma_2}{2} - x_2 \leq z_{12}^* \leq -\frac{\gamma_1}{2} + x_1$. Consequently, if $k_{12} = 0$ and $k_{21} > 0$, $\gamma_2 \geq 2x_2$ and $\gamma_1 + \gamma_2 \leq 2x_1 + 2x_2$, then

$$p_i^* = \frac{\gamma_i}{2\alpha_i} \; i = 1,2 \quad \frac{\gamma_2}{2} - x_2 \leq z_{12}^* \leq -\frac{\gamma_1}{2} + x_1 \text{ and } z_{21}^* = 0$$

   with $\lambda_i^* = \beta_i^* = 0$ $i = 1,2$, and $u_{12}^* = 0$ and $u_{21}^* = k_{21}$.

   ii) If $k_{12} > 0$, from (3.9)

$$-\gamma_1 + \alpha_1 p_1^* + x_1 - z_{12}^* \geq 0 \Rightarrow z_{12}^* \leq -\frac{\gamma_1}{2} + x_1 \tag{3.14}$$

$$-\gamma_2 + \alpha_2 p_2^* + x_2 + z_{12}^* = 0 \Rightarrow z_{12}^* = \frac{\gamma_2 - \alpha_2 k_{12} - 2x_2}{2} \tag{3.15}$$

   Combining (3.14) and (3.15), we obtain the condition that

$$\gamma_1 + \gamma_2 \leq 2x_1 + 2x_2 + \alpha_2 k_{12}.$$

Hence, if $\gamma_2 \geq \alpha_2 k_{12} + 2x_2$ and $\gamma_1 + \gamma_2 \leq 2(x_1 + 2x_2) + \alpha_2 k_{12}$ all K-K-T conditions are satisfied, and

$$p_1^* = \frac{\gamma_1}{2x_1} \quad p_2^* = \frac{\gamma_2 + k_{12}\alpha_2}{2\alpha_2} \quad z_{12}^* = \frac{\gamma_2 - \alpha_2 k_{12} - 2x_2}{2} \quad z_{21}^* = 0$$

with $\lambda_1^* = 0,\ \lambda_2^* = k_{12},\ \beta_1^* = \beta_2^* = u_{12}^* = 0$ and $u_{21}^* = k_{12} + k_{21}$.

Observing subcases i) and ii), we can conclude that the result of i) is the same as ii) with $k_{12} = 0$. As a result, we can use the above result for both cases.

b) When $\lambda_1^* > 0$ and $\lambda_2^* = \lambda_1^* + k_{12}$, then since $\lambda_1^*,\ \lambda_2^* > 0$ and $z_{21}^* = 0$, from (3.9) we have

$$-\gamma_1 + \alpha_1 p_1^* + x_1 - z_{12}^* = 0 \Rightarrow p_1^* = \frac{\gamma_1 - x_1 + z_{12}^*}{\alpha_1} \tag{3.16}$$

$$-\gamma_2 + \alpha_2 p_2^* + x_2 + z_{12}^* = 0 \Rightarrow p_2^* = \frac{\gamma_2 - 2x_2 - z_{12}^*}{\alpha_2} \tag{3.17}$$

We will consider two cases with $\beta_1^* = 0$ and $\beta_1^* > 0$, respectively.

i) When $\beta_1^* = 0$, from (3.7), $p_1^* = \frac{\gamma_1 + \alpha_1 \lambda_1^*}{2\alpha_1}$ and $p_2^* = \frac{\gamma_2 + \alpha_2(\lambda_1^* + k_{12})}{2\alpha_2}$.

Combining these expressions with (3.16 and (3.17), and solving for λ_1^* and z_{12}^*, we obtain

$$\lambda_1^* = \frac{\gamma_1 - 2x_1 + \gamma_2 - 2x_2 - \alpha_2 k_{12}}{\alpha_1 + \alpha_2} > 0$$

$$z_{12}^* = \frac{\alpha_2(2x_1 - \gamma_1) + \alpha_1(\gamma_2 - 2x_2) - \alpha_1\alpha_2 k_{12}}{2(\alpha_1 + \alpha_2)} \geq 0$$

Then, we obtain,

$$p_1^* = \frac{\gamma_1}{2\alpha_1} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{2(\alpha_1 + \alpha_2)}$$

$$p_2^* = \frac{\gamma_2}{2\alpha_2} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_1 k_{12}}{2(\alpha_1 + \alpha_2)}$$

K-K-T condition (3.10) is satisfied if $\gamma_i - \alpha_i p_i^* \geq 0,\ i = 1,2$. Plugging in the values of p_1^* and p_2^* in this set of inequalities, we observe that they are satisfied if

$$\frac{\gamma_1}{\alpha_1} \geq \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2}.$$

As a result, if

$$\gamma_1 + \gamma_2 > 2x_1 + 2x_2 + \alpha_2 k_{12}$$

$$\frac{\gamma_2 - 2x_2}{\alpha_2} - \frac{\gamma_1 - 2x_1}{\alpha_2} \geq k_{12}$$

and

$$\frac{\gamma_1}{\alpha_1} \geq \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2}$$

then

$$p_1^* = \frac{\gamma_1}{2\alpha_1} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{2(\alpha_1 + \alpha_2)}$$

$$p_2^* = \frac{\gamma_2}{2\alpha_2} + \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_1 k_{12}}{2(\alpha_1 + \alpha_2)}$$

$$z_{12}^* = \frac{\alpha_2(2x_1 - \gamma_1) + \alpha_1(\gamma_2 - 2x_2) - \alpha_1\alpha_2 k_{12}}{2(\alpha_1 + \alpha_2)}, \quad z_{21}^* = 0$$

with

$$\lambda_1^* = \frac{\gamma_1 - 2x_1 + \gamma_2 - 2x_2 - \alpha_2 k_{12}}{\alpha_1 + \alpha_2}, \quad \lambda_2^* = \lambda_1 + k_{12}$$

$$\beta_1^* = \beta_2^* = 0, \quad u_{12}^* = 0, \quad u_{21}^* = k_{12} + k_{21}.$$

ii) When $\beta_1^* > 0$, from (3.10), $\gamma_1 - \alpha_1 p_1^* = 0 \Rightarrow p_1^* = \frac{\gamma_1}{\alpha_1}$. Using $p_1^* = \frac{\gamma_1}{\alpha_1} = \frac{\gamma_1 - x_1 + z_{12}^*}{\alpha_1} \Rightarrow z_{12}^* = x_1$.

Then,

$$p_2^* = \frac{\gamma_2 - x_2 - z_{12}^*}{\alpha_2} = \frac{\gamma_2 - x_1 - x_2}{\alpha_2} \geq 0 \tag{3.18}$$

From (3.7),

$$p_2^* = \frac{\gamma_2 + \alpha_2(\lambda_1^* + k_{12})}{2\alpha_2} \tag{3.19}$$

Using (3.18) and (3.19),

$$\lambda_1^* = \frac{\gamma_2 - 2x_2 - 2x_1 - \alpha_2 k_{12}}{\alpha_2} > 0$$

From (3.7), $p_1^* = \frac{\gamma_1+\alpha_1(\lambda_1^*-\beta_1)}{2\alpha_1}$. Using $p_1^* = \frac{\gamma_1}{\alpha_1}$,

$$\beta_1^* = \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2} - \frac{\gamma_1}{\alpha_1} > 0.$$

Consequently, if

$$\frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2} > \frac{\gamma_2}{\alpha_2} \text{ then } p_1^* = \frac{\gamma_1}{\alpha_1}, \;\; p_2^* = \frac{\gamma_2 - x_1 - x_2}{\alpha_2}$$

$z_{12}^* = x_1, \;\; z_{21}^* = 0$ with

$$\lambda_1^* = \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{2\alpha_1}, \;\; \lambda_2^* = \lambda_1 + k_{12}$$

$$\beta_1^* = \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2} - \frac{\gamma_1}{\alpha_1}, \;\; \beta_2^* = 0$$

$$u_{12}^* = 0, \quad u_{21}^* = k_{12} + k_{21}. \;\; \blacksquare$$

Recall that γ_i denotes the market potential for resource i, and x_i is the capacity of resource i. We interpret the results of Proposition 3 in terms of the relationship between demand and supply as follows:

1. In region Ω_0, demands for resources 1 and 2 are both less than the available supplies. The capacity constraints are not binding. In other words, the optimal selling prices are exactly the optimal solution of optimization problem without the capacity constraints and there is no reallocation between resources 1 and 2.

2. In region Ω_1, the demand for resource 1 is at most equal to the supply. Demand for resource 2 is a little higher than the supply, but is lower than some level, which makes it not worth reallocating any capacity from resource 1 to resource 2. Hence, in this region, there is no reallocation between resources 1 and 2.

3. Region Ω_2 is the same as region Ω_1 if resources 1 and 2 are interchanged.

4. In region Ω_3, demands for both resources are higher than the supply, but are less than some level, which makes it not worth reallocating any capacity from either resource.

5. In region Ω_4, the demand for resource 1 is at most equal to the supply. Demand for resource 2 is higher than the supply at a level such that it is worth reallocating some capacity from resource 1 to demand for resource 2. However, the demand for resource 2 is still under some level such that the supply from resource 1 is enough to cover the deficit of supply of resource 2.

6. In region Ω_5, the total demand for the two resources is higher than the total available capacity at a level such that both resources require additional capacity. Furthermore, the deficit of the supply of resource 2 is larger than that of resource 1 at a level such that it is worth reallocating part of resource 1 capacity to demand for resource 2 by sacrificing some demand for resource 1. However, the demand for resource 2 is still lower than some level such that it is not worth sacrificing all the demand for resource 1.

7. In region Ω_6, the total demand for the two resources is higher than the total available capacity at a level such that both resources require additional capacity. Furthermore, the deficit of the supply of resource 2 is larger than that of resource 1 at a level such that it is worth sacrificing all the demand for resource 1.

8. Region Ω_7 is the same as Ω_4 if resources 1 and 2 are interchanged.

9. Region Ω_8 is the same as region Ω_5 if resources 1 and 2 are interchanged.

10. Region Ω_9 is the same as region Ω_6 if resources 1 and 2 are interchanged.

3.4 Optimal Solution of the Stage 1 Model

The stage 1 investment problem is a stochastic, nonlinear optimization problem. In this section, we investigate the structure of the optimal solution to the stage 1 problem. $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is the optimal objective of the operational stage problem (P_2). Its property directly affects the investment decision.

Lemma 1. $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ *is a continuous and differentiable function with respect to* $\overrightarrow{x}$.

Proof: Based on proposition 2, the demand space is divided into 10 disjoint regions. Inside each region, $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is continuous, and the partial derivative of $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ with respect to $x_i,\ i = 1,2$, i.e., the shadow price of resource i exists and it is a continuous function. At the boundary of two adjacent regions, since the concavity of P_2, there exists an unique optimal shadow price of resource i. Therefore, $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is a continuous and differentiable function with respect to $\overrightarrow{x}$. ■

Theorem 1. *The stage 1 problem is jointly concave in x_1 and x_2.*

Proof: Based on proposition 2, the demand space is decomposed into 10 regions as $\Omega_k,\ k = 0,...,9$. Let $f(\gamma_1,\gamma_2)$ denote the joint density function of the demand.

$$\begin{aligned}\Pi(\overrightarrow{x}) &= E[\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})] - \sum_{i=1}^{2} c_i x_i \\ &= \sum_{i=0}^{9} \int_{\Omega_i} \Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma}) f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - \sum_{i=1}^{2} c_i x_i\end{aligned}$$

In order to prove the concavity of the stage 1 problem, we will show that the Hessian matrix of the objective function of stage 1 problem is strictly negative definite. Taking the derivative of $\Pi(\overrightarrow{x})$ with respect to x_i, we obtain:

$$\frac{\partial \Pi(\overrightarrow{x})}{\partial x_i} = \sum_{j=0}^{9} \int_{\Omega_j} \lambda_i f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - c_i,$$

where $\lambda_i = \frac{\partial \Phi}{\partial x_i}$, which is the shadow price of resource i. Based on Proposition 2,

$$\begin{aligned}\frac{\partial \Pi(\overrightarrow{x})}{\partial x_1} &= \int_{\Omega_2+\Omega_3} \frac{\gamma_1 - 2x_1}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &+ \int_{\Omega_5} \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &+ \int_{\Omega_6} \frac{\gamma_2 - 2x_1 - 2x_2 - \alpha_2 k_{12}}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &+ \int_{\Omega_7} k_{21} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &+ \int_{\Omega_8} \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_2 k_{21}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\ &+ \int_{\Omega_9} \frac{\gamma_1 - 2x_1 - 2x_2}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - c_1,\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \Pi(\overrightarrow{x})}{\partial x_2} &= \int_{\Omega_1+\Omega_3} \frac{\gamma_2 - 2x_2}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_4} k_{12} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_5} \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 + \alpha_1 k_{12}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_6} \frac{\gamma_2 - 2x_1 - 2x_2}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_8} \frac{\gamma_1 + \gamma_2 - 2x_1 - 2x_2 - \alpha_1 k_{21}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_9} \frac{\gamma_1 - 2x_1 - 2x_2 - \alpha_1 k_{21}}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - c_2.
\end{aligned}
$$

Note that since $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is continuous on the boundaries, the effects of the change in x_i on the boundaries are cancelled out. When we take the second derivative of $\Pi(\overrightarrow{x})$ with respect to x_i, we obtain:

$$
\begin{aligned}
\frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_1^2} &= -\frac{2}{\alpha_1} \int_{\Omega_2+\Omega_3+\alpha_9} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&- \frac{2}{\alpha_1+\alpha_2} \int_{\Omega_5+\Omega_8} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - \frac{2}{\alpha_2} \int_{\Omega_6} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
\frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_2^2} &= -\frac{2}{\alpha_2} \int_{\Omega_1+\Omega_3+\alpha_6} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&- \frac{2}{\alpha_1+\alpha_2} \int_{\Omega_5+\Omega_8} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 - \frac{2}{\alpha_2} \int_{\Omega_9} f(\gamma_1,\gamma_1) d\gamma_1 d\gamma_2
\end{aligned}
$$

Taking the cross derivative of $\Pi(\overrightarrow{x})$ with respect to x_1 and x_2, we obtain:

$$
\frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_1 \partial x_2} = \frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_2 \partial x_1} = -\frac{2}{\alpha_1+\alpha_2} \int_{\Omega_5+\Omega_8} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 < 0. \tag{3.20}
$$

Since

$$
\frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_1^2} < 0, \; \frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_2^2} < 0, \; \frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_1^2} \frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_2^2} - \left(\frac{\partial^2 \Pi(\overrightarrow{x})}{\partial x_1 \partial x_2}\right)^2 > 0, \tag{3.21}
$$

the Hessian matrix of $\Pi(\overrightarrow{x})$ is strictly negative definite. Hence, $\Pi(\overrightarrow{x})$ is concave. ■

Let v_i denote the Lagrange multiplier of the nonnegativity constraint of $x_i, i = 1, 2$. Using the fact that $\Pi(\overrightarrow{x})$ is concave and the K-K-T conditions for P_1, the following proposition gives the necessary and sufficient conditions for optimal values of x_1 and x_2.

Proposition 4. The optimal capacity investment vector $\overrightarrow{x}^* = (x_1^*, x_2^*)$ satisfies the following equations:

$$\begin{aligned}
& E\begin{bmatrix} 0 \\ \frac{\Gamma_2-2x_2^*}{\alpha_2} \end{bmatrix}| \Omega_1 \Big] P(\Omega_1) + E\begin{bmatrix} \frac{\Gamma_1-2x_1^*}{\alpha_1} \\ 0 \end{bmatrix}| \Omega_2 \Big] P(\Omega_2) + E\begin{bmatrix} \frac{\Gamma_1-2x_1^*}{\alpha_1} \\ \frac{\Gamma_2-2x_2^*}{\alpha_2} \end{bmatrix}| \Omega_3 \Big] P(\Omega_3) \\
+ \; & E\begin{bmatrix} 0 \\ k_{12} \end{bmatrix}| \Omega_4 \Big] P(\Omega_4) + E\begin{bmatrix} \frac{\Gamma_1+\Gamma_2-2x_1^*-2x_2^*-\alpha_2 k_{12}}{\alpha_1+\alpha_2} \\ \frac{\Gamma_1+\Gamma_2-2x_1^*-2x_2^*+\alpha_1 k_{12}}{\alpha_1+\alpha_2} \end{bmatrix}| \Omega_5 \Big] P(\Omega_5) + E\begin{bmatrix} k_{21} \\ 0 \end{bmatrix}| \Omega_7 \Big] P(\Omega_7) \\
+ \; & E\begin{bmatrix} \frac{\Gamma_2-2x_1^*-2x_2^*-\alpha_2 k_{12}}{\alpha_2} \\ \frac{\Gamma_2-2x_1^*-2x_2^*}{\alpha_2} \end{bmatrix}| \Omega_6 \Big] P(\Omega_6) + E\begin{bmatrix} \frac{\Gamma_1+\Gamma_2-2x_1^*-2x_2^*+\alpha_2 k_{21}}{\alpha_1+\alpha_2} \\ \frac{\Gamma_1+\Gamma_2-2x_1^*-2x_2^*-\alpha_1 k_{21}}{\alpha_1+\alpha_2} \end{bmatrix}| \Omega_8 \Big] P(\Omega_8) \\
+ \; & E\begin{bmatrix} \frac{\Gamma_1-2x_1^*-2x_2^*}{\alpha_1} \\ \frac{\Gamma_1-2x_1^*-2x_2^*-\alpha_1 k_{21}}{\alpha_1} \end{bmatrix}| \Omega_9 \Big] P(\Omega_9) = \begin{bmatrix} c_1 - v_1^* \\ c_2 - v_2^* \end{bmatrix} \qquad (3.22)
\end{aligned}$$

$x_i^* v_i^* = 0, v_i^* \geq 0 \;\; \forall i = 1,2$

Proof: Based on Theorem 1, there exists a unique optimal solution $\overrightarrow{x}^*$ which satisfies the K-K-T conditions, i.e., $\frac{\partial \Pi(\overrightarrow{x})}{\partial x_1}|_{x_1=x_1^*} + v_1^* = 0$, $\frac{\partial \Pi(\overrightarrow{x})}{\partial x_2}|_{x_2=x_2^*} + v_2^* = 0$ and $x_i^* v_i^* = 0, v_i^* \geq 0 \;\forall i = 1,2$. In the proof of Theorem 1, we obtain $\frac{\partial \Pi(\overrightarrow{x})}{\partial x_1}$ and $\frac{\partial \Pi(\overrightarrow{x})}{\partial x_2}$. The result directly follows. ■

Next, we will investigate the structure of the optimal investment strategy. An optimal investment strategy should take one of the following forms:

(a) Do not invest at all

(b) Invest in resource 1 only

(c) Invest in resource 2 only

(d) Invest in both resources.

Before we explicitly describe the conditions under which forms (a) - (d) are observed, we define four threshold values as $\bar{c}_1$, $\underline{c}_1$, $\bar{c}_2$ and $\underline{c}_2$.

When $\overrightarrow{x}^* = (0,0)$, equation (3.22) in Proposition 4 reduces to

$$\frac{E(\Gamma_1)}{\alpha_1} + E(\frac{\Gamma_2}{\alpha_2} - \frac{\Gamma_1}{\alpha_1} - k_{12}|\, \Omega_6)P(\Omega_6) = c_1 - v_1^*,$$

$$\frac{E(\Gamma_2)}{\alpha_2} + E(\frac{\Gamma_1}{\alpha_1} - \frac{\Gamma_2}{\alpha_2} - k_{21}|\, \Omega_9)P(\Omega_9) = c_2 - v_2^*.$$

Let $\bar{c}_1 = \frac{E(\Gamma_1)}{\alpha_1} + E(\frac{\Gamma_2}{\alpha_2} - \frac{\Gamma_1}{\alpha_1} - k_{12} | \Omega_6) P(\Omega_6)$ and $\bar{c}_2 = \frac{E(\Gamma_2)}{\alpha_2} + E(\frac{\Gamma_1}{\alpha_1} - \frac{\Gamma_2}{\alpha_2} - k_{21} | \Omega_9) P(\Omega_9)$. If $c_i > \bar{c}_i$, then $v_i^* > 0$, i.e., it is not optimal to invest in resource i, $i = 1, 2$. Note that here Ω_6 and Ω_9 are the demand regions when both resource capacities are 0. When $\vec{x}^* = (x_1^*, 0)$ with $x_1^* > 0$, from equation (3.22), we obtain:

$$
\begin{aligned}
c_1 &= \int_{\Omega_3+\Omega_9} \frac{\gamma_1 - 2x_1^*}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_5} \frac{\gamma_1 + \gamma_2 - 2x_1^* - \alpha_2 k_{12}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_6} \frac{\gamma_2 - 2x_1^* - \alpha_2 k_{12}}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2.
\end{aligned}
$$

Since $x_2^* = 0$, regions Ω_2, Ω_7 and Ω_8 are empty. The optimality equation for resource 2 reduces to $v_2^* = c_2 - \underline{c}_2 > 0$ where

$$
\begin{aligned}
\underline{c}_2 &= \int_{\Omega_1+\Omega_3} \frac{\gamma_2}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_4} k_{12} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_5} \frac{\gamma_1 + \gamma_2 - 2x_1^* + \alpha_1 k_{12}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_6} \frac{\gamma_2 - 2x_1^*}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_9} \frac{\gamma_1 - 2x_1^* - \alpha_1 k_{21}}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2.
\end{aligned}
$$

When $\vec{x}^* = (0, x_2^*)$ with $x_2^* > 0$, from equation (3.22),

$$
\begin{aligned}
c_2 &= \int_{\Omega_3+\Omega_6} \frac{\gamma_2 - 2x_2^*}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_8} \frac{\gamma_1 + \gamma_2 - 2x_2^* - \alpha_1 k_{21}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_9} \frac{\gamma_1 - 2x_2^* - \alpha_1 k_{21}}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2.
\end{aligned}
$$

Since $x_1^* = 0$, regions Ω_1, Ω_4 and Ω_5 are empty. The optimality equation for resource 1 reduces to $v_1^* = c_1 - \underline{c}_1 > 0$ where

$$
\begin{aligned}
\underline{c}_1 &= \int_{\Omega_2+\Omega_3} \frac{\gamma_1}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_6} \frac{\gamma_2 - 2x_2^* - \alpha_2 k_{12}}{\alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_7} k_{21} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 + \int_{\Omega_8} \frac{\gamma_1 + \gamma_2 - 2x_2^* + \alpha_2 k_{21}}{\alpha_1 + \alpha_2} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_9} \frac{\gamma_1 - 2x_2^*}{\alpha_1} f(\gamma_1,\gamma_2) d\gamma_1 d\gamma_2.
\end{aligned}
$$

Based on these observations, we can state the following proposition that outlines the structure of the optimal investment strategy.

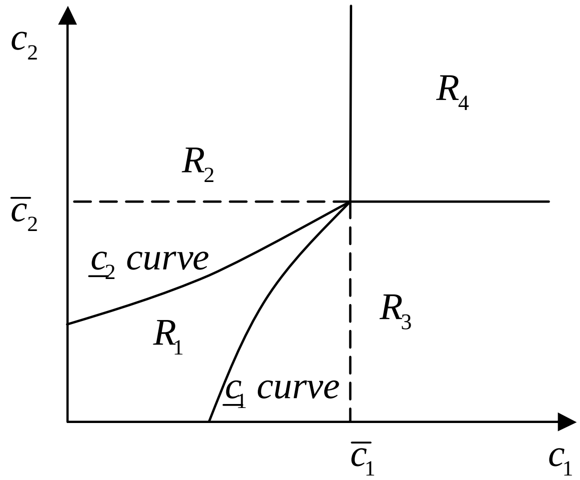

Figure 3.3: Impact of investment costs on the optimal investment strategies

Proposition 5. The optimal investment strategy has one of four distinct forms depending on the costs of resources c_1 and c_2:

(a) If $c_1 > \overline{c}_1$ and $c_2 > \overline{c}_2$, it is not optimal to invest in any of the resources.

(b) If $c_1 < \overline{c}_1$ and $c_2 > \underline{c}_2$, it is optimal to invest in resource 1 only.

(c) If $c_2 < \overline{c}_2$ and $c_1 > \underline{c}_1$, it is optimal to invest in resource 2 only.

(d) If $c_1 < \underline{c}_1$ and $c_2 < \underline{c}_2$, it is optimal to invest in both resources.

Proof: The proof of the proposition directly follows from the definitions of $\overline{c}_1$, $\overline{c}_2$, $\underline{c}_1$ and $\underline{c}_2$. ■

Figure 3.3 provides the intuition for the result of Proposition 5. Given the distributions of the demands and the corresponding α values, $\overline{c}_1$ and $\overline{c}_2$ are constants. On the other hand, $\underline{c}_1$ and $\underline{c}_2$ values depend on the values of c_1 and c_2, which are illustrated by $\underline{c}_1$ and $\underline{c}_2$ curves shown in Figure 3.3. First, observe that if $c_1 = 0$, then $\underline{c}_2 < \overline{c}_2$, and if $c_2 = 0$, then $\underline{c}_1 < \overline{c}_1$.

That is, if $c_1 = 0$, we can invest in resource 1 as much as we want. Therefore,

$$\begin{aligned}
\underline{c}_2 &= \int_{\Omega_1+\Omega_3} \frac{\gamma_2}{\alpha_2} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 + \int_{\Omega_4} k_{12} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_5} \frac{\gamma_1+\gamma_2-2x_1^*+\alpha_1 k_{12}}{\alpha_1+\alpha_2} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 + \int_{\Omega_6} \frac{\gamma_2-2x_1^*}{\alpha_2} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 \\
&+ \int_{\Omega_9} \frac{\gamma_1-2x_1^*-\alpha_1 k_{21}}{\alpha_1} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2. \\
&= \int_{\Omega_1+\Omega_3} \frac{\gamma_2}{\alpha_2} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 + \int_{\Omega_4} k_{12} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2.
\end{aligned}$$

Note that in demand regions Ω_5, Ω_6, Ω_9, all demand is satisfied since $c_1 = 0$, and the shadow price of resource 2 is zero. Based on the definition of demand region Ω_4, $k_{12} \le \frac{\gamma_2}{\alpha_2}$. We have $\underline{c}_2 \le \int_{\Omega_1+\Omega_3+\Omega_4} \frac{\gamma_2}{\alpha_2} f(\gamma_1,\gamma_2)d\gamma_1 d\gamma_2 \le E(\frac{\Gamma_2}{\alpha_2}) < \bar{c}_2$. We can show that if $c_2 = 0$, then $\underline{c}_1 < \bar{c}_1$ in a similar way. Furthermore, it is easy to show that if $c_1 = \bar{c}_1$, then $\underline{c}_2 = \bar{c}_2$, and if $c_2 = \bar{c}_2$, then $\underline{c}_1 = \bar{c}_1$.

Based on these observations, Proposition 5 can be explained as follows: The costs of the resources can be divided into four disjoint sets as R_1, R_2, R_3 and R_4. As shown in Figure 3.3, if (c_1, c_2) is in region R_1, R_2, R_3 and R_4, the optimal investment strategy is of form (a) (b), (c) and (d), respectively. The concavity of the stage 1 problem guarantees that the four regions in Figure 3.3 are disjoint because if there exists an overlapping area (except for the boundaries where the solutions are the same) between any two of the regions, there exists two distinct solutions satisfying the optimality condition given in Proposition 4, which is a contradiction.

Next proposition will address the sensitivity analysis for P_1 and P_2. Let Π^* and Φ^* denote the optimal objective values of P_1 and P_2, respectively. We have:

Proposition 6. For i=1,2,

1. Φ^* decreases in α_i.

2. Π^* decreases in α_i and c_i.

3. If $x_i^* > 0,\ i = 1,2$, x_i^* decreases in c_i, and $x_j^*,\ j \neq i$ increases in c_i.

Remark: α_i is the slope of the demand function for market segment i. As α_i increases, the optimal objective function value of P_2 decreases. As a result, the optimal objective of P_1 decreases as well. c_i is the unit cost of resource i. Intuitively, as c_i increases, the optimal objective function value of P_1 decreases and the optimal investment level in resource i decreases.

Proof:

1. In order to conduct the sensitivity analysis of P_2 in α_i, we consider the optimal solution and optimal objective function value as the functions of α_i, $i = 1,2$. Let $p_l^*(\alpha_i)$, $z_{lj}^*(\alpha_i)$, $l, j = 1,2$ be the optimal solution of $P_2(\alpha_i)$. Let $y_l^*(\alpha_i) = x_l + \sum_{j\neq l} z_{jl}^*(\alpha_i) - \sum_{j\neq l} z_{lj}^*(\alpha_i)$ be the optimal total available capacity of resource l after reallocation.$\Phi^*(\alpha_i) = \sum_{j=1}^{n} p_j^*(\gamma_j - \alpha_j p_j^*) - \sum_l \sum_{j\neq l} k_{lj} z_{lj}^*$. is the optimal objective function value of $P_2(\alpha_i)$. Consider that α_i decreases to $\alpha_i - \delta$, where δ is a small positive real number. We will show $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$. Let us construct a feasible solution for $P_2(\alpha_i - \delta)$. Let $p_l^*(\alpha_i - \delta) = p_l^*(\alpha_i)$, $l \neq i$, $l \in \{1,2\}$, $z_{lj}^*(\alpha_i - \delta) = z_{lj}^*(\alpha_i), l, j = 1,2$. All decision variables of $P_2(\alpha_i - \delta)$ except $p_i^*(\alpha_i - \delta)$ are named values. Then we determine the value of $p_i^*(\alpha_i - \delta)$ as follows:

   Consider the optimal solution of $P_2(\alpha_i)$,

   (a) If constraint (3.5) is binding for resource i, i.e., $\gamma_i - \alpha_i p_i^*(\alpha_i) = 0$,

      let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i}{\alpha_i - \delta}$. As far as δ is small enough, it can be easily verified the constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates the same objective value as $P_2(\alpha_i)$. The optimal solution of $P_2(\alpha_i - \delta)$ is at least as large as $\Phi^*(\alpha_i)$. Therefore, $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq 0$.

   (b) If constraints (3.5) and (3.2) or (3.3) are not binding for resource i.

      $p_i^*(\alpha_i) = \frac{\gamma_i}{2\alpha_i}$. Let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i}{2(\alpha_i - \delta)}$. The constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates a larger objective value than $P_2(\alpha_i)$. We have $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq \frac{\partial(\frac{\gamma_i^2}{4\alpha_i})}{\partial \alpha_i} = -\frac{\gamma_i^2}{4\alpha_i^2} < 0$.

   (c) If constraint (3.5) is not binding, and constraint (3.2) or (3.3) is binding for resource i. Let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i - y_i^*}{\alpha_i - \delta}$. The constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates a larger objective value than $P_2(\alpha_i)$. We have $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq \frac{\partial (y_i^* \frac{\gamma_i - y_i^*}{\alpha_i})}{\partial \alpha_i} = -\frac{(\gamma_i - y_i^*) y_i^*}{\alpha_i^2} < 0$ (both y_i^* and $\gamma_i - y_i^*$ are positive).

Therefore $\Phi^*(\alpha_i)$ is decreases in α_i for all $i = 1, 2$.

2. Since $\Phi^*(\alpha_i)$ is decreases in α_i, $\Pi^* = E[\Phi^*(\overrightarrow{x}^*, \overrightarrow{\Gamma})] - \sum_{i=1}^n c_i x_i^*$ decreases in α_i. $\frac{\partial \Pi^*}{\partial c_i} = -x_i^*$, Φ^* decreases in c_i with rate x_i^*.

3. When $x_i^* > 0,\ i = 1, 2$, the first order condition $\frac{\partial \Pi}{\partial x_1}|_{x_1 = x_1^*,\ x_2 = x_2^*} = F_1(x_1^*, x_2^*, c_1) = 0$ and $\frac{\partial \Pi}{\partial x_2}|_{x_1 = x_1^*,\ x_2 = x_2^*} = F_2(x_1^*, x_2^*, c_1) = 0$ implicitly define x_1^* and x_2^* as a function of c_1.

$$\frac{\partial F_1}{\partial x_1^*}\frac{\partial x_1^*}{\partial c_1} + \frac{\partial F_1}{\partial x_2^*}\frac{\partial x_2^*}{\partial c_1} + \frac{\partial F_1}{\partial c_1} = 0 \tag{3.23}$$

$$\frac{\partial F_2}{\partial x_1^*}\frac{\partial x_1^*}{\partial c_1} + \frac{\partial F_2}{\partial x_2^*}\frac{\partial x_2^*}{\partial c_1} + \frac{\partial F_2}{\partial c_1} = 0 \tag{3.24}$$

Since $\frac{\partial F_1}{\partial c_1} = -1$ and $\frac{\partial F_2}{\partial c_1} = 0$, by solving (3.23) and (3.24), we have,

$$\frac{\partial x_1^*}{\partial c_1} = \frac{\frac{\partial^2 \Pi}{\partial x_2^2}}{\frac{\partial^2 \Pi}{\partial x_1^2}\frac{\partial^2 \Pi}{\partial x_2^2} - (\frac{\partial^2 \Pi}{\partial x_1 \partial x_2})^2}, \quad \frac{\partial x_2^*}{\partial c_1} = \frac{-\frac{\partial^2 \Pi}{\partial x_1 \partial x_2}}{\frac{\partial^2 \Pi}{\partial x_1^2}\frac{\partial^2 \Pi}{\partial x_2^2} - (\frac{\partial^2 \Pi}{\partial x_1 \partial x_2})^2}.$$

Based on (3.20) and (3.21), we have $\frac{\partial x_1^*}{\partial c_1} < 0$ and $\frac{\partial x_2^*}{\partial c_1} > 0$. Similarly, we can show that $\frac{\partial x_2^*}{\partial c_2} < 0$ and $\frac{\partial x_1^*}{\partial c_2} > 0$. ■

3.5 Numerical Analysis

To carry on the numerical experiments, we use normal distribution with mean μ_i and standard deviation σ_i as the underlying distribution of the market size of the demand for resource i. Since we assume the sizes of the market demands are nonnegative, the negative portion of the underlying normal distribution is truncated when we conduct the numerical computation. The demands for different resources are correlated. The correlation coefficient between demand i and demand j is $\rho_{ij}\ \ \forall i \neq j$. Let x_i^{NR} denote the optimal capacity of resource i when there is no reallocation between the resources. $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is the expected objective function value of P_2 given a random market size vector $\overrightarrow{\Gamma}$ and a capacity vector $\overrightarrow{x}$.

We compute the optimal resource capacities by the following algorithm:

1. Let $x_i = x_i^{NR}$ $\forall i$ and $l = 1$.

2. Fixing other capacities, compute the capacity of resource l that maximizes the objective function of P_1, which is given by $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x})) - \sum_{i=1}^{n} c_i x_i$. The computation of $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is based on Monte Carlo simulation as explained in detail below. A new value of x_l, x_e^n, which maximizes the objective function of P_1, is obtained based on binary search in internal $[0, \sum_{i=1}^{n} x_i^{NR}]$. Note that the optimal value of x_l can not be larger than $\sum_{i=1}^{n} x_i^{NR}$.

3. If $|\overrightarrow{x} - \overrightarrow{x^n}| < \varepsilon$, return $\overrightarrow{x^n}$ as the optimal solution. Otherwise, let $l = l + 1$. If $l > n$, $l = 1$. Go to step 2.

In step 2, $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is obtained by Monte Carlo simulation. We generate M independent realizations of the market size $\overrightarrow{\Gamma}$. For each realization i, i=1,2,...,M, and a capacity vector $\overrightarrow{x}$, we compute $\Phi^*(\overrightarrow{\gamma}, \overrightarrow{x})$based on Proposition 3. Then, $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is approximated by the average over all realizations, i.e., $\sum_{i=1}^{M} \Phi^*(\overrightarrow{\gamma}, \overrightarrow{x})/M$. In order to generate a realization of the demand vector $\overrightarrow{\Gamma}$, we first generate a vector $\overrightarrow{Z}$ with size n, where $E(z_1) = E(z_2) = ... = E(z_n) = 0$, $Var(z_1) = Var(z_2) = ... = Var(z_n) = 1$ and $z_1, z_2, ..., z_n$ are independent. Suppose that Σ is the covariance matrix for the demands, and $\Sigma = A^T A$ after conducting the Cholesky decomposition where A is an upper triangular matrix. Let $\overrightarrow{\mu}$ denote the mean vector of the market sizes. Then $\overrightarrow{\Gamma} = \overrightarrow{\mu} + A\overrightarrow{Z}$ is the correlated market size vector, which has mean $\overrightarrow{\mu}$ and covariance matrix Σ.

Since Monte Carlo simulation follows square root (of the sample size) convergence, in our numerical analysis, we choose the sample size $40,000$ which gives a standard error of 0.5%.

With this numerical study, we would like to investigate how optimal capacities of the two resources change with respect to demand correlation.

Resource	c_i	α_i	μ_i	σ_i	k_{i1}	k_{i2}
1	55	1.2	120	40	0	0
2	40	2.0	200	80	5000	0

Table 3.1: Parameter values for the two-resource system

Table 3.1 shows the parameter values that we use. In this study, resource 1 can be used to satisfy demands from both markets at no cost. On the other hand, $k_{21} = 5000$ ensures that resource 2 can only serve its own market. This is a setting similar to the one given by Netessine et. al. (2002). The model presented in Netessine et. al. (2002) assumes that the prices are fixed whereas our model considers a postponed pricing strategy.

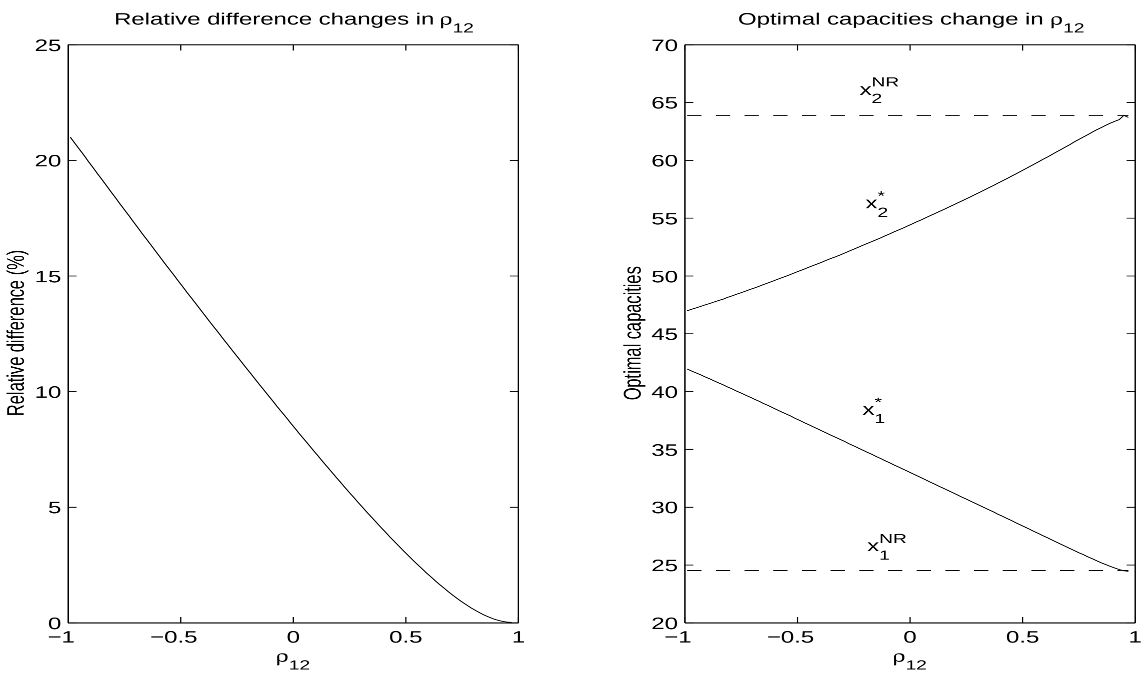

Figure 3.4: Optimal expected profit and resource capacities with respect to ρ_{12}

The left chart of Figure 3.4 shows the percent increase in the optimal expected profit obtained by reallocation as compared to the system no reallocation. When the demands are perfectly negatively correlated, the relative percent increase can be as high as 21%. As ρ_{12} increases, the relative percent increase decreases. When the demands are perfectly positively correlated, the benefit of reallocation diminishes.

The right chart of Figure 3.4 shows the optimal capacities as a function of the correlation coefficient between the demands. When the correlation increases from -1 to 1, the optimal capacities get closer and closer to the optimal capacities without reallocation. We also note that when the correlation between the two demands increases, the optimal expected profit (i.e., optimal objective function value of P_1) decreases. These conclusions are the same with conclusions reported in Netessine et. al. (2002).

3.6 Conclusion

In this chapter, we considered a capacity investment, resource allocation and pricing decision problem faced by a central decision maker that manages two resources which can be used flexibly to satisfy demands from two market segments. We formulated it as a two-stage stochastic programming problem, and explicitly solved the second stage problem. The analysis of the first stage problem showed that the optimal capacity investment strategy takes one of the following three forms: (1) Do not invest in any of the resources; (2) Invest in one of the resources; (3) Invest in both resources.

Chapter 4

Capacity Investment Strategies for Systems with Multiple Resources

In the previous chapter, we considered the two-resource model and provided the structural properties for the optimal capacity investment decision by solving the stage 2 model explicitly. In this chapter, we further investigate the multi-resource version (i.e., systems with more than two resources) of the model.

This chapter is organized as follows: In section 4.1, we introduce the two-stage optimization model to address the multi-resource capacity investment problem. In section 4.2, for the multi-resource stage 2 model, we study the structural properties of the optimal solution, and then present the heuristic methods to solve the problem approximately. In section 4.3, based on the analytical results of the second stage model, we further study the properties of the optimal solution of the first stage model. Finally, in section 4.4, we present the numerical experiment results.

4.1 Model Formulation

We use the same notation as the two-resource system except that we consider a firm that serves n markets instead of just two. p_i denotes the unit selling price of resource i, and $D_i = \Gamma_i - \alpha_i p_i$ is demand function. Γ_i is the intercept that denotes the market size of demand i, $i = 1,...,n$. We assume that Γ_i is a nonnegative continuous random variable, $i = 1,...,n$. Market i is primarily served by resource i, but it can also be served by resource j ($i \neq j$) at a nonnegative reallocation (i.e., substitution) cost $k_{ji},\ i,j=1,...,n$. We assume that $k_{ji}+k_{ij} > 0$ to avoid a trivial case that resource i and j can replace each other with no cost so that they can be aggregated into a single resource. Let c_j denote the unit cost of investing in resource $j,\ j=1,...,n$. The company commits to resource capacities $\overrightarrow{x}=(x_1,x_2,...,x_n)$ long before nonnegative market sizes of demands $\overrightarrow{\Gamma}=(\Gamma_1,\Gamma_2,...,\Gamma_n)$ are realized, in order to maximize the expected total profit. Let x_j denote units of capacity invested in resource $j,\ j=1,...,n$. We denote a realization of $\overrightarrow{\Gamma}=(\Gamma_1,\Gamma_2,...,\Gamma_n)$ by $\overrightarrow{\gamma}=(\gamma_1,\gamma_2,...,\gamma_n)$. Once the realization $\overrightarrow{\gamma}$ of $\overrightarrow{\Gamma}$ is observed, the company makes its pricing and resource allocation decisions so as to maximize its total expected profit under the resource investment decisions made earlier. Let $z_{ij}\ \ (i,j=1,...,n.\ \ i\neq j)$ denote the amount of demand j satisfied by resource i once demand is observed, and $Z=[z_{ij}]$ is the resource allocation matrix. Then, the model can be formulated as a two-stage optimization problem. Stage 1 problem P_1 makes the investment decisions as follows:

Stage 1 (P_1):

$$\max_{\overrightarrow{x}} \Pi(\overrightarrow{x}) = E[\Phi^*(\overrightarrow{x},\overrightarrow{\Gamma})] - \sum_{i=1}^{n} c_i x_i$$

subject to:

$$x_1,\ x_2,\ ...,\ x_n \geq 0$$

$E[\Phi^*(\overrightarrow{x},\overrightarrow{\Gamma})]$ is the expected revenue for a resource capacity vector $\overrightarrow{x}$, where $\Phi^*(\overrightarrow{x},\overrightarrow{\gamma})$ is the optimal objective function value of the stage 2 problem (P_2) which decides the optimal prices and allocates the resources optimally to fulfill the demand based on an observed demand, $d_i=\overrightarrow{\gamma_i}-\alpha_i p_i,\ i=1,...,n.$

Stage 2 (P_2):

$$\begin{aligned}
\Phi^*(\overrightarrow{x},\overrightarrow{\gamma}) \quad &= \quad \max_{Z,\overrightarrow{p}} \sum_{i=1}^{n} p_i(\gamma_i-\alpha_i p_i) - \sum_{i}\sum_{j\neq i} k_{ji}z_{ji} && (4.1)\\
s.t.\ \ & \gamma_i-\alpha_i p_i \leq x_i + \sum_{j\neq i} z_{ji} - \sum_{j\neq i} z_{ij} \quad \forall i && (4.2)\\
& z_{ji} \geq 0 \quad \forall j\neq i && (4.3)\\
& \gamma_i-\alpha_i p_i \geq 0 \quad \forall i && (4.4)\\
& p_i \geq 0 \quad \forall i && (4.5)
\end{aligned}$$

The stage 2 model (P_2) maximizes the profit given the resource investment vector $\overrightarrow{x}$ and the realized market size vector $\overrightarrow{\gamma}$. In (P_2), inequality (4.2) ensures that the demand from market i does not exceed the total available capacity. Inequalities (4.3), (4.4) and (4.5) are the nonnegativity constraints on the resource allocations, demands and prices, respectively. The objective function of P_2 is concave with respect to p_i, and it is linear in z_{ij}. Moreover, all the constraints are linear. Therefore, P_2 is a concave problem.

This model is a generalization of the model in Bish and Wang (2003). When $n = 3$, $k_{12}, k_{21}, k_{13}, k_{23} \to \infty$ and $k_{31}, k_{32} = 0$, resources 1 and 2 do not substitute for other resources because of the high reallocation cost (e.g., dedicated resources). Resource 3 can substitute for other resources at zero reallocation cost (e.g., flexible resource). Bish and Wang (2003) analyzes the two-stage optimization problem under this setting, and studies the impact of demand correlation on the investment strategy.

4.2 Characterization of the Stage 2 Model

4.2.1 Structural Properties

In section 3.3 of the previous chapter, we obtained the optimal solution to the stage 2 model with two resources. However, systems with more than two resources are complicated to analyze in a similar manner, and the optimal solution cannot be easily obtained. Therefore, in this section, we focus on studying some of the structural properties of the optimal solution for multi-resource models. Although a closed form solution cannot be obtained, we can get enough insight to develop efficient algorithms to solve the problem. Furthermore, the structural properties of the optimal solution of P_2 are useful in proving the concavity of P_1.

First, we provide the following definitions: If a resource is used to fulfill the demands for other resources, we call it a “supplier”. If demand for a resource is satisfied by other resources, we call that resource a “consumer”. If in a group of resources, each of the resources is connected to all the others either being a consumer or a supplier, directly or through other resources, we call this group a “sharing group”. A resource that is neither a consumer nor a supplier is a single element sharing group. In the optimal solution, the whole set of resources can be divided into several sharing groups.

Let $\lambda_i,\ u_{ij},\ \beta_i,\ i,j=1,2,...,n,\ i \neq j$ be the Lagrange multipliers of (4.2), (4.3), and (4.4) respectively, and we use $p_i^*, z_{ij}^*, \lambda_i^*,\ u_{ij}^*,\ \beta_i^*$ to denote the corresponding optimal values of the decision variables and the Lagrange multipliers. The optimal solution to P_2 with multiple resources (i.e., $n > 2$) satisfies the following K-K-T conditions:

$$p_i^* = \frac{\gamma_i}{2\alpha_i} + \frac{\lambda_i^* - \beta_i^*}{2} \quad \forall i = 1,2,...,n \tag{4.6}$$

$$\lambda_j^* - \lambda_i^* = k_{ij} - u_{ij}^* \quad \forall j \neq i \tag{4.7}$$

$$\lambda_i^*(-\gamma_i + \alpha_i p_i^* + x_i + \sum_{j\neq i} z_{ji}^* - \sum_{j\neq i} z_{ij}^*) = 0 \quad \forall i = 1,2,...,n \tag{4.8}$$

$$u_{ij}^* z_{ij}^* = 0 \quad \forall j \neq i \tag{4.9}$$

$$\beta_i^*(\gamma_i - \alpha_i p_i^*) = 0 \quad \forall i = 1,...,n \tag{4.10}$$

$$\beta_i^* \geq 0, \lambda_i^* \geq 0, u_{ij}^* \geq 0 \quad \forall i,j \tag{4.11}$$

Note that we omit the last constraint (4.5) in P_2 because p_i is always nonnegative in an optimal solution since the demand for resource i is nonnegative.

In the next three propositions, we present several basic properties of the optimal solution. Essentially, these properties are supported by transportation (minimum cost flow) theorems which can be found in Ahuja et. al. (1993). The minimum cost flow problem aims to reallocate multiple resources optimally (i.e., minimize the total reallocation cost) to fulfill the demands for the resources. In the minimum cost flow problem, the demand for each resource is given, and there is no optimization with respect to the prices which determines the quantity of the demand. Therefore, the minimum cost flow problem only needs to decide the optimal reallocation quantities where our model needs to decide the optimal reallocation quantities and the optimal selling prices simultaneously. However, since our model has the same reallocation structure as the minimum cost flow problem, the objective functions and constraints of two models are very similar except that our model includes the pricing factor, and therefore has a quadratic objective function instead if a linear objective function. It is not a surprise that the optimal solution of our model has similar properties

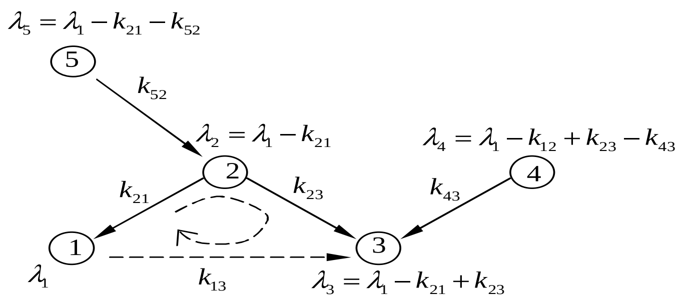

Figure 4.1: Optimal shadow prices of the resources in a sharing group

as those possessed by the optimal solution of the minimum cost flow problem. Theorem 11.1 in Ahuja et. al. (1993) shows the cycle free property of the optimal solution of minimum cost flow problem, and it is related to our Proposition 9. Theorem 11.3 introduces the minimum cost flow optimality conditions which are related to our Propositions 7 and 8. We provide the following properties for our model.

Proposition 7. In an optimal solution, if resource i is a consumer of resource j, then $\lambda_i^* = \lambda_j^* + k_{ji}$.

Proof: Based on (4.7), in an optimal solution, $\forall i \neq j$ we have

$$\lambda_j^* - \lambda_i^* = k_{ij} - u_{ij}^*$$

$$\lambda_i^* - \lambda_j^* = k_{ji} - u_{ji}^*.$$

Summing up these two equations, we obtain $u_{ij}^* + u_{ji}^* = k_{ij} + k_{ji}$. Since we assume $k_{ij} + k_{ji} > 0$, at least one of u_{ij}^* and u_{ji}^* is positive, i.e., if i is a consumer of j, it cannot be a supplier of j at the same time. Since resource j is a supplier of resource i, $z_{ji}^* > 0$ leads to $u_{ji}^* = 0$. Based on (4.7), $u_{ij}^* = k_{ij} + k_{ji} \Rightarrow \lambda_i^* = \lambda_j^* + k_{ji}$ ■

The above result shows that when resource i is a consumer of j, the difference between their shadow prices is equal to the unit reallocation cost from j to i, which is an intuitive result. In a sharing group, every resource (i.e., node) pair is connected through a set of nodes and arcs. Let S_{ij}^{node} and S_{ij}^{arc} denote the ordered set of nodes and the ordered set of arcs through which resource j can be reached from resource i, respectively, where i is the starting node and j is the ending node and each node between i and j is visited only once. For example, in Figure 4.1, resource 2 is a supplier of resources 1 and 3, resource 4 is a supplier of resource 3, and resource 5 is a supplier of resource 2. Then, starting at node 1 and ending at node 4 , resources 1 and 4 can be connected by $S^{node}_{14} = \{1,2,3,4\}$ and $S^{arc}_{14} = \{(2,1),(2,3),(4,3)\}$. If we know the optimal shadow price of one of the resources, the optimal shadow prices of the other resources in this group can be obtained sequentially based on Proposition 7. Referring to Figure 4.1, $\lambda^*_2 = \lambda^*_1 - k_{21}$, $\lambda^*_3 = \lambda^*_1 - k_{21} + k_{23}$, $\lambda^*_4 = \lambda^*_1 - k_{21} + k_{23} - k_{43}$, $\lambda^*_5 = \lambda^*_1 - k_{21} - k_{52}$.

Suppose that (f,h) is an arc in S^{arc}_{ij}. We define v_{fh} as the coefficient of unit reallocation cost from resource f to h, where v_{fh} is equal to 1 if the direction of the arc matches the order of nodes f and h appearing in S^{node}_{ij}, otherwise, v_{fh} equals to -1. In Figure 4.1, nodes 1 and 4 are connected by $S^{node}_{14} = \{1,2,3,4\}$. Node 1 appears right in front of node 2, and the order does not match the direction of the arc connecting 1 and 2 which is $(2,1)$, so $v_{21} = -1$. Hence, the shadow prices of resources in a sharing group satisfy the following equation:

$$\lambda^*_j = \lambda^*_i + \sum_{(f,h)\in S^{arc}_{ij}} v_{fh} k_{fh}. \tag{4.12}$$

for each resource pair i and j in the sharing group. Let S denote an optimal sharing group.

Proposition 8. When S denotes an optimal sharing group, for $i,j \in S,\ i \neq j$,

$$\sum_{(f,h)\in S^{arc}_{ij}} v_{fh} k_{fh} \leq k_{ij}. \tag{4.13}$$

Proof: Based on equation (4.12), for $i,j \in S,\ i \neq j$,

$$\lambda^*_j = \lambda^*_i + \sum_{(f,h)\in S^{arc}_{ij}} v_{fh} k_{fh}.$$

Based on (4.7), we have

$$\lambda^*_j - \lambda^*_i = k_{ij} - u^*_{ij} = \sum_{(f,h)\in S^{arc}_{ij}} v_{fh} k_{fh}.$$

Since $u_{ij}^* \geq 0$,

$$\sum_{(f,h)\in S_{ij}^{arc}} v_{fh}k_{fh} \leq k_{ij}. \qquad \blacksquare$$

In Figure 4.1, S_{14}^{node} and S_{14}^{arc} are not unique. S_{14}^{node} can alternatively be defined as $\{1,3,4\}$, and S_{14}^{arc} can be defined as $\{(1,3),(4,3)\}$. The optimal shadow prices of the resources in this sharing group can also be computed by $\lambda_3^* = \lambda_1^* + k_{13}$, $\lambda_2^* = \lambda_1^* + k_{13} - k_{23}$, $\lambda_4^* = \lambda_1^* + k_{13} - k_{43}$, $\lambda_5^* = \lambda_1^* + k_{13} - k_{23} - k_{52}$. Hence, there is more than one set of equations to compute the shadow prices. As shown in Figure 4.1, $\{(2,1),(2,3),(1,3)\}$ is a cycle. There is a cycle because S_{14}^{node} and S_{14}^{arc} are not unique.

Proposition 9. P_2 has an optimal solution which is acyclic.

Proof: Suppose that there exists a cycle in the optimal solution of P_2. Let M be the total number of nodes in the cycle, and the set of the ordered nodes is $\{i_1, i_2, ..., i_M, i_1\}$. Let us use C_{i_1,i_1} to denote the cycle that starts and ends at node i_1. Without loss of generality, we define the direction of the cycle as $i_1 \rightarrow i_2 \rightarrow ... \rightarrow i_M \rightarrow i_1$. From equation (4.12), we have

$$\lambda_{i_1}^* = \lambda_{i_1}^* + \sum_{(f,h)\in C_{i_1 i_1}} v_{fh}k_{fh} \Rightarrow \sum_{(f,h)\in C_{i_1 i_1}} v_{fh}k_{fh} = 0.$$

The reallocation amount between two adjacent nodes in a cycle should be positive by definition. If we send a tiny flow δ through the cycle, and the direction of the cycle is the same as the direction of the arc which connects two adjacent nodes, the reallocation amount is changed by $+\delta$. If the direction of the cycle is different from the direction of the arc, the reallocation amount of is changed by $-\delta$. Considering constraint (4.2) in P_2, $\sum_{j\neq i} z_{ji}^* - \sum_{j\neq i} z_{ij}^*$ is the total reallocation amount to/from resource i. If i is one of the resources in $\{i_1, i_2, ..., i_M\}$, after sending out the tiny flow δ, $\sum_{j\neq i} z_{ji}^* - \sum_{j\neq i} z_{ij}^*$ does not change because the flow δ enters node i, and then goes out to the next node in the cycle. Since δ is a tiny amount, it is guaranteed that the reallocation amount between two adjacent nodes in the cycle is nonnegative after the change. The objective value of P_2 is changed by

$$(\sum_{(f,h)\in C_{i_1 i_1}} v_{fh}k_{fh})\delta.$$

On the other hand, we know that $\sum_{(f,h)\in C_{i_1 i_1}} v_{fh}k_{fh} = 0$, so the objective value does not change. Increasing δ until the amount allocated through one of the arcs becomes zero, we eliminate one cycle in the optimal solution. If there are more than one cycle in the optimal solution, we repeat this process until all the cycles are eliminated to achieve a cycle-free optimal solution. ■

For example, in Figure 4.1, there is a cycle that consists of arcs $(2,1),(1,3)$ and $(2,3)$. Without loss of generality, if we send a flow, say, through nodes 2, 3 and 1, we can eliminate arc $(2,1)$ if $z^*_{21} < z^*_{13}$ or eliminate arc $(1,3)$ if $z^*_{13} < z^*_{21}$. The proof of Proposition 9 tells that, the optimal solution of P_2 may include cycles. However, we can always transform it to an acyclic optimal solution. As a result, an optimal sharing group can always be constructed as a spanning tree, which defines an unique path from each resource to every other resource in the optimal sharing group. The optimal solution of P_2 is composed of a set of sharing groups that are spanning trees.

In the reminder of this section, we will first investigate the structure of the optimal solution (i.e., compute the optimal values of the shadow prices and decision variables) for an optimal sharing group. In the next sections, we will provide procedures to identify the optimal sharing groups for a given stage 2 problem.

For now, suppose that the optimal spanning tree for a sharing group is given, including all the information of the arcs (starting node, ending node, and the directions). Let us denote the set of nodes in the optimal spanning tree by S. The following proposition gives the expressions of the optimal shadow prices and the optimal selling prices of the resources in an optimal sharing group. Once one of the shadow prices of the other resources is given, other shadow prices of the resources in the sharing group can be determined by using equation (4.12). Let $\lambda^*_{min} = \min_{j\in S}\lambda^*_j$. Let L denote the set of resources for which $\lambda^*_j = \lambda^*_{min},\ j \in S$. If $|L| > 1$, then we choose an arbitrary resource, say $l \in L$, as the "base resource".

Proposition 10. In an optimal sharing group S, for $j \in S$

$$\frac{\gamma_j}{\alpha_j} - \lambda^*_j \geq 0 \quad \Longleftrightarrow \beta^*_j = 0 \tag{4.14}$$

$$\frac{\gamma_j}{\alpha_j} - \lambda_j^* < 0 \quad \Longleftrightarrow \beta_j^* > 0 \tag{4.15}$$

Proof: Let us first prove (4.14).

1. Suppose that $\frac{\gamma_j}{\alpha_j} - \lambda_j^* \geq 0$.

   If $\beta_j^* > 0$, based on (4.10), $\gamma_j - \alpha_j p_j^* = 0 \quad \Rightarrow p_j^* = \frac{\gamma_j}{\alpha_j}$. Plugging it into equation (4.6), we have

   $$\beta_j^* = \lambda_j^* - \frac{\gamma_j}{\alpha_j} \leq 0,$$

   which is a contradiction. Therefore $\beta_j^* = 0$.

2. Suppose that $\beta_j^* = 0$. Based on equation (4.6), $p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_j^*}{2}$. Based on constraint (4.4) (i.e., $\gamma_i - \alpha_i p_i \geq 0 \ \forall i$), we have

   $$\gamma_i - \alpha_i p_i = \gamma_i - \alpha_i(\frac{\gamma_i}{2\alpha_i} + \frac{\lambda_i^*}{2}) = \frac{\gamma_i}{2} - \frac{\alpha_i \lambda_i^*}{2} \geq 0 \Leftrightarrow \frac{\gamma_j}{\alpha_j} - \lambda_j^* \geq 0.$$

Next, we prove (4.15) as follows:

1. Suppose that $\frac{\gamma_j}{\alpha_j} - \lambda_j^* < 0$. If $\beta_j^* = 0$, based on equation (4.6), $p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_j^*}{2}$. Based on constraint (4.4) (i.e., $\gamma_i - \alpha_i p_i \geq 0 \ \forall i$), we have

   $$\gamma_i - \alpha_i p_i = \gamma_i - \alpha_i(\frac{\gamma_i}{2\alpha_i} + \frac{\lambda_i^*}{2}) = \frac{\gamma_i}{2} - \frac{\alpha_i \lambda_i^*}{2} \geq 0 \Leftrightarrow \frac{\gamma_j}{\alpha_j} - \lambda_j^* \geq 0$$

   which is a contradiction. Therefore $\beta_j^* > 0$.

2. Suppose that $\beta_j^* > 0$. Based on (4.10), $\gamma_j - \alpha_j p_j^* = 0 \quad \Rightarrow p_j^* = \frac{\gamma_j}{\alpha_j}$. Plugging it into equation (4.6), we have

   $$\beta_j^* = \lambda_j^* - \frac{\gamma_j}{\alpha_j} > 0. \text{ Therefore, } \frac{\gamma_j}{\alpha_j} - \lambda_j^* < 0. \quad \blacksquare$$

Proposition 11. Suppose that l is the "base resource" of an optimal sharing group S. Assume that $\sum_{j \in S} x_j > 0$.

1. If $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ \leq \sum_{j\in S} 2x_j$, then $\lambda_l^* = 0$, and

$$\lambda_j^* = \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \quad \forall j \neq l. \tag{4.16}$$

a) If $\gamma_j \geq \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}$,

$$\beta_j^* = 0,\ p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}}{2}.$$

b) Otherwise,

$$\beta_j^* > 0,\ p_j^* = \frac{\gamma_j}{\alpha_j}.$$

2. If $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ > \sum_{j\in S} 2x_j$, then $\lambda_l^* > 0$. Let T be the set of resources $j \in S$ with $\beta_j^* = 0$. In an optimal solution, $T \neq \emptyset$.

$$\lambda_l^* = \frac{\sum_{j\in T}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}) - 2\sum_{j\in S} x_j}{\sum_{j\in T} \alpha_j} > 0, \tag{4.17}$$

a) If $j \in T$,

$$\beta_j^* = 0,\ p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_l^* + \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}}{2}.$$

b) Otherwise,

$$\beta_j^* > 0,\ p_j^* = \frac{\gamma_j}{\alpha_j}.$$

Remark: We can see the quantity $(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+$ as the "effective demand" of resource j. Indeed, if $\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} < 0$, $\frac{\gamma_j}{\alpha_j} < \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} = \lambda_j^* - \lambda_l^* \leq \lambda_j^*$. Based on Proposition 10, $\beta_j^* > 0$. When $\beta_j^* > 0$, based on (4.10), $\gamma_j - \alpha_j p_j^* = 0$, i.e., no demand for resource j is satisfied. In other words, When $\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \leq 0$, the market size of the demand for resource j is so small that no resource in the sharing group will be assigned to fulfill it ($\beta_j^* > 0$). Only when $\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} > 0$, it is possible that other resources in the sharing group will be assigned to fulfill the demand for resource j. Therefore, we call the nonnegative part of $\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}$ the "effective demand".

When the sum of all "effective demands" of the resources in the sharing group is less than $\sum_{j\in S} 2x_j$, the capacity constraint of the sharing group is nonbinding, such that $\lambda_l^* = 0$. When the sum of all "effective demands" exceeds the available resources, the capacity constraint of the sharing group is binding, such that, even the smallest shadow price of the resources is positive.

Proof:

Let us first prove the following equation,

$$\gamma_j - \alpha_j p_j^* = \left(\frac{\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \alpha_j\lambda_l^*}{2}\right)^+. \tag{4.18}$$

Based on equation (4.12), $\lambda_j^* = \lambda_l^* + \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \quad \forall j \neq l$. The optimal selling price of resource $j \in S$ is given by equation (4.6) as

$$p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_l^* + \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \beta_j^*}{2}. \tag{4.19}$$

Based on Proposition 10,

$$\frac{\gamma_j}{\alpha_j} - \lambda_l^* - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \geq 0 \iff \beta_j^* = 0. \tag{4.20}$$

$$\frac{\gamma_j}{\alpha_j} - \lambda_l^* - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} < 0 \iff \beta_j^* > 0. \tag{4.21}$$

If $\frac{\gamma_j}{\alpha_j} - \lambda_l^* - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \geq 0$, based on (4.20),

$$\beta_j^* = 0 \Rightarrow p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_l^* + \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}}{2}.$$

Therefore,

$$\gamma_j - \alpha_j p_j^* = \frac{\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \alpha_j\lambda_l^*}{2}.$$

On the other hand, if $\frac{\gamma_j}{\alpha_j} - \lambda_l^* - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} < 0$, based on (4.21), $\beta_j^* > 0$. Based on (4.10),

$$\gamma_j - \alpha_j p_j^* = 0.$$

Therefore,

$$\gamma_j - \alpha_j p_j^* = (\frac{\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \alpha_j\lambda_l^*}{2})^+.$$

1. If $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ \leq \sum_{j\in S} 2x_j$, suppose $\lambda_l^* > 0$. We have, $\lambda_j^* = \lambda_l^* + \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} > 0 \quad \forall j \neq l$. When $\lambda_l^* > 0$, all the shadow prices of the resources in the sharing group are positive. The constraints in (4.2) hold as equalities $\forall j \in S$, i.e.,

$$\gamma_j - \alpha_j p_j = x_j + \sum_{i\neq j} z_{ij} - \sum_{i\neq j} z_{ji} \quad \forall j$$

Summing those equations up, we obtain

$$\sum_{j\in S}(\gamma_j - \alpha_j p_j^*) = \sum_{j\in S} x_j + \sum_{j\in S}(\sum_{i\neq j} z_{ij} - \sum_{i\neq j} z_{ji}).$$

Since we consider the resources in the same sharing group,

$$\sum_{j\in S}(\sum_{i\neq j} z_{ij} - \sum_{j\neq i} z_{ji}) = 0.$$

Therefore,

$$\sum_{j\in S}(\gamma_j - \alpha_j p_j^*) = \sum_{j\in S} x_j. \tag{4.22}$$

Based on equation (4.22) and (4.18),

$$\sum_{j\in S} x_j = \sum_{j\in S}(\gamma_j - \alpha_j p_j^*) = \sum_{j\in S}\left(\frac{\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \alpha_j\lambda_l^*}{2}\right)^+$$

$$< \sum_{j\in S}\left(\frac{\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}}{2}\right)^+,$$

which is a contradiction. Therefore, if $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ \leq \sum_{j\in S} 2x_j$, $\lambda_l^* = 0$.

When $\lambda_l^* = 0$, according to equation (4.12), $\lambda_j^* = \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \quad \forall j \neq l$.

1(a). When $\lambda_l^* = 0$, the optimal selling price of resource $j \in S$ is given by equation (4.6) as

$$p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} - \beta_j^*}{2}$$

Based on Proposition 10,

$$\frac{\gamma_j}{\alpha_j} - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} \geq 0 \iff \beta_j^* = 0 \Rightarrow p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\sum_{(f,h)\in S_{ij}^{arc}} v_{fh}k_{fh}}{2}$$

1(b).

$$\frac{\gamma_j}{\alpha_j} - \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh} < 0 \iff \beta_j^* > 0 \iff \gamma_j - \alpha_j p_j^* = 0 \Rightarrow p_j^* = \frac{\gamma_j}{\alpha_j}.$$

2. If $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ > \sum_{j\in S} 2x_j$, suppose $\lambda_l^* = 0$.

   Summing up constraints (4.2) $\forall j \in S$, we have

$$\sum_{j\in S}(\gamma_j - \alpha_j p_j) \leq \sum_{j\in S} x_j \tag{4.23}$$

   Based on (4.18),

$$2\sum_{j\in S}(\gamma_j - \alpha_j p_j) = \sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ \leq \sum_{j\in S} 2x_j,$$

   which is a contradiction. Therefore, if $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ > \sum_{j\in S} 2x_j$, $\lambda_l^* > 0$.

   Given $\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh})^+ > \sum_{j\in S} 2x_j$ and $\lambda_l^* > 0$, let T denote the set of resources $j \in S$ with $\beta_j = 0$. When $\lambda_l^* > 0$, equality (4.22) holds., i.e.,

$$\sum_{j\in S} x_j = \sum_{j\in S}(\gamma_j - \alpha_j p_j)$$

If $j \in S \setminus T$, $\beta_j^* > 0 \Rightarrow \gamma_j - \alpha_j p_j = 0$. Therefore,

$$\sum_{j \in S} x_j = \sum_{j \in S} (\gamma_j - \alpha_j p_j) = \sum_{j \in T} (\gamma_j - \alpha_j p_j) \tag{4.24}$$

When $j \in T$, $\beta_j^* = 0$, based on (4.19), $p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_l^* + \sum_{(f,h) \in S_{lj}^{arc}} v_{fh} k_{fh}}{2}$. Using (4.24),

$$\sum_{j \in S} x_j = \sum_{j \in T} \left(\frac{\gamma_j - \alpha_j \sum_{(f,h) \in S_{lj}^{arc}} v_{fh} k_{fh} - \alpha_j \lambda_l^*}{2} \right)$$

$$\Longrightarrow \quad \lambda_l^* = \frac{\sum_{j \in T} (\gamma_j - \alpha_j \sum_{(f,h) \in S_{lj}^{arc}} v_{fh} k_{fh}) - 2 \sum_{j \in S} x_j}{\sum_{j \in T} \alpha_j}. \tag{4.25}$$

In order to prove that $T \neq \emptyset$, we will use contradiction. Suppose that $T = \emptyset$, then, $\beta_j > 0,\ \forall j \in S$. Based on (4.10), $\gamma_j - \alpha_j p_j^* = 0,\ \forall j \in S$ Since $\lambda_l^* > 0$, based on (4.22),

$$\sum_{j \in S} (\gamma_j - \alpha_j p_j^*) = \sum_{j \in S} x_j = 0$$

which is a contradiction with assumption that $\sum_{j \in S} x_j > 0$.

In Appendix A, we present the procedure to determine T, given

$\sum_{j \in S} (\gamma_j - \alpha_j \sum_{(f,h) \in S_{lj}^{arc}} v_{fh} k_{fh})^+ > \sum_{j \in S} 2x_j$.

2(a). If $j \in T$, $\beta_j^* = 0$ by definition of T. Based on (4.6),

$$p_j^* = \frac{\gamma_j}{2\alpha_j} + \frac{\lambda_l^* + \sum_{(f,h) \in S_{lj}^{arc}} v_{fh} k_{fh}}{2}.$$

2(b). If $j \notin T$, $\beta_j^* > 0$, and based on (4.10), $p_j^* = \frac{\gamma_j}{\alpha_j}$. ■

Once we find the optimal shadow and selling prices, it is straightforward to compute the reallocation quantities among the resources. As we have defined earlier, L denotes the set of resources for which $\lambda_l^* = \lambda_{min}^*, l \in S$. In Proposition 12, we will show that, for an acyclic optimal sharing group, when $\lambda_l^* > 0$ or $\lambda_l^* = 0$ and $|L| = 1$, the optimal reallocation quantities (i.e., $z_{ij}^*,\ i, j = 1, ..., n,\ i \neq j$) can be uniquely determined. However, when $\lambda_l^* = 0$ and $|L| > 1$, the optimal reallocation quantities may not be unique. If $|L| > 1$ and $\lambda_l^* = 0$, we choose two resources in set L, say l_1, l_2 with $\lambda_{l_1}^* = 0$ and $\lambda_{l_2}^* = 0$, which means that the capacity constraint (4.2) may be nonbinding for $i = l_1, l_2$ in an optimal sharing group. When the capacity constraint (4.2) is nonbinding for $i = l_1, l_2$ (i.e., there exists extra available capacity for both resources l_1 and l_2), if we send a tiny flow δ from l_1 to l_2 through the arcs which connect l_1 and l_2, as far as the amount of the flow is small enough, we obtain a new feasible solution (i.e., new reallocation quantities) and the objective value does not change (the change on the objective value is $\delta \sum_{(f,h) \in S_{l_1 l_2}^{arc}} v_{fh} k_{fh}$ which is equal to 0 because $\lambda_{l_2}^* = \lambda_{l_1}^* + \sum_{(f,h) \in S_{l_1 l_2}^{arc}} v_{fh} k_{fh}$ and $\lambda_{l_1}^* = \lambda_{l_2}^* = 0$).

The speciality of an acyclic optimal sharing group with $\lambda_l^* = 0$ and $|L| = 1$ is that there is only one base resource, which means the capacity constraint (4.2) is binding for all resources in the sharing group except the base resource. On the other hand, when $\lambda_l^* = 0$ and $|L| > 1$, there are more than one base resource in the sharing group. Since it is good enough to find one of the optimal solutions, if we can pick one of the base resources and make the capacity constraint (4.2) binding for all the other base resources by sending flows from them to the base resource that we have picked, we can obtain the same optimal solution in the case with $\lambda_l^* = 0$ and $|L| = 1$. However, when the optimal solution is not unique and we transform the optimal solution by sending the flows, the following two situations can arise:

1. One of the arcs which connects two base resources, say l_1 and l_2, is broken, i.e., the reallocation quantity changes from positive to 0, and the capacity constraints of l_1 and l_2 are still nonbinding. When this happens, the original optimal sharing group is decomposed into two optimal sharing groups.

2. Capacity constraint of l_1 becomes binding and none of the arcs which connect l_1 and l_2 is broken.

When $\lambda_l^* = 0$ and $|L| > 1$, if for all pairs of the base resources in set L, none of them can decompose the original optimal sharing group into two optimal sharing groups by sending flows between the pair of resources until one of the capacity constraints is binding, we call the original optimal sharing group an optimal undecomposable sharing group. Therefore, the optimal solution of such an undecomposable sharing group can always be transformed to the case with $\lambda_l^* = 0$ and $|L| = 1$, while an optimal decomposable sharing group can be decomposed into several optimal undecomposable sharing groups.

In an optimal acyclic sharing group S, the arc between node i and j connects two distinct subsets of nodes. Let us denote the two subsets as S_i and S_j respectively, where $i \in S_i$ and $j \in S_j$. Similarly, the set of resources $m \in S$ with $\beta_m^* = 0$, say T, can be separated into two distinct subsets T_i and T_j, where $T = T_i \cup T_j$. Let us define functions

$$U(M,i) = \sum_{m \in M} (\gamma_m - \alpha_m \sum_{(f,h) \in S_{im}^{arc}} v_{fh} k_{fh}), \quad H(M) = \sum_{m \in M} \alpha_m, \quad X(M) = \sum_{m \in M} x_m.$$

These functions have the following properties that will be used in the proof of Proposition 12:

1. $U(M_1,i) + U(M_2,i) = U(M_1 \cup M_2, i)$,
2. $H(M_1) + H(M_2) = H(M_1 \cup M_2)$,
3. $X(M_1) + X(M_2) = X(M_1 \cup M_2)$,
4. $U(M,i) - H(M)k_{ji} = U(M,j)$.

Without loss of generality, assume that the direction of the arc is from i to j. z_{ij}^* values, $i, j \in S$, can be obtained based on the following proposition. Recall that L denotes the set of resources for which $\lambda_l^* = \lambda_{min}^*, l \in S$.

Proposition 12. In an acyclic optimal sharing group,

1. If $\lambda_l^* > 0$,

   (a) If $S_i = \{i\}$ and $\beta_i^* > 0$ then $z_{ij}^* = x_i$.

   (b) Otherwise,

$$z_{ij}^* = \frac{H(T_i)H(T \setminus T_i)}{2H(T)} (\widetilde{\lambda}_j - \widetilde{\lambda}_i - k_{ij}) \tag{4.26}$$

   where,

$$\widetilde{\lambda}_i = \frac{U(T_i,i) - 2X(S_i)}{H(T_i)}, \quad \widetilde{\lambda}_j = \frac{U(T \setminus T_i, j) - 2X(S \setminus S_i)}{H(T \setminus T_i)}$$

2. If $\lambda_l^* = 0$ and $|L| = 1$,

   (a) If $S_i = \{i\}$ and $\beta_i^* > 0$ then $z_{ij}^* = x_i$.

   (b) Otherwise,

      i. If $l \in S_i$,

$$z_{ij}^* = (\widetilde{\lambda}_j - \widetilde{\lambda}_i - k_{ij})\frac{H(T \setminus T_i)}{2} \tag{4.27}$$

      where,

$$\widetilde{\lambda}_i = \sum_{(f,h)\in S_{li}^{arc}} v_{fh}k_{fh}, \ \ \widetilde{\lambda}_j = \frac{U(T \setminus T_i, j) - 2X(S \setminus S_i)}{H(T \setminus T_i)}.$$

      ii. If $l \in S_j$,

$$z_{ij}^* = (\widetilde{\lambda}_j - \widetilde{\lambda}_i - k_{ij})\frac{H(T_i)}{2} \tag{4.28}$$

      where,

$$\widetilde{\lambda}_i = \frac{U(T_i, i) - 2X(S_i)}{H(T_i)}, \ \ \widetilde{\lambda}_j = \sum_{(f,h)\in S_{lj}^{arc}} v_{fh}k_{fh}.$$

3. If $\lambda_l^* = 0$, $|L| > 1$ and the optimal sharing group is undecomposable, the optimal reallocation quantities may not be unique, and the optimal solution presented in part 2 is one of them.

Remark: By eliminating the arc which connects node i and j, we consider S_i and S_j as two separate sharing groups. Then, the shadow prices of resource i and j can be calculated by the methodology shown in Proposition 11 as $\widetilde{\lambda}_i$ and $\widetilde{\lambda}_j$ within S_i and S_j, respectively. $\widetilde{\lambda}_j - \widetilde{\lambda}_i - k_{ij}$ is the marginal profit of reallocation from i to j when the reallocation quantity between resource i and j is 0. Proposition 12 tells us, (except the cases that resource i is the only resource in subset S_i and $\beta_i^* > 0$ where the reallocation between i and j reaches the upper bound x_i,) the reallocation amount between resources i and j is proportional to $\widetilde{\lambda}_j - \widetilde{\lambda}_i - k_{ij}$.

Proof:

1. $\lambda_l^* > 0$.

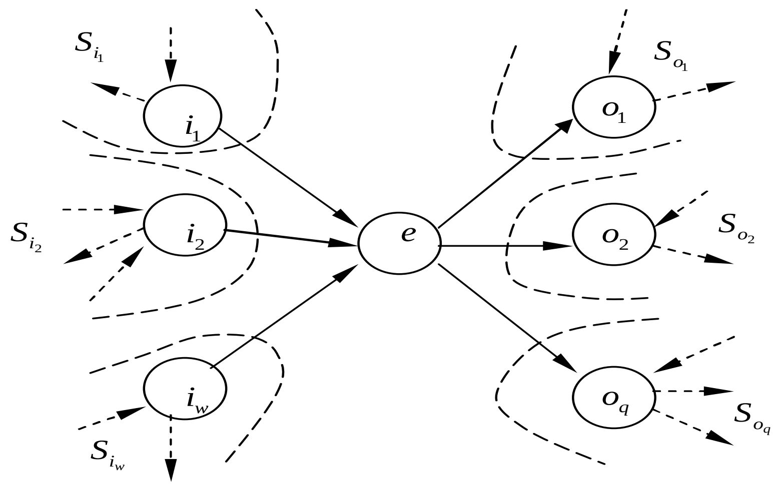

Figure 4.2: Reallocation among resource e and other resources

(a) When $\beta_i^* > 0$, $\gamma_i - \alpha_i p_i^* = 0$ from (4.10). Based on Proposition 10, $\lambda_i^* > \frac{\gamma_i}{\alpha_i} \geq 0$. Inequality (4.2) holds as equality for resource i. $S_i = \{i\}$ means that resource i is only connected to resource j in the sharing group. Therefore, from constraint (4.2),

$$\gamma_i - \alpha_i p_i^* = x_i - z_{ij}^* = 0 \Rightarrow z_{ij}^* = x_i$$

(b) Consider an arbitrary node e in the optimal sharing group. Suppose that there are w ingoing arcs into node e and q outgoing arcs from node e. These ingoing and outgoing arcs are shown in Figure 4.2. We define the subset S_{i_a} to denote the set of nodes (i.e., the tree originating at node i_a) that are connected to node e through node i_a such that there is an ingoing arc from node i_a into node e, where $a = 1,2,...,w$. Similarly, we define the subset S_{o_a} to denote the set of nodes (i.e., the tree originating at node o_a) that are connected to node e through node o_a such that there is an outgoing arc from node e into node o_a, where $a = 1,2,...,q$. These subsets are also illustrated in Figure 4.2.

Since $\lambda_e^* \geq \lambda_l^* > 0$, inequality (4.2) holds as equality for resource e, i.e.,

$$\gamma_e - \alpha_e p_e = x_e + \sum_{a=1}^{w} z_{i_a e}^* - \sum_{a=1}^{q} z_{e o_a}^*.$$

Let $A = \{a | a = 1,2,...,w\}$ (i.e., the set of the subscripts of the resources from i_1 to i_w), and $A_0 = \{a | a = 1,2,...,w,\ \beta_{i_a} > 0,\ S_{i_a} = i_a\}$ (i.e., the set of the subscripts of the resources which belong to resource set $\{i_1, i_2, ..., i_w\}$ and satisfy $\beta_{i_a} > 0$ and $S_{i_a} = i_a$ with $a \in \{1, 2, ..., w\}$).

Based on the definition, if $a \in A_0$, $\beta_{i_a} > 0, S_{i_a} = i_a$. Therefore $z_{i_a e} = x_{i_a}$, and

$$\gamma_e - \alpha_e p_e = x_e + \sum_{a \in A_0} x_{i_a} + \sum_{a \in A \setminus A_0} z^*_{i_a e} - \sum_{a=1}^{q} z^*_{e o_a}. \tag{4.29}$$

When we plug in the z^*_{ij} values given in equation (4.26) into the right hand side of equation (4.29), we obtain:

$$\begin{aligned}
& x_e + \sum_{a\in A_0} x_{i_a} + \sum_{a\in A\setminus A_0} z^*_{i_a e} - \sum_{a=1}^{q} z^*_{eo_a} \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \sum_{a\in A\setminus A_0} \frac{H(T_{i_a})H(T\setminus T_{i_a})}{2H(T)}\Big(\frac{U(T\setminus T_{i_a},e)-2X(S\setminus S_{i_a})}{H(T\setminus T_{i_a})} \\
& -\frac{U(T_{i_a},i_a)-2X(S_{i_a})}{H(T_{i_a})} - k_{i_a e}\Big) - \sum_{a=1}^{q} \frac{H(T_{o_a})H(T\setminus T_{o_a})}{2H(T)}\Big(\frac{U(T_{o_a},o_a)-2X(S_{o_a})}{H(T_{o_a})} \\
& -\frac{U(T\setminus T_{o_a},e)-2X(S\setminus S_{o_a})}{H(T\setminus T_{o_a})} - k_{eo_a}\Big) \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \sum_{a\in A\setminus A_0} \frac{1}{2H(T)}[(U(T\setminus T_{i_a},e)-2X(S\setminus S_{i_a}))H(T_{i_a}) \\
& -(U(T_{i_a},i_a)-2X(S_{i_a}))H(T\setminus T_{i_a}) - H(T_{i_a})H(T\setminus T_{i_a})k_{i_a e}] \\
& -\sum_{a=1}^{q} \frac{1}{2H(T)}[(U(T_{o_a},o_a)-2X(S_{o_a}))H(T\setminus T_{o_a}) \\
& -(U(T\setminus T_{o_a},e)-2X(S\setminus S_{o_a}))(H(T_{o_a}) - H(T_{o_a})H(T\setminus T_{o_a})k_{eo_a}] \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \sum_{a\in A\setminus A_0} \frac{1}{2H(T)}[(U(T,e)-2X(S))H(T_{i_a}) \\
& -(U(T_{i_a},e)-2X(S_{i_a}))H(T)] \\
& -\sum_{a=1}^{q} \frac{1}{2H(T)}[(U(T_{o_a},e)-2X(S_{o_a}))H(T) - (U(T,e)-2X(S))H(T_{o_a})] \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \frac{U(T,e)-2X(S)}{2H(T)}[\sum_{a\in A\setminus A_0} H(T_{i_a}) + \sum_{a=1}^{q} H(T_{o_a})] \\
& -\frac{1}{2}[\sum_{a\in A\setminus A_0}(U(T_{i_a},e)-2X(S_{i_a})) + \sum_{a=1}^{q}(U(T_{o_a},e)-2X(S_{o_a}))] \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \frac{U(T,e)-2X(S)}{2H(T)} H((\cup_{a\in A\setminus A_0} T_{i_a}) \cup (\cup_{a=1}^{q} T_{o_a})) \\
& -\frac{1}{2}[\sum_{a\in A\setminus A_0}(U(T_{i_a},e)-2X(S_{i_a})) + \sum_{a=1}^{q}(U(T_{o_a},e)-2X(S_{o_a}))] \\
= \;& \frac{U(T,e)-2X(S)}{2H(T)} H((\cup_{a\in A\setminus A_0} T_{i_a}) \cup (\cup_{a=1}^{q} T_{o_a})) \\
& -\frac{1}{2}U((\cup_{a\in A\setminus A_0} T_{i_a}) \cup (\cup_{a=1}^{q} T_{o_a}), e) + X(S) \qquad (4.30)
\end{aligned}$$

If $\beta^*_e > 0$, then $\gamma_e - \alpha_e p^*_e = 0$, $(\cup_{a\in A\setminus A_0} T_{i_a}) \cup (\cup_{a=1}^{q} T_{o_a}) = T$. Then, $(4.30) = 0 = \gamma_e - \alpha_e p^*_e$. Equation (4.29) holds.

If $\beta_e^* = 0$, $(\cup_{a\in A\setminus A_0} T_{i_a}) \cup (\cup_{a=1}^{q} T_{o_a}) = T \setminus \{e\}$.

$$\begin{aligned}
(4.30) \quad &= \frac{U(T,e)-2X(S)}{2H(T)} H(T\setminus\{e\}) - \frac{1}{2}U(T\setminus\{e\},e) + X(S) \\
&= \frac{1}{2}U(\{e\},e) - \frac{U(T,e)-2X(S)}{2H(T)} H(\{e\}) \\
&= \frac{1}{2}(\gamma_e - \alpha_e \lambda_e^*) \\
&= \gamma_e - \alpha_e p_e^*
\end{aligned}$$

Equation (4.29) holds.

In an acyclic optimal sharing group , each of the leaves of the spanning tree is connected to the tree by a single arc. Since equation (4.29) must be satisfied by every resource in the sharing group and we have shown that the equation holds by plugging the expression in Proposition 12. Therefore, the expression given in Proposition 12 is the unique solution for each of the leaves. If we cut those leaves, new leaves appear and the amount of the flow through the arc which connects each of the new leaves to the tree can be uniquely determined. Eventually, all the resources in the sharing group may become the leaves and the reallocation amounts can be uniquely determined.

2. $\lambda_l^* = 0$, $|L| = 1$

   (a) The proof is the same as $1(a)$.

   (b) We consider the same configuration as shown in Figure 4.2. Here, we suppose that $e \neq l$ (When $e = l$, the proof is similar). Without loss of generality, assume that $l \in S_{i1}$. Inequality (4.2) holds as equality for resource e. Therefore,

$$\gamma_e - \alpha_e p_e = x_e + \sum_{a\in A_0} x_{i_a} + z_{i_1 e} + \sum_{a\in A\setminus A_0\setminus\{1\}} z_{i_a e}^* - \sum_{a=1}^{q} z_{e o_a}^*. \tag{4.31}$$

Note that since $l \in S_{i_1}$, set A_0 can not include integer 1. Because if $1 \in A_0$, $i_1 = l \Rightarrow \beta_l^* > 0 \Rightarrow$ (based on Proposition 10) $\frac{\gamma_e}{\alpha_e} < \lambda_l^* = 0$. Since the demand γ_e is nonnegative, this results in contradiction.

We plug the expression of z_{ij} presented in the (4.28) into the right hand side of equation (4.31) to obtain

$$\begin{aligned}
& x_e + \sum_{a\in A_0} x_{i_a} + z_{i_1 e} + \sum_{a\in A\setminus A_0\setminus\{1\}} z^*_{i_a e} - \sum_{a=1}^{q} z^*_{eo_a} \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} + \frac{H(T\setminus T_{i_1})}{2}\left(\frac{U(T\setminus T_{i_1}, e) - 2X(S\setminus S_{i_1})}{H(T\setminus T_{i_1})} - \sum_{(f,h)\in S^{arc}_{li_1}} v_{fh}k_{fh} - k_{i_1 e}\right) \\
& + \sum_{a\in A\setminus A_0\setminus\{1\}} \frac{H(T_{i_a})}{2}\left(\sum_{(f,h)\in S^{arc}_{le}} v_{fh}k_{fh} - \frac{U(T_{i_a}, i_a) - 2X(S_{i_a})}{H(T_{i_a})} - k_{i_a e}\right) \\
& - \sum_{a=1}^{q} \frac{H(T_{o_a})}{2}\left(\frac{U(T_{o_a}, o_a) - 2X(S_{o_a})}{H(T_{o_a})} - \sum_{(f,h)\in S^{arc}_{le}} v_{fh}k_{fh} - k_{eo_a}\right) \\
= \;& x_e + \sum_{a\in A_0} x_{i_a} - X(S\setminus S_{i_1}) + \sum_{a\in A\setminus A_0\setminus\{1\}} X(S_{i_a}) + \sum_{a=1}^{q} X(S_{o_a}) \\
& + \frac{1}{2}\left(U(T\setminus T_{i_1}, e) - \sum_{a\in A\setminus A_0\setminus\{1\}} U(T_{i_a}, e) - \sum_{a=1}^{q} U(T_{o_a}, e)\right) \\
& - \left(H(T\setminus T_{i_1}) - \sum_{a\in A\setminus A_0\setminus\{1\}} H(T_{i_a}) - \sum_{a=1}^{q} H(T_{o_a})\right) \sum_{(f,h)\in S^{arc}_{le}} v_{fh}k_{fh} \qquad (4.32)
\end{aligned}$$

If $\beta^*_e > 0$, $e \notin T$.

$\Rightarrow (4.32) = 0 = \gamma_e - \alpha_e p^*_e$. Equation (4.31) holds.

If $\beta^*_e = 0$, $e \in T$.

$$\begin{aligned}
(4.32) &= \frac{1}{2}[U(\{e\}, e) - H(\{e\}) \sum_{(f,h)\in S^{arc}_{le}} v_{fh}k_{fh}] \\
&= \frac{1}{2}(\gamma_e - \alpha_e \lambda^*_e) \\
&= \gamma_e - \alpha_e p^*_e
\end{aligned}$$

Equation (4.31) holds.

By conducting similar computation procedure (leaves-tree) as shown at the end of the proof of $1(b)$ except for resource l, we can uniquely determine the reallocation amounts among the resources.

3. $\lambda^*_l = 0$, $|L| > 1$,

If $|L| > 1$, we pick one of the base resources, denoted by l. Since the optimal sharing group is undecomposable, by sending the flows among the resources in set L, we are able to concentrate all the extra available capacity to resource l that we have picked and the capacity constraint (4.2) is binding for all resources in the sharing group except resource l. Eventually, we transform the solution to the solution in the case with $\lambda_l^* = 0$ and $|L| = 1$. ■

In order to guarantee the optimality of a sharing group, the z_{ij}'s computed in Proposition 12 must be nonnegative. If some z_{ij}'s computed in Proposition 12 are negative, the assumption that a given sharing group is an undecomposable optimal sharing group does not hold.

The resources in a sharing group give rise to significant profits by balancing the asymmetry of demand and supply which can compensate the costs incurred as a result of reallocation. If the unit reallocation costs are relatively high, the resources are broken into several sharing groups.

Let us assume that the optimal solution to P_2 consists of m sharing groups, and denote the set of resources in each sharing group by $S_1, S_2, ..., S_m$, respectively. As a result, P_2 can be divided into m subproblems P_2^h, $h = 1, 2, ..., m$. Note that each optimal sharing group can be decomposable or undecomposable, and it can be cyclic or acyclic because Proposition 13 is only related to the optimal shadow prices which are always unique.

Subproblem (P_2^h):

$$\Phi^*(\vec{x}, \vec{\gamma}) = \max_{z_{ij}, \vec{p}} \sum_{i \in S_h} p_i(\gamma_i - \alpha_i p_i) - \sum_{i \in S_h} \sum_{j \neq i, j \in S_h} k_{ij} z_{ij} \tag{4.33}$$

$$s.t.: \quad \gamma_i - \alpha_i p_i \leq x_i + \sum_{j \neq i} z_{ji} - \sum_{j \neq i} z_{ij} \tag{4.34}$$

$$z_{ij} \geq 0 \quad \forall j \neq i, \ i, j \in S_h \tag{4.35}$$

$$\gamma_i - \alpha_i p_i \geq 0 \quad \forall i \in S_h \tag{4.36}$$

$$p_i \geq 0 \quad \forall i \in S_h \tag{4.37}$$

Let $\overrightarrow{Z}^*_h$ denote the optimal reallocation matrix for subproblem h. Let $\overrightarrow{p}^*_h$ denote the optimal price vector for subproblem h. Let $\overrightarrow{\lambda}^*_h$, $\overrightarrow{\beta}^*_h$, $\overrightarrow{u}^*_h$ denote the vectors and the matrix for the optimal values of Lagrange multipliers, respectively, for subproblem h.

Consider two subproblems $P_2^{h_1}$ and $P_2^{h_2}$. Let us denote the optimal of solution and the corresponding optimal Lagrange multipliers for subproblem $P_2^{h_i}$ by $(\overrightarrow{Z}^*_{h_i}, \overrightarrow{p}^*_{h_i}, \overrightarrow{\lambda}^*_{h_i}, \overrightarrow{\beta}^*_{h_i}, \overrightarrow{u}^*_{h_i})$ for $i = 1,2$.

Proposition 13. If S_{h_1} and S_{h_2} are combined into a single set of resources, $S_c = S_{h_1} + S_{h_2}$. $(\overrightarrow{Z}^*_{h_i}, \overrightarrow{p}^*_{h_i}, \overrightarrow{\lambda}^*_{h_i}, \overrightarrow{\beta}^*_{h_i}, \overrightarrow{u}^*_{h_i})$ for $i = 1,2$ are also optimal for S_c if and only if

$$\lambda^*_j - k_{ij} \leq \lambda^*_i \leq \lambda^*_j + k_{ji} \;\; \forall i \in S_{h_1}, \forall j \in S_{h_2}$$

Proof: If $(\overrightarrow{Z}^*_{h_i}, \overrightarrow{p}^*_{h_i}, \overrightarrow{\lambda}^*_{h_i}, \overrightarrow{\beta}^*_{h_i}, \overrightarrow{u}^*_{h_i})$ for $i = 1,2$ are optimal for S_c, the Lagrange multipliers $\overrightarrow{\lambda}^*_{h_i}$, $\overrightarrow{\beta}^*_{h_i}$ and $\overrightarrow{u}^*_{h_i}$ for $i = 1,2$ should satisfy the KKT conditions. It is straightforward to observe that the KKT conditions given by (4.6) and (4.8) – (4.11) are satisfied. Therefore, let us consider the KKT condition given by (4.7). If we consider two nodes $i, j \in S_{h_1}$ or $i, j \in S_{h_2}$, the KKT condition (4.7) is satisfied. If $i \in S_{h_1}$ and $j \in S_{h_2}$, based on (4.7),

$$u_{ij} = \lambda^*_i - \lambda^*_j + k_{ij} \geq 0$$

$$\lambda^*_i \geq \lambda^*_j - k_{ij}$$

$$u_{ji} = k_{ji} - \lambda_i + \lambda_j \geq 0$$

$$\Rightarrow \lambda^*_i \leq \lambda^*_j + k_{ji}$$

$$\Rightarrow \lambda^*_j - k_{ij} \leq \lambda^*_i \leq \lambda^*_j + k_{ji} \;\; \forall i \in S_{t_1}, j \in S_{t_2}$$

On the other hand, if

$$\lambda^*_j - k_{ij} \leq \lambda^*_i \leq \lambda^*_j + k_{ji} \;\; \forall i \in S_{h_1}, j \in S_{h_2}$$

If $i \in S_{h_1}, j \in S_{h_2}$, let $z^*_{ij} = 0, z^*_{ji} = 0$, $u^*_{ij} = \lambda^*_i - \lambda^*_j + k_{ij}$ and $u^*_{ji} = k_{ji} - \lambda^*_i + \lambda^*_j$.

Plugging the solution $(\overrightarrow{Z}^*_{h_i}, \overrightarrow{p}^*_{h_i}, \overrightarrow{\lambda}^*_{h_i}, \overrightarrow{\beta}^*_{h_i}, \overrightarrow{u}^*_{h_i})$ for $i = 1,2$ with the z^*_{ij}s and u^*_{ij}s defined above for the pairs of resources which belongs to two subsets into the combined problem, it is straightforward to observe that the KKT conditions given by (4.6) – (4.11) are satisfied. Therefore, $(\overrightarrow{Z}^*_{h_i}, \overrightarrow{p}^*_{h_i}, \overrightarrow{\lambda}^*_{h_i}, \overrightarrow{\beta}^*_{h_i}, \overrightarrow{u}^*_{h_i})$ for $i = 1,2$. are also optimal for S_c. ■

Proposition 13 gives us a more clear picture of the optimal solution. In the optimal solution, the set of resources is decomposed into several sharing groups. Each sharing group can be described by the results in Proposition 11 and 12. The resources in different sharing groups satisfy the condition presented in Propositions 13, i.e., relatively high reallocation cost prevent the resources in different sharing groups from merging into a single sharing group.

4.2.2 An Exact Procedure for Solving P_2

Based on previous results, in this section, we propose a procedure to solve the problem based on partitioning the demand space. The idea is to list all possible forms of the optimal solution, and determine the corresponding valid regions in the demand space. Although this procedure solves P_2 optimally, it works for problems that are small in size (e.g., problems with 3-4 resources). In the next section, we will present two heuristic procedures that can be used to solve realistic size problems. The exact method based on partitioning the demand space is summarized as follows where each step of the procedure is explained for a system with three resources.

1. List all the possible sharing groups. For example, the three-resource model has following combinations: $\{(1),(2),(3)\}$, $\{(1,2),(3)\}$, $\{(1),(2,3)\}$, $\{(1,3),(2)\}$, $\{(1,2,3)\}$. The numbers in () are the indexes of resources in the same sharing group.

2. For the group which has more than one resource, list all the possible combinations of reallocation. For example, the group $\{(1,2,3)\}$ has three resources. Let $\rightarrow$ and $\leftarrow$ denote the direction of reallocation. The cycle-free combinations include $\{(1 \rightarrow 2 \rightarrow 3)\}$, $\{(1 \rightarrow 2 \leftarrow 3)\}$, $\{(1 \leftarrow 2 \rightarrow 3)\}$, $\{(1 \leftarrow 2 \leftarrow 3)\}$, $\{(1 \rightarrow 3 \rightarrow 2)\}$, $\{(1 \rightarrow 3 \leftarrow 2)\}$,

$\{(1 \leftarrow 3 \rightarrow 2)\}$, $\{(1 \leftarrow 3 \leftarrow 2)\}$. According to Proposition 8, inequality (4.13) must be satisfied by every pair of the resources in the combinations.

3. Each group may work in capacity constraint binding or nonbinding status. Each resource i may have a positive β_i or $\beta_i = 0$. Thus, corresponding combinations are generated. For example, group $\{(1 \rightarrow 2 \rightarrow 3)\}$ has the following combinations:

   (a) Capacity constraint binding, $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = 0$.

   (b) Capacity constraint binding, $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 > 0$.

   (c) Capacity constraint binding, $\beta_1 = 0$, $\beta_2 > 0$, $\beta_3 = 0$.

   (d) Capacity constraint binding, $\beta_1 = 0$, $\beta_2 > 0$, $\beta_3 > 0$.

   (e) Capacity constraint binding, $\beta_1 > 0$, $\beta_2 = 0$, $\beta_3 = 0$.

   (f) Capacity constraint binding, $\beta_1 > 0$, $\beta_2 = 0$, $\beta_3 > 0$.

   (g) Capacity constraint binding, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 = 0$.

   (h) Capacity constraint binding, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$.

   (i) Capacity constraint nonbinding, $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 = 0$.

   (j) Capacity constraint nonbinding, $\beta_1 = 0$, $\beta_2 = 0$, $\beta_3 > 0$.

   (k) Capacity constraint nonbinding, $\beta_1 = 0$, $\beta_2 > 0$, $\beta_3 = 0$.

   (l) Capacity constraint nonbinding, $\beta_1 = 0$, $\beta_2 > 0$, $\beta_3 > 0$.

   (m) Capacity constraint nonbinding, $\beta_1 > 0$, $\beta_2 = 0$, $\beta_3 = 0$.

   (n) Capacity constraint nonbinding, $\beta_1 > 0$, $\beta_2 = 0$, $\beta_3 > 0$.

   (o) Capacity constraint nonbinding, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 = 0$.

   (p) Capacity constraint nonbinding, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$.

4. Apply the result of Proposition 11 to find the expressions of the optimal shadow prices and selling prices, then further obtain the optimal reallocation quantities by Proposition 12. The resources must satisfy the corresponding inequalities given in Propositions 10 and 11. The amounts of reallocations must be nonnegative.

5. The resources in different sharing groups must satisfy the inequalities given in Proposition 13. (If one or more of the inequalities are not satisfied, the corresponding grouping can not be optimal.)

6. Summarizing the inequalities obtained in steps 2, 4 and 5 for each combination (which represents each possible combination of steps 1, 2 and 3) gives rise to a region of the demand space in which the combination is optimal. For example, that $\{(1),(3 \to 2)\}$ with $\beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 0$, the capacity constraint of group (1) is binding and the capacity constraint of group $(3 \to 2)$ is nonbinding is one of the combinations.

The algorithm above solves P_2 by obtaining all the valid regions of the optimal solution. However, as the dimension of the problem increases, the number of regions increases exponentially. Therefore, this procedure is good for problems with small dimension. For moderate or large dimensional problems, it is necessary to seek other methods which solves the problem efficiently.

In the remainder of this section, we focus on a special case which allows us decompose P_2 into smaller independent subproblems. As a result, the above procedure, based on partitioning the demand space, may be applied to each subproblem. The structure of P_2 is complicated because of the reallocation imposed by flexibility. If we add more constraints to the model in a way that the reallocation will be less flexible, the problem can be decomposed into small subproblems as explained below.

Let $k_{ij} + k_{jl} \geq k_{il}\ \forall i, j, l$. This inequality indicates that the minimum unit reallocation cost between resource i and l can be obtained by reallocating one unit resource from i to l directly. We will call this inequality as the "triangle assumption". This is a reasonable assumption in many cases. For example, consider that the reallocation cost is the transportation cost between two locations. Suppose that there are three locations, A, B and C. Generally, the transportation cost between A and C can be assumed to be less than or equal to the sum of transportation costs from city A to B and from B to C. Under the "triangle assumption", we have the following result:

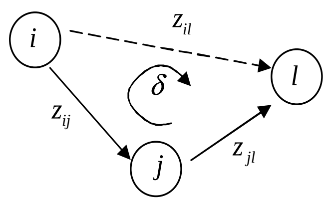

Figure 4.3: Pure consumer or supplier

Proposition 14. If $k_{ij}+k_{jl} \geq k_{il}$ $\forall i,j,l$, P_2 has an optimal solution such that each resource is either a pure supplier or a pure consumer in an optimal solution.

Proof: Suppose P_2 has an optimal solution in which resource j is a supplier of l and a consumer of i as shown is Figure 4.3. Consider i is a supplier of l with $z_{il}=0$. Let A denote the optimal objective value of P_2 excluding the reallocation cost among resources i, j and l. Then, the optimal objective value is $A+z^*_{ij}k_{ij}+z^*_{jl}k_{jl}+z^*_{il}k_{il}$. If $k_{ij}+k_{jl}>k_{il}$, if we send a flow $\delta \leq \min(z_{ij},z_{jl})$ with the direction as shown in Figure 4.3, the objective value becomes

$$\begin{aligned} & A+(z^*_{ij}-\delta)k_{ij}+(z^*_{jl}-\delta)k_{jl}+(z^*_{il}+\delta)k_{il} \\ = \quad & A+z^*_{ij}k_{ij}+z^*_{jl}k_{jl}+z^*_{il}k_{il}+\delta(k_{il}-k_{ij}-k_{jl}) \\ < \quad & A+z^*_{ij}k_{ij}+z^*_{jl}k_{jl}+z^*_{il}k_{il}. \end{aligned}$$

which is a contradiction with the assumption of the optimality. Therefore, $k_{ij}+k_{jl}>k_{il}$ can not be true. So, only when $k_{ij}+k_{jl}=k_{il}$, resource j can be a supplier and a consumer at the same time. Without loss of generality, suppose that $z^*_{ij} \leq z^*_{jl}$. If we send a flow $\delta = z^*_{ij}$ with the direction as shown in Figure 4.3, each resource becomes either a pure supplier or a pure consumer. That is, resources i and j are suppliers, and resource l is a consumer. ■

Proposition 15. If $k_{ij}+k_{jl} \geq k_{il}$ $\forall i,j,l$ $i \neq j \neq l$, and if for $j \in S$, $(\frac{\gamma_i-2x_i}{\alpha_i})^+ - (\frac{\gamma_j-2x_j}{\alpha_j})^+ < k_{ji}$, resource j cannot be a supplier of resource i in an optimal solution.

Remark: Proposition 15 provides a necessary condition to check if some resource could be a supplier or consumer of another resource in the optimal solution.

Proof: Suppose that in an optimal solution, resource j is a supplier of resource i. Resource j must be a pure supplier and resource i must be a pure consumer. $(\frac{\gamma_j-2x_j}{\alpha_j})^+$ is the optimal shadow price of j (i.e., λ_j^*) when it is neither a supplier nor a consumer. When resource j is a pure supplier, then the shadow price of j must be greater than $(\frac{\gamma_j-2x_j}{\alpha_j})^+$. Similarly, as a pure consumer, the shadow price λ_i^* of i must be less than $(\frac{\gamma_i-2x_i}{\alpha_i})^+$. That is, $\lambda_i^* < (\frac{\gamma_i-2x_i}{\alpha_i})^+$, $\lambda_j^* > (\frac{\gamma_j-2x_j}{\alpha_j})^+$. Then

$$\lambda_i^* - \lambda_j^* < (\frac{\gamma_i - 2x_i}{\alpha_i})^+ - (\frac{\gamma_j - 2x_j}{\alpha_j})^+ < k_{ji},$$

where the second inequality follows from the assumption of the proposition. This result is in contradiction based on Proposition 7. Therefore, resource j cannot be a supplier of resource i in an optimal solution. ■

When the dimension of the problem, n, increases, the number of partitions explodes. The procedure to solve P_2 by partitioning the demand space becomes computationally infeasible. The assumption $k_{ij} + k_{jl} > k_{il}$ gives the solution of P_2 a nicer structure, and enable us to decompose the problem into several subproblems. The algorithm to decompose the problem into subproblems is given as follows:

Decomposition under the "triangle assumption"

1. Define $S = \{1, 2, ..., n\}$.

2. Let $i = 0$.

3. If $S = \emptyset$, set $i^* = i$ and stop the algorithm. Otherwise, let $i := i + 1$ and define $G_i = \emptyset$. Choose a resource $j \in S$, update S to $S \setminus \{j\}$, G_i to $G_i \cup \{j\}$ and mark resource j as not visited.

4. If all the resources in G_i are visited, go to step 3. Otherwise, choose a resource $j \in G_i$ which is not visited. Let S_j be the set of resources that can be suppliers or consumers of resource j. Update S as $S \setminus S_j$ and update G_i as $G_i \cup S_j$.

5. Mark resource j as visited and go to step 4.

When the algorithm stops, i^* is the smallest number of optimal sharing groups in P_2. The sets $G_1, G_2, ..., G_{i^*}$ are mutually exclusive. P_2 can be solved separately for each of these sets as i^* distinct subproblems.

4.2.3 Heuristic Algorithms for Solving P_2

The procedure presented in the previous section is inefficient even when the problem can be decomposed into subproblems under the "triangle" assumption. In this section, we propose two heuristic algorithms in order to solve P_2 (i.e., the second stage problem) approximately. Both algorithms follow the same procedure: Beginning with a feasible solution (i.e., the solution with no reallocation among the resources), we iteratively choose a pair of resources which may generate significant profit with reallocation between them. The process continues until there does not exist any pair of resources that can generate profit by reallocating. Both algorithms try to take the advantage of reallocation as much as possible to approach the optimum. The difference between the two heuristics is the reallocation strategies among the resources. Numerical results show that both algorithms provide very good approximations to the optimal solution. Let us denote the marginal reallocation profit between two resources i and j, $i,\ j = 1,2,...,n$, as MRP_{ij}. Since marginal reallocation profit (MRP) plays the key role in both algorithms, we call them MRP algorithms. The inputs of the MRP algorithms include market potential vector $\overrightarrow{\gamma} = \{\gamma_1, \gamma_2, ..., \gamma_n\}$, demand function coefficient vector $\overrightarrow{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, unit reallocation cost matrix K and the initial resource capacity vector $\overrightarrow{x} = \{x_1, x_2, ..., x_n\}$. In the algorithms, since we keep updating the reallocation amount among the resources, the available capacity at each resource is changing accordingly. We use vector $\overrightarrow{y} = \{y_1, y_2, ..., y_n\}$ to denote the actual available capacity vector, i.e., $y_i = x_i + \sum_{j \neq i} z_{ji} - \sum_{j \neq i} z_{ij}, \forall i$. Z denotes the reallocation amount matrix. The outputs include the approximate optimal selling price vector $\overrightarrow{\widehat{p}^*}$, the approximate optimal reallocation amount matrix $\widehat{Z}^*$ and the approximate optimal objective value $\widehat{\Phi}^*$.

MRP Algorithm 1

Let ε denote a small positive real number, which is used to control the accuracy of the algorithm, and δ denote the step length of each reallocation.

1. Start with a solution without reallocation among the resources. Set $\overrightarrow{y} = \overrightarrow{x}$, $z_{ij} = 0,\ i, j = 1,2,...,n$.

2. Compute the marginal reallocation profit from resource i to resource j, denoted by MRP_{ij}, for all $i \neq j$ as follows:

   (a) If $z_{ji} = 0$, $MRP_{ij} = [(\frac{\gamma_j - 2y_j}{\alpha_j})^+ - (\frac{\gamma_i - 2y_i}{\alpha_i})^+ - k_{ij}]\mathbb{1}(y_i > 0)$.

   (b) If $z_{ji} > 0$, $MRP_{ij} = -[((\frac{\gamma_i - 2y_i}{\alpha_i})^+ - (\frac{\gamma_j - 2y_j}{\alpha_j})^+ - k_{ji})]\mathbb{1}(y_i > 0)$.

3. Choose the pair of the resources, say i^*, j^*, with the largest MRP.

4. (a) If $MRP_{i^*j^*} \leq \varepsilon$, compute the output as follows:

$$\begin{aligned} \widehat{z}_{ij}^* &= z_{ij},\ i,j = 1,2,...,n \\ \widehat{p}_i^* &= \frac{\gamma_i}{2\alpha_i} + (\frac{\gamma_i - 2y_i}{2\alpha_i})^+,\ i = 1,2,...,n. \\ \widehat{\Phi}^* &= \sum_{i=1}^{n} \widehat{p}_i^*(\gamma_i - \alpha_i \widehat{p}_i^*) + \sum_{i=1}^{n} \sum_{j \neq i} k_{ij} \widehat{z}_{ij}^* \end{aligned}$$

   Return the output and stop the algorithm.

   (b) Otherwise, go to the next step.

5. Reallocate a small amount of capacity δ from i^* to j^* and update y_{i^*} and y_{j^*} with $y_{i^*} - \delta$, $y_{j^*} + \delta$, respectively.

   (a) If the reallocation is a forward reallocation ($z_{j^*i^*} = 0$), update $z_{i^*j^*}$ with $z_{i^*j^*} + \delta$.

   (b) If the reallocation is a backward reallocation ($z_{j^*i^*} > 0$), update $z_{j^*i^*}$ with $z_{j^*i^*} - \delta$.

   Then, go to step 2.

For a single-resource system with input γ_i, y_i, α_i, the optimal solution can be easily obtained as follows:

$$\lambda_i^* = (\frac{\gamma_i - 2y_i}{\alpha_i})^+,\ p_i^* = \frac{\gamma_i}{2\alpha_i} + (\frac{\gamma_i - 2y_i}{2\alpha_i})^+.$$

When there is more than one resource in the system, and we send one unit of capacity from resource i ($y_i > 0$, i.e., resource i is available) to j, the increase in profit of resource j is

$(\frac{\gamma_j-2y_j}{\alpha_j})^+$, the decrease in profit of resource i is $(\frac{\gamma_i-2y_i}{\alpha_i})^+$, and the reallocation cost is k_{ij}. Therefore, the profit that can be obtained from the reallocation is

$$(\frac{\gamma_j-2y_j}{\alpha_j})^+ - (\frac{\gamma_i-2y_i}{\alpha_i})^+ - k_{ij}. \tag{4.38}$$

Step 2 calculates the marginal reallocation profit of each pair of the resources. Consider two resources i and j. If there is no reallocation from j to i and resource i is available ($y_i > 0$), the marginal reallocation profit is $(\frac{\gamma_j-2y_j}{\alpha_j})^+ - (\frac{\gamma_i-2y_i}{\alpha_i})^+ - k_{ij}$, and we call it "forward" marginal reallocation profit. On the other hand, if there is reallocation from j to i and resource i is available, sending back one unit capacity from i to j will generate profit $(\frac{\gamma_j-2y_j}{\alpha_j})^+ - (\frac{\gamma_i-2y_i}{\alpha_i})^+ + k_{ji}$, and we call this case "backward" reallocation.

In step 5, we reallocate a small amount δ from resource i^* to resource j^*. After the reallocation, y_{i^*} is updated by $y_{i^*} - \delta$ and y_{j^*} is updated by $y_{j^*} + \delta$. Therefore, the marginal reallocation profit from i^* to j^*, $((\frac{\gamma_{j^*}-2y_{j^*}-2\delta}{\alpha_{j^*}})^+ - (\frac{\gamma_{i^*}-2y_{i^*}+2\delta}{\alpha_{i^*}})^+ - k_{i^*j^*})\mathbb{1}(y_{i^*} - \delta > 0)$, decreases at each iteration. Note that here we only discuss forward reallocation. The backward reallocation is similar. By iterating the steps of the algorithm, eventually the largest marginal reallocation profit becomes less than or equal to ε. Since we define $MRP_{i^*j^*}$ as the product of $(\frac{\gamma_{j^*}-2y_{j^*}}{\alpha_{j^*}})^+ - (\frac{\gamma_{i^*}-2y_{i^*}}{\alpha_{i^*}})^+ - k_{i^*j^*}$ and $\mathbb{1}(y_{i^*} > 0)$, when $MRP_{i^*j^*} \leq \varepsilon$ and ε is small enough, it means that either no profit can be obtained by reallocating resource between any pair of resources or there is no available resource capacity.

In step 4, when the algorithm stops, since the available capacity of resource i is y_i, based on the result of the single-resource system, we have

$$p_i^* = \frac{\gamma_i}{2\alpha_i} + (\frac{\gamma_i - 2y_i}{2\alpha_i})^+.$$

Next, we will show the following: (1) The algorithm presented above converges for $\varepsilon > 0$; (2) When the algorithm stops, a feasible solution is obtained and no profit can be obtained by further reallocation between any pair of resources; (3) Under some conditions, the output of the algorithm is the optimal solution.

Let y^t_i, z^t_{ij} and MRP^t_{ij} denote the available capacity of resource i, reallocation amount from resource i to resource j, and marginal reallocation profit from resource i to resource j, respectively, just before the t^{th} iteration of the algorithm for $i,j = 1,2,...,n,\ i \neq j,\ t = 1,2,...$. Let i^*_t and j^*_t denote the indexes of the resources which have the largest marginal reallocation profit just before the t^{th} iteration. Also, let $\lambda^t_m = (\frac{\gamma_m - 2y^t_m}{\alpha_m})^+,\ m = 1,2,...,n.$

In order to guarantee the convergence of the algorithm, we need to define an appropriate step length δ. Let $\alpha_{min} = \min\{\alpha_i | i = 1,2,...,n\}$ and $\overline{\delta} = \frac{\varepsilon\alpha_{min}}{2}$. The following proposition shows that when the step length is less than $\overline{\delta}$, after the t^{th} iteration, the marginal reallocation profit between i^*_t and j^*_t decreases but always keeps nonnegative.

Proposition 16. If $\delta \leq \overline{\delta}$, for iterations $t = 1,2,...,\ MRP^t_{i^*_t j^*_t} > MRP^{t+1}_{i^*_t j^*_t} \geq 0$.

Proof:

For any iteration t, $MRP^t_{i^*_t j^*_t} > \varepsilon$. Otherwise, the algorithm would have stopped. When the reallocation is a forward reallocation,

$$\begin{aligned}
MRP^{t+1}_{i^*_t j^*_t} &= \left[\left(\frac{\gamma_{j^*_t} - 2y^t_{j^*_t} - 2\delta}{2\alpha_{j^*_t}}\right)^+ - \left(\frac{\gamma_{i^*_t} - 2y^t_{i^*_t} + 2\delta}{2\alpha_{i^*_t}}\right)^+ - k_{i^*_t j^*_t}\right] \mathbb{1}(y_{i^*_t} - \delta > 0) \\
&< \left[\left(\frac{\gamma_{j^*_t} - 2y^t_{j^*_t}}{2\alpha_{j^*_t}}\right)^+ - \left(\frac{\gamma_{i^*_t} - 2y^t_{i^*_t}}{2\alpha_{i^*_t}}\right)^+ - k_{i^*_t j^*_t}\right] (\mathbb{1}(y_{i^*_t} > 0)) = MRP^t_{i^*_t j^*_t} \\
MRP^{t+1}_{i^*_t j^*_t} &= \left[\left(\frac{\gamma_{j^*_t} - 2y^t_{j^*_t} - 2\delta}{2\alpha_{j^*_t}}\right)^+ - \left(\frac{\gamma_{i^*_t} - 2y^t_{i^*_t} + 2\delta}{2\alpha_{i^*_t}}\right)^+ - k_{i^*_t j^*_t}\right] \mathbb{1}(y_{i^*_t} - \delta > 0) \\
&\geq \left[\left(\frac{\gamma_{j^*_t} - 2y^t_{j^*_t}}{2\alpha_{j^*_t}}\right)^+ - \left(\frac{\gamma_{i^*_t} - 2y^t_{i^*_t}}{2\alpha_{i^*_t}}\right)^+ - k_{i^*_t j^*_t} - \frac{\delta}{\alpha_{i^*_t}} - \frac{\delta}{\alpha_{j^*_t}}\right] \mathbb{1}(y_{i^*_t} - \delta > 0) \\
&\geq \left[\varepsilon - \frac{2\delta}{\alpha_{min}}\right] \mathbb{1}(y_{i^*_t} - \delta > 0) = \left[\frac{2}{\alpha_{min}}(\overline{\delta} - \delta)\right] \mathbb{1}(y_{i^*_t} - \delta > 0)
\end{aligned}$$

If $\delta \leq \overline{\delta}$, $MRP^{t+1}_{i^*_t j^*_t} \geq 0$.

When the reallocation is a backward reallocation, the proof is similar. ■

Proposition 17. If $\delta < \overline{\delta}$ and $\frac{x_i}{\delta}$ is an integer for $i = 1,2,...,n$, the output of each iteration of the MRP algorithm 1 is a feasible solution of P_2 and MRP algorithm 1 will stop after a finite number of iterations.

Proof: First, we will show that when $\frac{x_i}{\delta}$ is an integer for $i = 1, 2, ..., n$, the output of each iteration generated by the MRP algorithm 1 is a feasible solution. Recall that $y_i = x_i + \sum_{j \neq i} z_{ji} - \sum_{j \neq i} z_{ij}, \forall i$. The constraints of P_2 include: (1) $\gamma_i - \alpha_i p_i \leq y_i \ \forall i$; (2) $z_{ji} \geq 0 \ \forall j \neq i$; (3) $\gamma_i - \alpha_i p_i \geq 0 \ \forall i$; (4) $p_i \geq 0 \ \forall i$. First, note that no matter how many reallocations have been conducted, $\frac{y_i}{\delta}$ is integral for $i = 1, 2, ..., n$ because in each iteration of MRP algorithm 1, y_i will be updated with $y_i + \delta$ or $y_i - \delta$ and the initial value of y_i is x_i. Since $\frac{y_i}{\delta}$ is an integer, when $y_i > 0$, $y_i - \delta \geq 0$. Since each reallocation in MRP algorithm 1 happens only when there is available capacity at the supplier resource, i.e., $y_i > 0$, after each iteration, $y_i \geq 0, \forall i$. Similarly, in each iteration of MRP algorithm 1, z_{ij} will be updated as $z_{ij} + \delta$ or $z_{ij} - \delta$, and the initial value of z_{ij} is 0, hence $\frac{z_{ij}}{\delta}$ is integral. Based on the algorithm, z_{ij} decreases only when $z_{ij} > 0$. Since $\frac{z_{ij}}{\delta}$ is integral, after each iteration, $z_{ij} \geq 0$. Since the MRP algorithm 1 defines

$$\widehat{p}_i^* = \frac{\gamma_i}{2\alpha_i} + (\frac{\gamma_i - 2y_i}{2\alpha_i})^+ \geq 0, \ i = 1, 2, ..., n.$$

Nonnegativity constraint of the selling prices is satisfied. Since

$$\gamma_i - \alpha_i \widehat{p}_i^* = \frac{\gamma_i - (\gamma_i - 2y_i)^+}{2} \geq 0,$$

The third constraint of P_2 is satisfied. We show that the first constraint of P_2 is satisfied by considering the following two cases:

1. If $\gamma_i - 2y_i \geq 0$, $\gamma_i - \alpha_i \widehat{p}_i^* = y_i$.

2. If $\gamma_i - 2y_i < 0$, $\gamma_i - \alpha_i \widehat{p}_i^* = \frac{\gamma_i}{2} < y_i$.

Since all constraints of P_2 are satisfied by the output of each iteration, the output of each iteration of the algorithm is a feasible solution.

Next, we will prove the algorithm will stop after finite step iterations by contradiction. Obviously, the optimal objective value of P_2 is finite. Therefore, the objective value of a feasible solution is finite. Suppose the algorithm will never stop, i.e., $MRP^t_{i_t^* j_t^*} > \varepsilon$, $t = 1, 2, ...$ Suppose the objective value of P_2 obtained from MRP algorithm 1 just before the

t^{th} iteration is Φ_t. After the t^{th} iteration, the marginal reallocation profit between resource i_t^* and resource j_t^* changes from $MRP^t_{i_t^* j_t^*}$ to $MRP^{t+1}_{i_t^* j_t^*}$, and the increase of the objective value is the integral of the marginal reallocation profit between resource i_t^* and resource j_t^*. Let θ denote the additional reallocation amount between i_t^* and resource j_t^* during the t^{th} iteration, and $M(\theta)$ denote the corresponding marginal reallocation profit function of θ. We have,

$$\Phi_{t+1} - \Phi_t = \int_0^\delta M(\theta) d\theta,$$

and $M(0) = MRP^t_{i_t^* j_t^*}$, $M(\delta) = MRP^{t+1}_{i_t^* j_t^*}$. Here we only discuss the forward reallocation. Based on the definition of the forward marginal reallocation profit,

$$M(\theta) = ((\frac{\gamma_{j_t^*} - 2y_{j_t^*} - 2\theta}{\alpha_{j_t^*}})^+ - (\frac{\gamma_{i_t^*} - 2y_{i_t^*} + 2\theta}{\alpha_{i_t^*}})^+ - k_{i_t^* j_t^*})\mathbb{1}(y_{i_t^*} - \theta > 0).$$

Actually, if it is a backward reallocation,

$$M(\theta) = ((\frac{\gamma_{j_t^*} - 2y_{j_t^*} - 2\theta}{\alpha_{j_t^*}})^+ - (\frac{\gamma_{i_t^*} - 2y_{i_t^*} + 2\theta}{\alpha_{i_t^*}})^+ + k_{j_t^* i_t^*})\mathbb{1}(y_{i_t^*} - \theta > 0).$$

Only the sign and the quantity of the unit reallocation cost are different, and the proof is similar. $M(\theta)$ is a decreasing function of θ, and when $\theta = \delta$, $M(\theta)$ reaches the minimum, $MRP^{t+1}_{i_t^* j_t^*}$. Based on Proposition 16, when $\delta < \bar{\delta}$, $MRP^{t+1}_{i_t^* j_t^*} > 0$. Since $M(\theta) > MRP^{t+1}_{i_t^* j_t^*} > 0$, we have $\frac{\gamma_{j_t^*} - 2y_{j_t^*} - 2\theta}{\alpha_{j_t^*}} > 0$ and $\mathbb{1}(y_{i_t^*} - \theta > 0) = 1$. Furthermore,

$$M(\theta) \geq \frac{\gamma_{j_t^*} - 2y_{j_t^*} - 2\theta}{\alpha_{j_t^*}} - (\frac{\gamma_{i_t^*} - 2y_{i_t^*}}{\alpha_{i_t^*}})^+ - \frac{2\theta}{\alpha_{i_t^*}} - k_{i_t^* j_t^*}.$$

Therefore,

$$M(\theta) \geq (MRP^t_{i_t^* j_t^*} - 2\theta(\frac{1}{\alpha_{i_t^*}} + \frac{1}{\alpha_{j_t^*}}))^+.$$

$$\Phi_{t+1} - \Phi_t = \int_0^\delta M(\theta) d\theta \geq \int_0^\delta (MRP^t_{i_t^* j_t^*} - 2\theta(\frac{1}{\alpha_{i_t^*}} + \frac{1}{\alpha_{j_t^*}}))^+ d\theta$$

If $MRP^t_{i_t^* j_t^*} - 2\delta(\frac{1}{\alpha_{i_t^*}} + \frac{1}{\alpha_{j_t^*}}) < 0$

$$\int_0^{\delta}(MRP^t_{i_t^* j_t^*}-2\theta(\frac{1}{\alpha_{i_t^*}}+\frac{1}{\alpha_{j_t^*}}))^+ d\theta = \frac{(MRP^t_{i_t^* j_t^*})^2}{4(\frac{1}{\alpha_{i_t^*}}+\frac{1}{\alpha_{j_t^*}})} > \frac{\varepsilon^2}{4(\frac{1}{\alpha_{i_t^*}}+\frac{1}{\alpha_{j_t^*}})}.$$

If $MRP^t_{i_t^* j_t^*}-2\delta(\frac{1}{\alpha_{i_t^*}}+\frac{1}{\alpha_{j_t^*}}) \geq 0$,

$$\int_0^{\delta}(MRP^t_{i_t^* j_t^*}-2\theta(\frac{1}{\alpha_{i^*}}+\frac{1}{\alpha_{j^*}}))^+ d\theta \geq \frac{\delta MRP^t_{i_t^* j_t^*}}{2} > \frac{\delta\varepsilon}{2}.$$

Therefore,

$$\Phi_{t+1}-\Phi_t > \min\{\frac{\varepsilon^2}{4(\frac{1}{\alpha_{i_t^*}}+\frac{1}{\alpha_{j_t^*}})}, \frac{\delta\varepsilon}{2}\}.$$

After the t^{th} iteration,

$$\Phi_{t+1} > \Phi_1 + t\min\{\frac{\varepsilon^2}{4(\frac{1}{\alpha_{i^*}}+\frac{1}{\alpha_{j^*}})}, \frac{\delta\varepsilon}{2}\}$$

If the algorithm does not converge, Φ_{t+1} will approach to $+\infty$ as t goes to ∞. It is a contradiction.

■

Algorithm 2 uses a different reallocation strategy which converges to a feasible solution at a faster rate.

MRP Algorithm 2

Let ε be a small positive real number.

1. Start with a solution without reallocation among the resources. $\overrightarrow{y}=\overrightarrow{x}$, $z_{ij}=0, i,j=1,2,...,n$.

2. Compute the marginal reallocation profit from resource i to resource j, denoted by MRP_{ij}, for all $i \neq j$ as follows:

   (a) If $z_{ji}=0$, $MRP_{ij}=[(\frac{\gamma_j-2y_j}{\alpha_j})^+-(\frac{\gamma_i-2y_i}{\alpha_i})^+-k_{ij}]\mathbb{1}(y_i>0)$.

   (b) If $z_{ji}>0$, $MRP_{ij}=-[((\frac{\gamma_i-2y_i}{\alpha_i})^+-(\frac{\gamma_j-2y_j}{\alpha_j})^+-k_{ji})]\mathbb{1}(y_i>0)$.

3. Choose the pair of the resources, say i^*, j^*, with the largest marginal reallocation profit.

4. If $MRP_{i^* j^*} \leq \varepsilon$, compute the output as follows:

$$
\begin{aligned}
\widehat{z}_{ij}^{*} &= z_{ij},\ i,j = 1,2,...,n \\
\widehat{p}_i^{*} &= \frac{\gamma_i}{2\alpha_i} + \left(\frac{\gamma_i - 2y_i}{2\alpha_i}\right)^{+},\ i = 1,2,...,n. \\
\widehat{T}^{*} &= \sum_{i=1}^{n} \widehat{p}_i^{*}(\gamma_i - \alpha_i \widehat{p}_i^{*}) + \sum_{i=1}^{n}\sum_{j\neq i} k_{ij}\widehat{z}_{ij}^{*}
\end{aligned}
$$

Return the output and stop the algorithm. Otherwise, go to the next step.

5. Reallocate from resource i^* to resource j^* based on the following:

Let $\Delta_{i^* j^*}$ denote the adjustment of the reallocation amount from i^* to j^*.

(a) If $z_{j^* i^*} = 0$.

i. If $\frac{\gamma_{j^*} - 2y_{i^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}}{\alpha_{i^*}} - k_{i^* j^*} \geq 0$, $\Delta_{i^* j^*} = y_{i^*}$.

ii. Else If $\gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} - \alpha_{j^*} k_{i^* j^*} \geq 0$,

$$\Delta_{i^* j^*} = \frac{\alpha_{i^*}\alpha_{j^*}}{2(\alpha_{i^*} + \alpha_{j^*})}\left(\frac{\gamma_{j^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*} - 2y_{i^*}}{\alpha_{i^*}} - k_{i^* j^*}\right)$$

iii. Else, $\Delta_{i^* j^*} = \frac{\gamma_{j^*} - 2y_{j^*} - \alpha_{j^*} k_{i^* j^*}}{2}$

(b) If $z_{j^* i^*} > 0$.

i. If $\left(\left(\frac{\gamma_{i^*} - 2y_{i^*} + 2\min(y_{i^*}, z_{j^* i^*})}{\alpha_{i^*}}\right)^{+} - \left(\frac{\gamma_{j^*} - 2y_{j^*} - 2\min(y_{i^*}, z_{j^* i^*})}{\alpha_{j^*}}\right)^{+} - k_{j^* i^*}\right) < 0$, $\Delta_{i^* j^*} = \min(y_{i^*}, z_{j^* i^*})$.

ii. Else

A. If $\frac{\gamma_{i^*}}{\alpha_{i^*}} - \frac{\gamma_{j^*} - 2y_{j^*} - 2y_{i^*}}{\alpha_{j^*}} - k_{j^* i^*} < 0$, $\Delta_{i^* j^*} = y_{i^*}$.

B. Else if $\gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} - \alpha_{i^*} k_{j^* i^*} \geq 0$,

$$\Delta_{i^* j^*} = \frac{\alpha_{i^*}\alpha_{j^*}}{2(\alpha_{i^*} + \alpha_{j^*})}\left(\frac{\gamma_{j^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*} - 2y_{i^*}}{\alpha_{i^*}} + k_{j^* i^*}\right).$$

C. Else, $\Delta_{i^* j^*} = -\frac{\gamma_{i^*} - 2y_{i^*} - \alpha_{i^*} k_{j^* i^*}}{2}$.

Update y_{i^*}, y_{j^*} and $z_{i^* j^*}$ with $y_{i^*} - \Delta_{i^* j^*}$, $y_{j^*} + \Delta_{i^* j^*}$ and $z_{i^* j^*} + \Delta_{i^* j^*}$ respectively, and go to step 2.

All the steps of the MRP algorithm 2 are the same as the MRP algorithm 1 except for step 5. In step 5, MRP algorithm 2 uses the solution of the 2-resource problem studied in chapter 2 to determine the reallocation from resource i^* to resource j^* determined in step 3 of the algorithm. The idea is to make the maximum amount of reallocation from resource i^* to resource j^* based on the analysis in Proposition 3. The decisions made in step 5 of the algorithm can be explained as follows:

(a) When $z_{j^*i^*} = 0$, there is a forward reallocation profit from i^* to j^*. Based on the results of the two-resource system (see Figure 3.2), no reallocation can generate profit in regions Ω_0, Ω_1, Ω_2 and Ω_3. Therefore, we just need to examine the three possible regions Ω_4, Ω_5 and Ω_6 (or Ω_7, Ω_8 and Ω_9) because we know that there is reallocation profit from i^* to j^*. Based on Proposition 3,

i If
$$\frac{\gamma_{j^*} - 2y_{i^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}}{\alpha_{i^*}} - k_{i^*j^*} \geq 0,$$
resources i^* and j^* fit in region Ω_6. As a supplier, resource i^* sends all available capacity y_{i^*} to resource j^*. After this reallocation, no further reallocation can be made from i^* to j^*.

ii If
$$\frac{\gamma_{j^*} - 2y_{i^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}}{\alpha_{i^*}} - k_{i^*j^*} < 0, \;\; \gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} - \alpha_{j^*}k_{i^*j^*} \geq 0,$$
resources i^* and j^* fit in region Ω_5. After reallocating
$$\frac{\alpha_{i^*}\alpha_{j^*}}{2(\alpha_{i^*} + \alpha_{j^*})}\left(\frac{\gamma_{j^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*} - 2y_{i^*}}{\alpha_{i^*}} - k_{i^*j^*}\right)$$
from i^* to j^*, no profit can be generated by further reallocation between i^* to j^*.

iii Otherwise, resource i^* and j^* fit in region Ω_4. After reallocating $\frac{\gamma_{j^*} - 2y_{j^*} - \alpha_{j^*}k_{i^*j^*}}{2}$ from i^* to j^*, no profit can be generated by further reallocation between i^* to j^*.

(b) When $z_{j^*i^*} > 0$, there is a positive backward reallocation profit from i^* to j^* because $MRP_{i^*j^*} > \varepsilon > 0$. Recall that we define the backward reallocation follows: If there is reallocation from j^* to i^*, i.e., $z_{j^*i^*} > 0$, and resource i^* is available, sending back one unit capacity from i^* to j^*, i.e., $y_{i^*} = y_{i^*} - 1,\ y_{j^*} = y_{j^*} + 1$ and $z_{j^*i^*} = z_{j^*i^*} - 1$, will generate profit $(\frac{\gamma_{j^*} - 2y_{j^*}}{\alpha_{j^*}})^+ - (\frac{\gamma_{i^*} - 2y_{i^*}}{\alpha_{i^*}})^+ + k_{j^*i^*}$, and we call this case "backward" reallocation and the generated profit as "backward reallocation profit". Since the reallocation amount from j^* to i^* is $z_{j^*i^*}$ and the available capacity of i^* is y_{i^*}, at most $\min\{z_{j^*i^*}, y_{i^*}\}$ can be send back from i^* to j^*.

i If after the reallocation with amount of $\min\{z_{j^*i^*}, y_{i^*}\}$ from i^* to j^* $MRP_{i^*j^*}$ is still positive, i.e.,

$$-((\frac{\gamma_{i^*} - 2y_{i^*} + 2\min(y_{i^*}, z_{j^*i^*})}{\alpha_{i^*}})^+ - (\frac{\gamma_{j^*} - 2y_{j^*} - 2\min(y_{i^*}, z_{j^*i^*})}{\alpha_{j^*}})^+ - k_{j^*i^*}) > 0,$$

we send $\min\{z_{j^*i^*}, y_{i^*}\}$ from i^* to j^* as the adjustment.

ii Otherwise,

A. If $\frac{\gamma_{i^*}}{\alpha_{i^*}} - \frac{\gamma_{j^*} - 2y_{j^*} - 2y_{i^*}}{\alpha_{j^*}} - k_{j^*i^*} < 0$, after reallocating y_{i^*} from i^* to j^*, the backward reallocation profit becomes $-(\frac{\gamma_{i^*}}{\alpha_{i^*}} - \frac{\gamma_{j^*} - 2y_{j^*} - 2y_{i^*}}{\alpha_{j^*}} - k_{j^*i^*})$ which is still positive but resource i^* has 0 capacity. Therefore, in this case we send y_{i^*} from i^* to j^* as the adjustment.

B. Else if $\gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} - \alpha_{i^*}k_{j^*i^*} \geq 0$, after reallocating

$$\Delta_{i^*j^*} = \frac{\alpha_{i^*}\alpha_{j^*}}{2(\alpha_{i^*} + \alpha_{j^*})}(\frac{\gamma_{j^*} - 2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*} - 2y_{i^*}}{\alpha_{i^*}} + k_{j^*i^*})$$

from i^* to j^*, the backward reallocation profit becomes

$$
\begin{aligned}
& -\left(\frac{\gamma_{i^*} - 2y_{i^*} + \frac{\alpha_{i^*}\alpha_{j^*}}{(\alpha_{i^*}+\alpha_{j^*})}\left(\frac{\gamma_{j^*}-2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}-2y_{i^*}}{\alpha_{i^*}} + k_{j^*i^*}\right)}{\alpha_{i^*}}\right)^+ \\
+ & \left(\frac{\gamma_{j^*} - 2y_{j^*} - \frac{\alpha_{i^*}\alpha_{j^*}}{(\alpha_{i^*}+\alpha_{j^*})}\left(\frac{\gamma_{j^*}-2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}-2y_{i^*}}{\alpha_{i^*}} + k_{j^*i^*}\right)}{\alpha_{j^*}}\right)^+ + k_{j^*i^*} \\
= & \frac{-1}{(\alpha_{i^*} + \alpha_{j^*})}(\gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} + \alpha_{j^*}k_{j^*i^*})^+ \\
+ & \frac{1}{(\alpha_{i^*} + \alpha_{j^*})}(\gamma_{i^*} + \gamma_{j^*} - 2y_{i^*} - 2y_{j^*} - \alpha_{i^*}k_{j^*i^*})^+ + k_{j^*i^*} \\
= & 0.
\end{aligned}
$$

Therefore we send $\frac{\alpha_{i^*}\alpha_{j^*}}{2(\alpha_{i^*}+\alpha_{j^*})}\left(\frac{\gamma_{j^*}-2y_{j^*}}{\alpha_{j^*}} - \frac{\gamma_{i^*}-2y_{i^*}}{\alpha_{i^*}} + k_{j^*i^*}\right)$ from resource i^* to resource j^* to eliminate the backward reallocation profit.

C. Else, after reallocating

$$
\Delta_{i^*j^*} = -\frac{\gamma_{i^*} - 2y_{i^*} - \alpha_{i^*}k_{j^*i^*}}{2}
$$

from i^* to j^*, the backward reallocation profit becomes

$$
\begin{aligned}
& -\left(\frac{\gamma_{i^*} - 2y_{i^*} - (\gamma_{i^*} - 2y_{i^*} - \alpha_{i^*}k_{j^*i^*})}{\alpha_{i^*}}\right)^+ \\
+ & \left(\frac{\gamma_{j^*} - 2y_{j^*} + (\gamma_{i^*} - 2y_{i^*} - \alpha_{i^*}k_{j^*i^*})}{\alpha_{j^*}}\right)^+ + k_{j^*i^*} \\
= & 0.
\end{aligned}
$$

Proposition 18. MRP algorithm 2 will stop after a finite number of iterations, and the output of MRP algorithm 2 is a feasible solution of P_2.

Proof: In each iteration of the MRP algorithm 2, the capacity of the supplier and the reallocation quantity between the supplier and consumer are kept nonnegative. Therefore, $y_i \geq 0, \forall i$ and $z_{ij} \geq 0,\ \forall i,j,\ \ i \neq j$. Based on the definition of the $\widehat{p_i^*}$, $\widehat{z_{ij}}^*$, we obtain a feasible solution at each iteration. The optimal objective value of P_2 is finite. If the algorithm will never stop, $MRP^t_{i_t^* j_t^*} > \varepsilon,\ \ t = 1,2,....$ In each iteration, the increase of the objective function value is the integral of the marginal reallocation profit between resource i_t^* and resource j_t^*, and is positive. If the algorithm does not stop, the objective value of P_2 will go to infinity which is a contradiction. Therefore, MRP algorithm 2 will stop after a finite number of iterations. ■

4.2.4 Discussion on the Complexity of the Heuristics

1. MRP algorithm 1 requires $O(n^2)$ computations to find the pair of resources which have the largest marginal reallocation profit in each iteration. There are $O(n)$ arcs will be involved in the computation and it takes at most $O(M)$ steps to decrease the reallocation profit associated with each arc to 0, where $M = MRP^1_{i^*j^*}$ is the maximum marginal reallocation profit before the 1^{st} reallocation. Therefore, the number of iterations is $O(nM)$. The complexity of algorithm 1 is $O(n^3M)$

2. MRP algorithm 2 is similar as algorithm 1 except that it takes $O(\ln M)$ steps to decrease the reallocation profit associated with each arc to 0 because the step length of each reallocation is proportional to M. Therefore, the number of iterations of algorithm 2 is $O(n \ln M)$. The complexity of algorithm 2 is $O(n^3 \ln M)$. Therefore, it is much faster algorithm than algorithm 1.

Further investigation about the performance of the algorithms will be presented in Section 3.5.

4.2.5 I-MRP Algorithm

In this section, we propose an extension for the MRP algorithms. This extension may further improve the accuracy of the MRP algorithms. Therefore, we call it as I-MRP (i.e., "I" stands for "improvement").

Before we introduce the I-MRP algorithm, let us give the following definitions: We define a set of resources as S_{zero} if $y_i = 0$ for all $i \in S_{zero}$ when the MRP algorithms stop, and all the resources in set S_{zero} are connected to each other directly or indirectly through other resources in the set. We call S_{zero} a zero-capacity set. If $\exists j \notin S_{zero}$ and j is directly connected to one of the resources in S_{zero}, we call j a leaf of S_{zero}. As shown in Figure

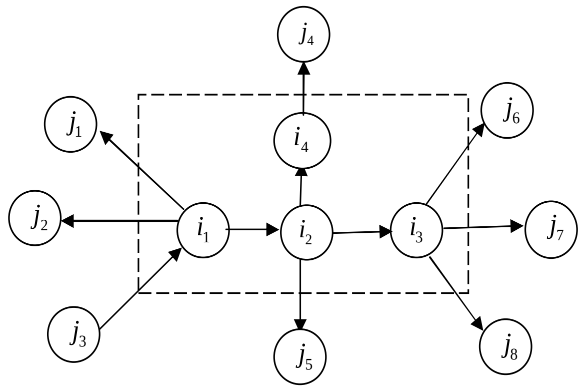

Figure 4.4: Zero-capacity set and its leaves

4.4, the resources in the dashed rectangle, $i_1,\ i_2,\ i_3$ and i_4 form a zero-capacity set S_{zero} and $y_{i_m} = 0,\ m = 1,2,...,4.$, and $\{j_m | m = 1,2,...,8\}$ is the set of the leaves of S_{zero} and $y_{j_m} > 0,\ m = 1,2,...,8.$

MRP algorithms 1 and 2 will stop when there is no reallocation profit between any pair of resources or the remaining capacity of the supplier is zero. However, we may still obtain an increase in the profit by conducting reallocation between the resources through the resources which have zero remaining capacity after the algorithm stops. For example, when MRP algorithms stop, suppose $y_i = 0$ and resource i is the supplier of resource j and l, such that

$$\widehat{\lambda}_j^* - \widehat{\lambda}_i^* - k_{ij} > \widehat{\lambda}_l^* - \widehat{\lambda}_i^* - k_{il} > 0.$$

The MRP algorithms stop because $y_i = 0$. Hence,

$$MRP_{ij} = (\widehat{\lambda}_j^* - \widehat{\lambda}_i^* - k_{ij})\mathbb{1}(y_i > 0) = 0$$

$$MRP_{il} = (\widehat{\lambda}_l^* - \widehat{\lambda}_i^* - k_{il})\mathbb{1}(y_i > 0) = 0.$$

On the other hand, the marginal reallocation profit from l to j through i is $\widehat{\lambda}_j^* - \widehat{\lambda}_i^* - k_{ij} - (\widehat{\lambda}_l^* - \widehat{\lambda}_i^* - k_{il})$. Therefore, the performance of the algorithms can be improved by conducting the reallocation. I-MRP algorithm aims to identify the zero-capacity sets and extract the profit associated with them. We state the I-MRP algorithm as follows:

1. Run MRP algorithm 1 (2) until it stops. Let ε denote the same small number used in MRP algorithm 1 (2).

2. Starting from the output of the MRP algorithm 1 (2), find all the zero-capacity sets, denoted by S^i_{zero}, $i = 1,2,...,m$. Let flag:= 0. If $m < 1$, go to step 4. Otherwise, let $j := 1$.

3. Choose zero-capacity set S^j_{zero}, and find the leaves of S^j_{zero}.

   (a) i. If the total number of the leaves of S^j_{zero} is less than 2, let $j := j+1$.

      A. If $j = m+1$, go to step 4.

      B. Otherwise go to step 3.

   ii. Otherwise, choose a pair of the leaves of S^j_{zero}.

   (b) Suppose the chosen pair of resources are g and l. Identify S^{arc}_{gl} which is the set of arcs that connects resources g and l.

   i. A. If $(\frac{\gamma_l-2y_l}{\alpha_l})^+ - (\frac{\gamma_g-2y_g}{\alpha_g})^+ - \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh} > \varepsilon$, there is positive reallocation profit from g to l through S^{arc}_{gl}. Let $z^{min}_{gl} = \min\{z_{fh}|(f,h) \in S^{arc}_{gl},\ v_{fh} = -1\}$ and $\bar{\delta} = \min\{y_g,\ z^{min}_{gl}\}$.

   - If $(\frac{\gamma_l-2y_l-2\bar{\delta}}{\alpha_l})^+ - (\frac{\gamma_g-2y_g+2\bar{\delta}}{\alpha_g})^+ - \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh} \geq 0$, $\Delta_{gl} = \bar{\delta}$.
   - If $(\frac{\gamma_l-2y_l-2\bar{\delta}}{\alpha_l})^+ - (\frac{\gamma_g-2y_g+2\bar{\delta}}{\alpha_g})^+ - \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh} < 0$,

     □ If $(\frac{\gamma_l-2y_l-2y_g}{\alpha_l})^+ - \frac{\gamma_g}{\alpha_g} - \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh} \geq 0$, $\Delta_{gl} = y_g$.

     □ Else If $\gamma_l + \gamma_g - 2y_l - 2y_g - \alpha_l \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh} > 0$,
     $\Delta_{gl} = \frac{\alpha_g\alpha_l}{2(\alpha_g+\alpha_l)}((\frac{\gamma_l-2y_l}{\alpha_l})^+ - (\frac{\gamma_g-2y_g}{\alpha_g})^+ - \sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh})$.

     □ Else $\Delta_{gl} = \frac{\gamma_l-2y_l-\alpha_l\sum_{(f,h)\in S^{arc}_{gl}} v_{fh}k_{fh}}{2}$

   Update y_g and y_l with $y_g - \Delta_{gl}$ and $y_l + \Delta_{gl}$, respectively. For each $(f,h) \in S^{arc}_{gl}$, if $v_{fh} = 1$, update z_{fh} with $z_{fh} + \Delta_{gl}$. If $v_{fh} = -1$, update z_{fh} with $z_{fh} - \Delta_{gl}$. Let flag= 1 and go to step 3(b)(iii).

   B. If $(\frac{\gamma_g-2y_g}{\alpha_g})^+ - (\frac{\gamma_l-2y_l}{\alpha_l})^+ - \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh} > \varepsilon$, there is positive reallocation profit from l to g through S^{arc}_{lg}. Let $z^{min}_{lg} = \min\{z_{fh}|(f,h) \in S^{arc}_{lg},\ v_{fh} = -1\}$ and $\bar{\delta} = \min\{y_l,\ z^{min}_{lg}\}$.

- If $(\frac{\gamma_g-2y_g-2\bar{\delta}}{\alpha_g})^+ - (\frac{\gamma_l-2y_l+2\bar{\delta}}{\alpha_l})^+ - \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh} \geq 0$, $\Delta_{lg} = \bar{\delta}$.
- If $(\frac{\gamma_g-2y_g-2\bar{\delta}}{\alpha_g})^+ - (\frac{\gamma_l-2y_l+2\bar{\delta}}{\alpha_l})^+ - \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh} < 0$,

  □ If $(\frac{\gamma_g-2y_g-2y_l}{\alpha_g})^+ - \frac{\gamma_l}{\alpha_l} - \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh} \geq 0$, $\Delta_{lg} = y_l$.

  □ Else If $\gamma_l + \gamma_g - 2y_l - 2y_g - \alpha_g \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh} > 0$,

  $\Delta_{lg} = \frac{\alpha_g\alpha_l}{2(\alpha_g+\alpha_l)}((\frac{\gamma_g-2y_g}{\alpha_g})^+ - (\frac{\gamma_l-2y_l}{\alpha_l})^+ - \sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh})$.

  □ Else $\Delta_{lg} = \frac{\gamma_g-2y_g-\alpha_g\sum_{(f,h)\in S^{arc}_{lg}} v_{fh}k_{fh}}{2}$

  Update y_g and y_l with $y_g + \Delta_{gl}$ and $y_l - \Delta_{gl}$, respectively. For each $(f,h) \in S^{arc}_{lg}$, if $v_{fh} = 1$, update z_{fh} with $z_{fh} + \Delta_{lg}$. If $v_{fh} = -1$, update z_{fh} with $z_{fh} - \Delta_{lg}$. Let flag= 1 and go to step 3(b)(iii).

ii. If no reallocation profit exists between g and l. Go to step 3(b)(iii).

iii. Choose another pair of the leaves of S^j_{zero} and go to step 3(b). If all the pairs of leaves of S^j_{zero} have been visited, let $j := j+1$. If $j = m+1$, stop. Otherwise go to step 3.

4. If flag= 1, go to step 1 (Note that the first step of MRP algorithm 1 (2) needs to be skipped when we revisit the step 1). Otherwise, stop.

The idea the I-MRP algorithm is to explore the profit associated with the zero-capacity sets which may not be captured by the MRP 1 (2) algorithm. In the algorithm, "flag" is the variable to flag the improvement. If there happens any reallocation related to the zero-capacity sets, "flag" will be set as 1 and the MRP 1 (2) needs to be revisited because the resource capacities have been changed so that the MRP 1(2) algorithm may generate more profit. The step 3(b)(i) is similar as the step 5 of MRP algorithm 2 which is explained in detail. The only difference is that in step 3(b)(i), the supplier and the consumer are connected by multiple arcs (no loop). Therefore, we need to consider updating each arc in the set. By similar argument presented in the proof of Proposition 18, the convergence of I-MRP algorithm is guaranteed.

4.2.6 Comparison of Heuristic Algorithms with Interior Point Methods

Interior point methods represent a significant development in the theory and practice of linear and nonlinear programming. The idea of interior point method is to find an optimal solution by moving in the interior of the feasible set. In each step of an implementation of the interior point method, the algorithm solves a system of multiple linear equations to obtain the Newton direction which is the most computationally intensive step of an interior point method. The algorithm stops when the duality gap (the difference in objective values between the primal solution and the dual solution) is less than a given positive small value ε. Then, a near-optimal ($\varepsilon-$optimal) solution is obtained. MRP and I-MRP algorithms apply a similar idea. The advantage of MRP and I-MRP is that they provide explicit and simple ways to obtain the moving direction of the solution. There are n^2+n decision variables in the model where n is the total number of resources. Based on the special structure of the model, MRP algorithms 1 and 2 find an appropriate moving direction within $O(n^2)$ operations, where the interior point method needs to solve a linear system with n^2+n equations and n^2+n unknown variables. Therefore, MRP algorithms (used in conjunction with I-MRP) are much more efficient. On the other hand, interior point methods can always obtain an near-optimal solution with higher computational complexity.

4.3 Optimal Solution of the Stage 1 Model

The stage 1 investment problem is a stochastic, nonlinear optimization problem. In this section, we investigate the structure of the optimal solution to the stage 1 problem. $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is the optimal objective function value of the operational stage problem (P_2). Its property directly affects the investment decision.

Lemma 2. $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ *is a continuous and differentiable function with respect to* $\overrightarrow{x}$.

Proof: The demand space is divided into multiple disjoint regions based on the forms of the optimal solution. According to Proposition 11, inside each region, $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is continuous, and the partial derivative of $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ with respect to $x_i, i = 1, 2, ..., n$, i.e., the shadow price of resource i exists and it is a continuous function. At the boundary of two adjacent regions, since the concavity of P_2, there exists an unique optimal shadow price of resource i. Therefore, $\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})$ is a continuous and differentiable function with respect to $\overrightarrow{x}$. ■

Theorem 2. *The stage 1 problem is jointly concave with respect to* $\overrightarrow{x}$.

Proof: Observe that function $\Phi(\overrightarrow{x}, \overrightarrow{\gamma})$ is jointly concave in $\overrightarrow{\gamma}$ and $\overrightarrow{x}$ because $\overrightarrow{\gamma}$ and $\overrightarrow{x}$ determine a quadratic program with concave objective function and linear constraints. The expectation of a concave function, e.g., $E[\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma})]$ is concave in $\overrightarrow{x}$ and hence $\Pi(\overrightarrow{x})$ is concave as it is sum of concave and linear functions. ■

According to the concavity of $\Pi(\overrightarrow{x})$ and the K-K-T condition of P_1, the shadow price vector λ^* of the optimal capacity investment $\overrightarrow{x}$ satisfies, $\forall i = 1, ..., n$

$$\sum_j E(\lambda_i^* | \Omega_j) P(\Omega_j) = c_i - v_i \tag{4.39}$$

$$x_i v_i = 0.$$

If there is a positive investment in resource i, the expected shadow price of resource i should equal to its unit resource cost c_i when the investment strategy is optimal. On the other hand, if x_i decreases to 0, $E(\lambda_i^*)$ reaches the maximum and if the maximum value is less than c_i then $v_i > 0$ indicating that it is not optimal to invest in resource i.

Proposition 19. When $x_i^* > 0$, $E(D_i) = E(\Gamma_i - \alpha_i p_i^*) \geq \frac{E(\Gamma_i)}{2} - \frac{c_i \alpha_i}{2}$

Proof:

$E(D_i) = E(\Gamma_i - \alpha_i p_i^*)$ is the expected demand satisfied from market segment i when the optimal price is p_i^*. Based on (4.6) and (4.39), when $x_i^* > 0$

$$E(D_i) = \frac{E(\Gamma_i)}{2} - \frac{c_i \alpha_i}{2} + \frac{\alpha_i E(\beta_i^*)}{2} \geq \frac{E(\Gamma_i)}{2} - \frac{c_i \alpha_i}{2} \quad ■$$

$\frac{E(\Gamma_i)}{2} - \frac{c_i \alpha_i}{2}$ is the expected amount of satisfied type i demand if there is no substitution allowed in the model. So, we can expect a higher demand satisfaction with the substitution.

Let Π^* and Φ^* denote the optimal objective values of P_1 and P_2, respectively. We have

Proposition 20. (Sensitivity analysis for P_1 and P_2) For all i=1,2,...,n,

1. Φ^* decreases in α_i.

2. Π^* decreases in α_i and c_i.

3. If $x_i^* > 0$ for all $i = 1,2,...,n$, x_i^* decreases in c_i.

Remark: α_i is the slope of the demand function for market segment i. As α_i increases, the optimal objective function value of P_2 decreases. As a result, the optimal objective of P_1 decreases as well. c_i is the unit cost of resource i. Intuitively, as c_i increases, the optimal objective function value of P_1 decreases and the optimal investment level in resource i decreases.

Proof:

1. In order to conduct the sensitivity analysis of P_2 in α_i, we consider the optimal solution and optimal objective function value as functions of α_i, $i = 1,2,...,n$. Let $p_l^*(\alpha_i)$, $z_{lj}^*(\alpha_i)$, $l, j = 1,2,...,n$ be the optimal solution of $P_2(\alpha_i)$. Let $y_l^*(\alpha_i) = x_l + \sum_{j \neq l} z_{jl}^*(\alpha_i) - \sum_{j \neq l} z_{lj}^*(\alpha_i)$ be the optimal total available capacity of resource l after reallocation. $\Phi^*(\alpha_i) = \sum_{j=1}^{n} p_j^*(\gamma_j - \alpha_j p_j^*) - \sum_l \sum_{j \neq l} k_{lj} z_{lj}^*$. is the optimal objective function value of $P_2(\alpha_i)$. Consider that α_i decreases to $\alpha_i - \delta$, where δ is a small positive real number. We will show $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$. Let us construct a feasible solution for $P_2(\alpha_i - \delta)$. Let $p_l^*(\alpha_i - \delta) = p_l^*(\alpha_i)$, $l \neq i$, $l \in \{1,2,...,n\}$, $z_{lj}^*(\alpha_i - \delta) = z_{lj}^*(\alpha_i), l, j = 1,2,...,n$. All decision variables of $P_2(\alpha_i - \delta)$ except $p_i^*(\alpha_i - \delta)$ are named values. Then we determine the value of $p_i^*(\alpha_i - \delta)$ as follows:

   Consider the optimal solution of $P_2(\alpha_i)$,

   (a) If constraint (4.4) is binding for resource i, i.e., $\gamma_i - \alpha_i p_i^*(\alpha_i) = 0$,

   let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i}{\alpha_i - \delta}$. As far as δ is small enough, it can be easily verified the constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates the same objective value as $P_2(\alpha_i)$. The optimal solution of $P_2(\alpha_i - \delta)$ is at least as large as $\Phi^*(\alpha_i)$. Therefore, $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq 0$.

(b) If both constraints (4.4) and (4.2) are not binding for resource i.

$p_i^*(\alpha_i) = \frac{\gamma_i}{2\alpha_i}$. Let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i}{2(\alpha_i-\delta)}$. The constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates a larger objective value than $P_2(\alpha_i)$. We have $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq \frac{\partial(\frac{\gamma_i^2}{4\alpha_i})}{\partial \alpha_i} = -\frac{\gamma_i^2}{4\alpha_i^2} < 0$.

(c) If constraint (4.4) is not binding, and constraint (4.2) is binding for resource i. Let $p_i^*(\alpha_i - \delta) = \frac{\gamma_i - y_i^*}{\alpha_i - \delta}$. The constructed solution for $P_2(\alpha_i - \delta)$ is feasible and generates a larger objective value than $P_2(\alpha_i)$. We have $\Phi^*(\alpha_i - \delta) \geq \Phi^*(\alpha_i)$ and $\frac{\partial \Phi^*}{\partial \alpha_i} \leq \frac{\partial(y_i^* \frac{\gamma_i - y_i^*}{\alpha_i})}{\partial \alpha_i} = -\frac{(\gamma_i - y_i^*)y_i^*}{\alpha_i^2} < 0$ (both y_i^* and $\gamma_i - y_i^*$ are positive).

Therefore $\Phi^*(\alpha_i)$ is decreases in α_i for all $i = 1, 2, ..., n$.

2. Since $\Phi^*(\alpha_i)$ is decreases in α_i, $\Pi^* = E[\Phi^*(\overrightarrow{x}^*, \overrightarrow{\Gamma})] - \sum_{i=1}^n c_i x_i^*$ decreases in α_i. $\frac{\partial \Pi^*}{\partial c_i} = -x_i^*$, Φ^* decreases in c_i with rate x_i^*.

3. When $x_i^* > 0,\ i = 1, 2, ..., n$, the first order condition

$\frac{\partial \Pi}{\partial x_i}|_{x_1 = x_1^*,\ x_2 = x_2^*, ...,\ x_n = x_n^*} = F_i(x_1^*, x_2^*, ..., x_n^*, c_1) = 0,\ i = 1, 2, ..., n$ implicitly define

$x_1^*,\ x_2^*, ...,\ x_n^*$ as a function of c_1.

$$\frac{\partial F_1}{\partial x_1^*}\frac{\partial x_1^*}{\partial c_1} + \frac{\partial F_1}{\partial x_2^*}\frac{\partial x_2^*}{\partial c_1} + ... + \frac{\partial F_1}{\partial x_n^*}\frac{\partial x_n^*}{\partial c_1} + \frac{\partial F_1}{\partial c_1} = 0 \tag{4.40}$$

$$\frac{\partial F_i}{\partial x_1^*}\frac{\partial x_1^*}{\partial c_1} + \frac{\partial F_i}{\partial x_2^*}\frac{\partial x_2^*}{\partial c_1} + ... + \frac{\partial F_i}{\partial x_n^*}\frac{\partial x_n^*}{\partial c_1} + \frac{\partial F_i}{\partial c_1} = 0 \quad i = 2, 3, ..., n \tag{4.41}$$

Let define the Hessian matrix of the objective function of stage 1 problem $\Pi(\overrightarrow{x}^*)$ as Q. Since $\frac{\partial F_1}{\partial c_1} = -1$ and $\frac{\partial F_i}{\partial c_1} = 0,\ i = 2, 3, ..., n$, by solving (4.40) and (4.41), we have,

$$\left[\frac{\partial x_1^*}{\partial c_1}, \frac{\partial x_2^*}{\partial c_1}, ..., , \frac{\partial x_n^*}{\partial c_1}\right] = [1, 0, ..., 0]Q^{-1}.$$

Due to the concavity of $\Pi(\overrightarrow{x}^*)$, $\frac{\partial x_1^*}{\partial c_1} < 0$. Similarly, we can show $\frac{\partial x_i^*}{\partial c_i} < 0$ for $i = 2, 3, ..., n$ ■ .

4.4 Numerical Experiments

In this section, we first investigate the performance of the heuristics in solving the stage 2 problem by comparing their performance to the optimal for a wide range of parameter settings. Second, we provide a solution procedure for the stage 1 problem based on Monte Carlo simulation. Finally, we investigate the impact of various system parameters such as the slope of the demand function, unit investment cost, mean and variance of the demand, and the demand correlation on the optimal objective function value and the solution of the stage 1 problem.

To carry on the numerical experiments, we assume that the market size for resource i, Γ_i, follows a normal distribution with mean μ_i and standard deviation $\sigma_i,\ i = 1,...,n$. The demands for different resources may be correlated. The correlation coefficient between market size i and j is $\rho_{ij},\ \forall i \neq j$.

4.4.1 Performance of the Heuristics

We first evaluate the performances of MRP_1, MRP_2, and I-MRP heuristics to solve the stage 2 problem only. In Section 4.2.3, MRP algorithm 1 and MRP algorithm 2 use a fixed small number ε, as the error tolerance, i.e., when the $MRP_{i^*j^*} < \varepsilon$, the algorithms stop. In our implementation of the algorithms, we use $\varepsilon = \frac{\text{current objective function value}}{M}$ as the stopping criterion, where M is a large positive number. We choose appropriate value M so that the results of the heuristics are accurate enough, and they are also efficient. Based on extensive numerical experiments, we found that $M = 10000n^2$ is a good choice. In order to evaluate the performances of the three heuristics, i.e., MRP_1, MRP_2 and I-MRP, to solve P_2, we considered a wide range of parameter settings as follows:

1) Set n=3, 6, 12.

2) Form the reallocation cost matrix by generating the reallocation costs based on $k_{ij} \sim$ Uniform (0,100) $i, j = 1,...,n,\ i \neq j$.

3) Form the capacity vector by generating each resource capacity based on $x_i \sim$ Uniform (0,C) where C=50,100.

4) Form the slope vector by generating each slope based on $\alpha_i \sim$ Uniform(0,S), S=1,10.

5) Form the mean demand vector by generating each mean demand based on $\mu_i \sim$ Uniform(0,M), M=50,500.

6) Form the standard deviation vector by generating each standard deviation $\sigma_i \sim$ Uniform(0,50).

7) Form the correlation coefficient matrix in three forms as follows:

   (a) Generate $\rho_{ij} \sim$ Uniform (0,1) (All demands are positively correlated).

   (b) Generate $\rho_{ij} \sim$ Uniform (-1,0) (All demands are negatively correlated).

   (c) Generate $\rho_{ij} \sim$ Uniform (0,1) (Demands can be negatively or positively correlated).

As a result, we used three variables for the number of resources (i.e., n=3, 6, 12), two sets of resource capacity levels (i.e., c=50, 100), two sets for the demand slope vector (i.e., S=1, 10), two sets for mean demand values (i.e., M=50, 500), and three sets for the demand correlation coefficient matrix, resulting in a total of 72 scenarios. For each of the 72 scenarios given above, we generated 100 data sets randomly, which resulted in 7,200 experiments. We evaluated the optimal profit and profits of the heuristics for all 7,200 cases. We used CPLEX to compute the optimal profit for the stage 2 problem, and used a code written in C++ to compute the profits for the heuristics. Tables 4.1 and 4.2 present the results. In Tables 4.1 and 4.2, the first five columns indicate the number of facilities, the upper bound for the slope of the demand functions, the upper bound for the resource capacities, the upper bound for the mean demands, and the form of the demand correlations, respectively.

In the fifth column of Tables 4.1 and 4.2, +, -, and +/- indicate that all demands are positively correlated, all demands are negatively correlated, and demands can be negatively or positively correlated, respectively. In Tables 4.1 and 4.2, column 6 to column 11 give the maximum and average percent errors of the MRP_1, MRP_2 and I-MRP heuristics from the optimal, respectively.

We observe that overall performance of the algorithms is good. Most of the time, the average percent errors of MRP_1 and MRP_2 are below 3% .The average percent errors of I-MRP are less than 1.2%. On the other hand, when the demand is much larger than the resource capacity, the performances of MRP_1 and MRP_2 may be bad. As shown in Table 4.2, when $n = 12,\ S = 10,\ C = 50,\ M = 500$, the demand is about ten times larger than the capacity, and the maximum percent error can be as large as 29%. As we discussed in Section 4.2.5, the appearance of "zero-capacity" sets can make the performances of the MRP_1 and MRP_2 algorithms bad. When the demand is much larger than the resource capacity, there is better chance that the optimal solution contains "zero-capacity" sets. I-MRP algorithm addresses this issue and can improve the performance. As shown in the Table 4.2, for the case with $n = 12,\ S = 10,\ C = 50$ and $M = 500$, the maximum percent error drops to 5.59% from 29% when I-MRP is used, and the average percent error drops from 6.71% to 1.16%.

					MRP_1		MRP_2		I-MRP	
n	S	C	M	cor	e_{max}	e_{ave}	e_{max}	e_{ave}	e_{max}	e_{ave}
3	1	50	50	+	2.12%	0.05%	0.00%	0.00%	0.00%	0.00%
3	1	50	50	+/-	0.67%	0.01%	0.66%	0.00%	0.60%	0.00%
3	1	50	50	-	19.2%	0.30%	18.9%	0.28%	0.00%	0.00%
3	1	100	50	+	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	1	100	50	+/-	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	1	100	50	-	1.23%	0.01%	1.14%	0.01%	0.00%	0.00%
3	10	50	50	+	0.06%	0.00%	0.00%	0.00%	0.00%	0.00%
3	10	50	50	+/-	0.04%	0.00%	0.00%	0.00%	0.00%	0.00%
3	10	50	50	-	0.25%	0.00%	0.00%	0.00%	0.00%	0.00%
3	10	100	50	+	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	10	100	50	+/-	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
3	10	100	50	-	0.09%	0.00%	0.00%	0.00%	0.00%	0.00%
3	1	50	500	+	8.52%	0.39%	8.44%	0.43%	3.59%	0.09%
3	1	50	500	+/-	8.45%	0.39%	8.40%	0.33%	2.68%	0.06%
3	1	50	500	-	7.69%	0.38%	7.17%	0.40%	4.61%	0.14%
3	1	100	500	+	10.2%	0.37%	10.1%	0.23%	1.33%	0.03%
3	1	100	500	+/-	9.30%	0.37%	9.94%	0.24%	1.48%	0.04%
3	1	100	500	-	8.36%	0.38%	8.31%	0.26%	2.36%	0.07%
3	10	50	500	+	16.9%	0.50%	16.9%	0.50%	0.63%	0.00%
3	10	50	500	+/-	17.3%	0.5%	17.2%	0.50%	0.58%	0.00%
3	10	50	500	-	19.2%	0.6%	19.1%	0.6%	0.65%	0.01%
3	10	100	500	+	18.4%	0.39%	18.4%	0.36%	0.48%	0.00%
3	10	100	500	+/-	18.4%	0.38%	18.4%	0.34%	0.51%	0.01%
3	10	100	500	-	19.3%	0.45%	19.5%	0.44%	0.53%	0.01%
6	1	50	50	+	4.81%	0.11%	3.27%	0.00%	0.32%	0.00%
6	1	50	50	+/-	8.80%	0.13%	9.07%	0.15%	1.69%	0.02%
6	1	50	50	-	4.92%	0.07%	4.90%	0.07%	0.74%	0.00%
6	1	100	50	+	0.7%	0.01%	0.00%	0.00%	0.00%	0.00%
6	1	100	50	+/-	3.24%	0.04%	1.84%	0.02%	0.00%	0.00%
6	1	100	50	-	0.39%	0.00%	0.38%	0.00%	0.00%	0.00%
6	10	50	50	+	0.07%	0.00%	0.00%	0.00%	0.00%	0.00%
6	10	50	50	+/-	14.0%	0.14%	14.0%	0.14%	0.00%	0.00%
6	10	50	50	-	3.89%	0.05%	3.76%	0.05%	0.00%	0.00%
6	10	100	50	+	0.02%	0.00%	0.00%	0.00%	0.00%	0.00%
6	10	100	50	+/-	1.18%	0.01%	1.16%	0.01%	0.00%	0.00%
6	10	100	50	-	5.37%	0.05%	5.29%	0.05%	0.00%	0.00%
6	1	50	500	+	9.29%	1.00%	9.23%	1.00%	1.73%	0.18%
6	1	50	500	+/-	7.89%	1.09%	5.77%	1.13%	2.33%	0.24%
6	1	50	500	-	10.5%	1.19%	10.5%	1.22%	2.16%	0.27%
6	1	100	500	+	7.44%	0.80%	7.43%	0.75%	2.09%	0.19%
6	1	100	500	+/-	5.94%	0.89%	5.69%	0.78%	4.69%	0.23%
6	1	100	500	-	6.49%	0.87%	6.47%	0.74%	1.73%	0.20%
6	10	50	500	+	16.0%	2.25%	16.4%	2.24%	5.03%	0.40%
6	10	50	500	+/-	16.8%	2.83%	16.7%	2.78%	9.28%	0.40%
6	10	50	500	-	23.3%	2.92%	23.3%	2.94%	9.29%	0.38%
6	10	100	500	+	8.51%	1.02%	7.99%	0.89%	3.62%	0.17%
6	10	100	500	+/-	12.1%	0.95%	11.1%	0.93%	3.59%	0.23%
6	10	100	500	-	16.7%	1.12%	16.5%	1.05%	7.04%	0.24%

Table 4.1: Performance of MRP algorithm 1, MRP algorithm 2 and I-MRP algorithm

					MRP_1		MRP_2		I-MRP	
n	S	C	M	cor	e_{max}	e_{ave}	e_{max}	e_{ave}	e_{max}	e_{ave}
12	1	50	50	+	5.68%	0.32%	5.66%	0.24%	1.90%	0.02%
12	1	50	50	+/-	5.51%	0.26%	5.14%	0.18%	0.04%	0.00%
12	1	50	50	-	3.35%	0.21%	3.33%	0.16%	0.00%	0.00%
12	1	100	50	+	4.39%	0.07%	4.38%	0.07%	0.14%	0.00%
12	1	100	50	+/-	2.50%	0.03%	1.87%	0.02%	0.04%	0.00%
12	1	100	50	-	1.80%	0.03%	0.44%	0.005%	0.00%	0.00%
12	10	50	50	+	19.2%	0.35%	19.1%	0.35%	0.29%	0.00%
12	10	50	50	+/-	9.59%	0.11%	9.54%	0.11%	2.79%	0.03%
12	10	50	50	-	13.2%	0.33%	6.64%	0.14%	0.00%	0.00%
12	10	100	50	+	12.5%	0.13%	12.4%	0.13%	0.00%	0.00%
12	10	100	50	+/-	0.11%	0.00%	0.11%	0.00%	0.00%	0.00%
12	10	100	50	-	0.11%	0.00%	0.11%	0.00%	0.00%	0.00%
12	1	50	500	+	6.68%	1.37%	6.19%	1.37%	2.43%	0.37%
12	1	50	500	+/-	5.80%	1.74%	5.76%	1.70%	1.91%	0.40%
12	1	50	500	-	7.34%	1.89%	8.09%	1.89%	2.53%	0.40%
12	1	100	500	+	4.29%	0.89%	4.4%	0.08%	1.95%	0.23%
12	1	100	500	+/-	3.68%	0.84%	4.06%	0.72%	1.07%	0.13%
12	1	100	500	-	4.36%	0.79%	4.23%	0.71%	1.09%	0.12%
12	10	50	500	+	27.0%	5.31%	27.0%	5.20%	5.70%	1.13%
12	10	50	500	+/-	28.2%	6.0%	28.2%	6.0%	5.75%	1.05%
12	10	50	500	-	29.1%	6.43%	29.1%	6.71%	5.59%	1.16%
12	10	100	500	+	12.5%	2.55%	11.1%	2.27%	3.31%	0.46%
12	10	100	500	+/-	12.3%	2.59%	12.2%	2.42%	5.82%	0.38%
12	10	100	500	-	12.3%	2.35%	11.5%	2.23%	5.82%	0.39%

Table 4.2: Continue: Performance of MRP algorithm 1, MRP algorithm 2 and I-MRP algorithm

Since the results presented in Tables 4.1 and 4.2 show that MRP_2 and I-MRP algorithms are consistently better than MRP_1, we conduct more experiments to further investigate the performances of MRP_2 and I-MRP algorithms with the following parameter settings:

1) Set n=4, n=10, n=16.

2) Form the reallocation cost matrix by generating the reallocation costs based on $k_{ij} \sim$ Uniform (0,200) $i, j = 1, ..., n,\ i \neq j$.

3) Form the capacity vector by generating each resource capacity as follows:

   $X_1)$ $x_i = \lfloor 20\%\mu_i \rfloor$ $i = 1, ..., n$.

   $X_2)$ $x_i = \lfloor 50\%\mu_i \rfloor$ $i = 1, ..., n$.

   $X_3)$ $x_i = \lfloor 20\%\mu_i \rfloor$ for $i = 1, ..., n/2$ and $x_i = \lfloor 50\%\mu_i \rfloor$ for $i = n/2+1, ..., n$.

4) Form the slope vector by generating each slope based on $\alpha_i \sim$ Uniform(0,S), S=1,10.

5) Form the mean demand vector by generating each mean demand based on $\mu_i \sim$ Uniform(100,M), M=500,1000.

6) Form the standard deviation vector as follows:

   $V_1)$ $\sigma_i = 10\%\mu_i$ $i = 1, ..., n$.

   $V_2)$ $\sigma_i = 30\%\mu_i$ $i = 1, ..., n$.

   $V_3)$ $\sigma_i = 10\%\mu_i$ for $i = 1, ..., n/2$ and $\sigma_i = 30\%\mu_i$ for $i = n/2+1, ..., n$.

7) Form the correlation coefficient matrix in three forms as follows:

   (a) Generate $\rho_{ij} \sim$ Uniform (0,1) (All demands are positively correlated).

   (b) Generate $\rho_{ij} \sim$ Uniform (-1,0) (All demands are negatively correlated).

   (c) Generate $\rho_{ij} \sim$ Uniform (-1,1) (Demands can be negatively or positively correlated).

Based on this experimental setup, we have a total of 576 different scenarios. We replicate each scenario with different random number seeds 2000 times, which results in a total of 1,152,000 experiments. In Tables 4.3 to 4.14, the first three columns present the values for the standard deviation of market size vector, resource capacity vector and the demand correlation matrix, respectively. Columns 4 to 6 present the total time required to run 2000 replication, average percent error from the optimal, and the maximum percent error from the optimal over 2000 replications for MRP_2, respectively. Columns 7 to 9 present the same quantities for I-MRP, respectively. Column 10 presents the time in seconds required to solve the stage 2 problem optimally by CPLEX for 2000 replications.

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	0	0.236%	5.493%	0	0.067%	3.194%	16
V_1	X_1	-	0	0.234%	5.499%	0	0.071%	3.186%	16
V_1	X_1	+/-	1	0.237%	5.516%	0	0.068%	3.195%	16
V_1	X_2	+	0	0.001%	0.154%	0	0.000%	0.016%	16
V_1	X_2	-	0	0.001%	0.154%	0	0.000%	0.020%	16
V_1	X_2	+/-	0	0.001%	0.129%	0	0.000%	0.020%	16
V_1	X_3	+	0	0.093%	7.704%	0	0.017%	2.984%	16
V_1	X_3	-	0	0.089%	7.796%	1	0.015%	3.031%	15
V_1	X_3	+/-	1	0.092%	7.743%	0	0.016%	3.011%	16
V_1	X_4	+	0	0.086%	6.293%	0	0.011%	1.377%	16
V_1	X_4	-	1	0.088%	6.532%	0	0.012%	1.374%	15
V_1	X_4	+/-	1	0.087%	6.306%	0	0.012%	1.381%	16
V_2	X_1	+	0	0.260%	13.251%	0	0.065%	5.033%	16
V_2	X_1	-	0	0.262%	14.043%	0	0.068%	4.727%	16
V_2	X_1	+/-	0	0.263%	13.746%	0	0.068%	5.346%	16
V_2	X_2	+	0	0.009%	2.834%	0	0.002%	0.663%	16
V_2	X_2	-	0	0.010%	2.924%	0	0.002%	0.292%	16
V_2	X_2	+/-	0	0.010%	2.875%	0	0.002%	0.517%	16
V_2	X_3	+	0	0.096%	8.325%	1	0.025%	4.160%	15
V_2	X_3	-	1	0.101%	8.325%	0	0.023%	4.160%	16
V_2	X_3	+/-	0	0.097%	8.325%	0	0.025%	4.160%	16
V_2	X_4	+	1	0.109%	11.432%	0	0.016%	1.220%	15
V_2	X_4	-	1	0.119%	12.340%	0	0.017%	1.713%	16
V_2	X_4	+/-	0	0.111%	11.941%	1	0.016%	1.194%	15
V_3	X_1	+	0	0.259%	12.384%	1	0.065%	3.172%	15
V_3	X_1	-	1	0.256%	13.106%	0	0.066%	3.186%	16
V_3	X_1	+/-	0	0.261%	12.835%	0	0.065%	3.251%	16
V_3	X_2	+	0	0.004%	0.776%	0	0.001%	0.066%	16
V_3	X_2	-	0	0.004%	0.838%	0	0.001%	0.174%	16
V_3	X_2	+/-	0	0.004%	0.762%	0	0.001%	0.083%	16
V_3	X_3	+	1	0.094%	9.591%	0	0.021%	2.769%	16
V_3	X_3	-	0	0.096%	9.354%	0	0.021%	2.812%	16
V_3	X_3	+/-	1	0.098%	9.448%	0	0.023%	3.038%	16
V_3	X_4	+	0	0.103%	9.833%	0	0.012%	1.345%	16
V_3	X_4	-	0	0.108%	10.701%	1	0.013%	1.281%	15
V_3	X_4	+/-	1	0.104%	10.373%	0	0.014%	1.327%	16

Table 4.3: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 4,\ \alpha = 1,\ M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	0	0.187%	4.854%	0	0.057%	1.617%	16
V_1	X_1	-	0	0.186%	4.852%	0	0.057%	1.597%	16
V_1	X_1	+/-	0	0.188%	4.854%	0	0.057%	1.608%	16
V_1	X_2	+	0	0.002%	0.125%	0	0.001%	0.032%	16
V_1	X_2	-	0	0.002%	0.130%	0	0.001%	0.056%	16
V_1	X_2	+/-	0	0.002%	0.123%	1	0.001%	0.228%	15
V_1	X_3	+	1	0.089%	4.357%	0	0.027%	1.636%	16
V_1	X_3	-	0	0.086%	4.392%	1	0.025%	1.656%	16
V_1	X_3	+/-	0	0.090%	4.373%	0	0.027%	1.643%	16
V_1	X_4	+	1	0.084%	5.123%	0	0.023%	1.356%	15
V_1	X_4	-	1	0.086%	5.184%	0	0.023%	1.351%	16
V_1	X_4	+/-	0	0.084%	5.116%	0	0.022%	1.359%	16
V_2	X_1	+	0	0.186%	8.607%	0	0.050%	2.587%	16
V_2	X_1	-	0	0.194%	8.770%	0	0.053%	2.433%	16
V_2	X_1	+/-	0	0.193%	8.786%	1	0.051%	2.744%	16
V_2	X_2	+	0	0.015%	1.742%	0	0.005%	0.376%	16
V_2	X_2	-	0	0.015%	1.339%	0	0.004%	0.277%	16
V_2	X_2	+/-	1	0.014%	1.880%	0	0.004%	0.277%	16
V_2	X_3	+	1	0.098%	5.720%	0	0.028%	1.993%	18
V_2	X_3	-	1	0.099%	5.713%	0	0.028%	1.994%	26
V_2	X_3	+/-	0	0.102%	5.717%	0	0.028%	2.138%	20
V_2	X_4	+	0	0.107%	7.508%	1	0.025%	1.134%	15
V_2	X_4	-	1	0.111%	7.762%	0	0.025%	1.141%	16
V_2	X_4	+/-	0	0.109%	7.702%	0	0.026%	1.053%	16
V_3	X_1	+	0	0.190%	8.085%	0	0.053%	1.632%	16
V_3	X_1	-	0	0.188%	8.214%	0	0.051%	1.609%	17
V_3	X_1	+/-	0	0.190%	8.188%	0	0.054%	1.806%	16
V_3	X_2	+	0	0.008%	0.907%	0	0.003%	0.288%	16
V_3	X_2	-	0	0.008%	0.924%	1	0.003%	0.207%	15
V_3	X_2	+/-	0	0.008%	0.910%	1	0.003%	0.212%	16
V_3	X_3	+	0	0.096%	5.806%	0	0.030%	2.351%	16
V_3	X_3	-	1	0.099%	5.792%	0	0.031%	2.370%	16
V_3	X_3	+/-	0	0.096%	5.806%	1	0.031%	2.490%	16
V_3	X_4	+	0	0.094%	6.969%	0	0.023%	0.995%	16
V_3	X_4	-	0	0.098%	7.135%	1	0.023%	0.973%	16
V_3	X_4	+/-	0	0.096%	7.093%	0	0.023%	1.251%	16

Table 4.4: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 4,\ \alpha = 1,\ M = 1000$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	0	0.164%	13.798%	1	0.019%	3.752%	16
V_1	X_1	-	0	0.168%	13.845%	0	0.021%	3.754%	16
V_1	X_1	+/-	0	0.163%	13.807%	0	0.020%	3.812%	16
V_1	X_2	+	0	0.000%	0.012%	0	0.000%	0.012%	16
V_1	X_2	-	0	0.000%	0.012%	0	0.000%	0.012%	16
V_1	X_2	+/-	1	0.000%	0.011%	0	0.000%	0.011%	16
V_1	X_3	+	0	0.062%	13.303%	0	0.005%	2.479%	16
V_1	X_3	-	0	0.062%	13.362%	0	0.005%	2.588%	16
V_1	X_3	+/-	0	0.062%	13.315%	0	0.006%	2.456%	16
V_1	X_4	+	1	0.054%	11.389%	0	0.003%	1.729%	16
V_1	X_4	-	0	0.052%	11.409%	0	0.003%	1.816%	16
V_1	X_4	+/-	0	0.053%	11.540%	0	0.003%	1.756%	17
V_2	X_1	+	0	0.235%	13.512%	0	0.031%	3.294%	16
V_2	X_1	-	0	0.237%	15.376%	0	0.030%	3.398%	16
V_2	X_1	+/-	1	0.238%	13.506%	0	0.031%	3.552%	16
V_2	X_2	+	0	0.003%	1.180%	0	0.000%	0.044%	16
V_2	X_2	-	0	0.003%	1.308%	0	0.000%	0.208%	16
V_2	X_2	+/-	0	0.003%	1.204%	0	0.000%	0.088%	17
V_2	X_3	+	0	0.078%	11.873%	0	0.006%	2.622%	16
V_2	X_3	-	0	0.081%	11.873%	0	0.005%	1.858%	16
V_2	X_3	+/-	1	0.077%	11.873%	0	0.005%	2.662%	16
V_2	X_4	+	0	0.086%	14.707%	0	0.004%	2.933%	16
V_2	X_4	-	0	0.089%	14.450%	0	0.006%	3.790%	16
V_2	X_4	+/-	0	0.088%	14.168%	1	0.004%	3.009%	16
V_3	X_1	+	0	0.203%	13.799%	0	0.024%	3.342%	16
V_3	X_1	-	0	0.206%	13.846%	0	0.024%	3.573%	16
V_3	X_1	+/-	0	0.203%	13.809%	0	0.024%	3.569%	16
V_3	X_2	+	0	0.001%	0.649%	0	0.000%	0.022%	16
V_3	X_2	-	0	0.001%	0.735%	0	0.000%	0.022%	16
V_3	X_2	+/-	0	0.001%	0.755%	1	0.000%	0.023%	15
V_3	X_3	+	1	0.064%	13.306%	0	0.006%	3.608%	16
V_3	X_3	-	0	0.064%	13.365%	0	0.006%	2.533%	16
V_3	X_3	+/-	0	0.064%	13.318%	0	0.006%	3.584%	16
V_3	X_4	+	1	0.086%	15.740%	0	0.002%	0.777%	16
V_3	X_4	-	0	0.087%	15.609%	0	0.002%	0.980%	16
V_3	X_4	+/-	0	0.087%	15.593%	0	0.001%	0.738%	17

Table 4.5: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 4$, $\alpha = 10$, $M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	0	0.201%	8.159%	0	0.039%	2.592%	16
V_1	X_1	-	0	0.198%	8.234%	0	0.038%	3.013%	16
V_1	X_1	+/-	0	0.200%	8.174%	1	0.039%	2.748%	16
V_1	X_2	+	0	0.000%	0.086%	0	0.000%	0.021%	16
V_1	X_2	-	0	0.000%	0.098%	0	0.000%	0.018%	16
V_1	X_2	+/-	0	0.000%	0.088%	0	0.000%	0.023%	17
V_1	X_3	+	0	0.078%	7.979%	0	0.006%	2.498%	16
V_1	X_3	-	0	0.078%	7.872%	0	0.006%	2.472%	16
V_1	X_3	+/-	1	0.078%	7.818%	0	0.006%	2.495%	16
V_1	X_4	+	0	0.078%	9.034%	0	0.004%	0.874%	16
V_1	X_4	-	0	0.075%	8.981%	0	0.004%	0.861%	16
V_1	X_4	+/-	0	0.078%	8.923%	0	0.004%	0.866%	16
V_2	X_1	+	1	0.277%	12.018%	0	0.052%	5.334%	16
V_2	X_1	-	0	0.276%	12.678%	0	0.048%	4.746%	16
V_2	X_1	+/-	0	0.273%	12.038%	0	0.053%	4.956%	16
V_2	X_2	+	0	0.005%	1.319%	0	0.001%	0.206%	16
V_2	X_2	-	0	0.006%	1.441%	0	0.001%	0.131%	17
V_2	X_2	+/-	0	0.006%	1.342%	0	0.001%	0.184%	16
V_2	X_3	+	0	0.102%	13.269%	0	0.009%	3.167%	16
V_2	X_3	-	0	0.105%	13.890%	1	0.010%	3.320%	16
V_2	X_3	+/-	0	0.104%	13.477%	0	0.010%	3.182%	16
V_2	X_4	+	0	0.099%	11.156%	0	0.010%	2.011%	16
V_2	X_4	-	1	0.102%	11.527%	0	0.010%	2.941%	16
V_2	X_4	+/-	0	0.107%	11.320%	0	0.011%	2.349%	16
V_3	X_1	+	0	0.237%	10.826%	0	0.051%	4.488%	16
V_3	X_1	-	1	0.238%	10.613%	0	0.043%	2.796%	16
V_3	X_1	+/-	0	0.244%	10.826%	0	0.051%	4.313%	16
V_3	X_2	+	0	0.003%	1.328%	0	0.000%	0.072%	17
V_3	X_2	-	0	0.003%	1.278%	0	0.000%	0.093%	16
V_3	X_2	+/-	0	0.003%	1.269%	0	0.000%	0.065%	16
V_3	X_3	+	1	0.087%	11.731%	0	0.008%	2.431%	16
V_3	X_3	-	0	0.087%	11.125%	0	0.008%	2.160%	16
V_3	X_3	+/-	0	0.086%	11.304%	0	0.008%	2.409%	16
V_3	X_4	+	0	0.095%	11.625%	0	0.007%	1.292%	16
V_3	X_4	-	0	0.095%	11.603%	0	0.007%	1.356%	17
V_3	X_4	+/-	0	0.099%	11.596%	0	0.007%	1.312%	16

Table 4.6: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 4$, $\alpha = 10$, $M = 1000$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	1	0.580%	3.102%	1	0.241%	1.801%	19
V_1	X_1	-	2	0.582%	3.125%	1	0.243%	1.770%	19
V_1	X_1	+/-	1	0.582%	3.151%	1	0.240%	1.812%	19
V_1	X_2	+	0	0.007%	0.231%	1	0.004%	0.072%	18
V_1	X_2	-	1	0.007%	0.239%	0	0.004%	0.069%	19
V_1	X_2	+/-	1	0.007%	0.236%	0	0.004%	0.069%	19
V_1	X_3	+	2	0.196%	2.665%	2	0.087%	0.984%	19
V_1	X_3	-	2	0.194%	2.687%	2	0.086%	0.978%	19
V_1	X_3	+/-	2	0.195%	2.658%	2	0.087%	0.981%	22
V_1	X_4	+	2	0.207%	2.359%	2	0.090%	1.462%	20
V_1	X_4	-	2	0.208%	2.421%	2	0.090%	1.527%	19
V_1	X_4	+/-	3	0.206%	2.367%	1	0.090%	1.458%	21
V_2	X_1	+	1	0.685%	5.521%	2	0.259%	1.764%	22
V_2	X_1	-	2	0.695%	5.150%	1	0.268%	2.255%	19
V_2	X_1	+/-	1	0.689%	5.725%	2	0.263%	2.357%	19
V_2	X_2	+	1	0.043%	1.154%	1	0.020%	0.323%	19
V_2	X_2	-	1	0.042%	1.185%	1	0.019%	0.287%	19
V_2	X_2	+/-	1	0.043%	1.162%	1	0.020%	0.357%	18
V_2	X_3	+	3	0.261%	5.747%	2	0.106%	2.054%	19
V_2	X_3	-	3	0.265%	5.859%	1	0.109%	1.821%	19
V_2	X_3	+/-	3	0.262%	5.814%	2	0.111%	2.173%	18
V_2	X_4	+	3	0.266%	3.268%	1	0.109%	1.461%	19
V_2	X_4	-	3	0.265%	3.344%	2	0.109%	1.742%	19
V_2	X_4	+/-	2	0.270%	3.726%	2	0.109%	1.141%	19
V_3	X_1	+	1	0.637%	4.738%	1	0.247%	1.955%	19
V_3	X_1	-	2	0.638%	4.484%	1	0.254%	1.949%	19
V_3	X_1	+/-	1	0.638%	3.741%	1	0.247%	1.963%	19
V_3	X_2	+	1	0.025%	0.784%	1	0.012%	0.174%	19
V_3	X_2	-	1	0.025%	0.403%	1	0.012%	0.165%	19
V_3	X_2	+/-	1	0.025%	0.381%	0	0.012%	0.164%	19
V_3	X_3	+	3	0.226%	5.993%	2	0.098%	1.045%	19
V_3	X_3	-	3	0.227%	5.052%	2	0.097%	1.097%	19
V_3	X_3	+/-	3	0.229%	6.038%	1	0.100%	1.083%	19
V_3	X_4	+	3	0.233%	2.809%	1	0.096%	1.149%	19
V_3	X_4	-	3	0.232%	2.825%	2	0.096%	1.072%	19
V_3	X_4	+/-	2	0.233%	2.849%	2	0.096%	1.149%	19

Table 4.7: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 10,\ \alpha = 1,\ M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	1	0.485%	2.574%	0	0.196%	1.294%	19
V_1	X_1	-	1	0.490%	2.610%	1	0.200%	1.367%	19
V_1	X_1	+/-	1	0.489%	2.583%	0	0.198%	1.319%	19
V_1	X_2	+	1	0.020%	0.261%	0	0.015%	0.128%	19
V_1	X_2	-	0	0.021%	0.283%	1	0.015%	0.125%	19
V_1	X_2	+/-	0	0.021%	0.284%	1	0.015%	0.126%	19
V_1	X_3	+	1	0.276%	1.725%	1	0.133%	0.825%	19
V_1	X_3	-	1	0.278%	1.771%	1	0.136%	0.754%	19
V_1	X_3	+/-	1	0.277%	1.853%	1	0.134%	0.799%	19
V_1	X_4	+	1	0.288%	1.780%	1	0.140%	0.985%	19
V_1	X_4	-	2	0.288%	1.804%	1	0.139%	1.023%	19
V_1	X_4	+/-	1	0.288%	1.748%	1	0.139%	0.972%	19
V_2	X_1	+	1	0.559%	3.538%	1	0.220%	1.761%	19
V_2	X_1	-	1	0.564%	3.022%	1	0.226%	1.520%	19
V_2	X_1	+/-	1	0.563%	3.572%	0	0.224%	1.814%	20
V_2	X_2	+	0	0.079%	0.861%	1	0.044%	0.437%	19
V_2	X_2	-	0	0.082%	0.888%	1	0.045%	0.461%	19
V_2	X_2	+/-	1	0.080%	0.926%	1	0.044%	0.430%	19
V_2	X_3	+	1	0.342%	3.034%	1	0.154%	1.286%	19
V_2	X_3	-	1	0.349%	3.187%	1	0.153%	1.181%	19
V_2	X_3	+/-	2	0.345%	3.088%	1	0.155%	1.197%	19
V_2	X_4	+	1	0.348%	2.124%	1	0.150%	1.253%	19
V_2	X_4	-	1	0.358%	2.162%	1	0.153%	1.303%	19
V_2	X_4	+/-	2	0.350%	2.135%	1	0.152%	1.241%	19
V_3	X_1	+	0	0.523%	3.050%	1	0.209%	1.717%	19
V_3	X_1	-	1	0.528%	3.043%	1	0.211%	1.710%	19
V_3	X_1	+/-	1	0.522%	3.045%	1	0.211%	1.780%	19
V_3	X_2	+	1	0.054%	0.540%	0	0.033%	0.431%	20
V_3	X_2	-	0	0.055%	0.585%	1	0.033%	0.448%	19
V_3	X_2	+/-	1	0.054%	0.554%	0	0.032%	0.428%	20
V_3	X_3	+	1	0.321%	3.077%	1	0.144%	1.060%	19
V_3	X_3	-	2	0.322%	3.108%	0	0.143%	0.892%	20
V_3	X_3	+/-	1	0.323%	3.088%	1	0.146%	1.062%	20
V_3	X_4	+	1	0.306%	1.797%	1	0.147%	1.159%	19
V_3	X_4	-	1	0.311%	1.944%	1	0.148%	1.312%	19
V_3	X_4	+/-	2	0.305%	1.768%	1	0.148%	1.154%	19

Table 4.8: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 10,\ \alpha = 1,\ M = 1000$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	1	0.581%	7.520%	0	0.148%	2.728%	19
V_1	X_1	-	1	0.584%	7.515%	1	0.150%	2.689%	19
V_1	X_1	+/-	1	0.581%	7.536%	1	0.150%	2.725%	19
V_1	X_2	+	0	0.001%	0.772%	0	0.001%	0.118%	19
V_1	X_2	-	0	0.001%	0.827%	1	0.001%	0.061%	19
V_1	X_2	+/-	0	0.001%	0.815%	1	0.001%	0.121%	18
V_1	X_3	+	2	0.163%	9.544%	0	0.021%	1.464%	20
V_1	X_3	-	1	0.164%	9.510%	1	0.021%	1.499%	18
V_1	X_3	+/-	2	0.164%	9.456%	0	0.021%	1.511%	19
V_1	X_4	+	1	0.169%	9.700%	1	0.022%	1.791%	19
V_1	X_4	-	1	0.172%	9.885%	1	0.022%	1.841%	19
V_1	X_4	+/-	1	0.170%	9.620%	1	0.022%	1.835%	19
V_2	X_1	+	1	0.816%	11.401%	1	0.206%	4.895%	19
V_2	X_1	-	1	0.835%	11.586%	0	0.207%	4.931%	19
V_2	X_1	+/-	1	0.823%	11.880%	1	0.206%	3.556%	19
V_2	X_2	+	1	0.014%	3.705%	0	0.003%	0.404%	19
V_2	X_2	-	1	0.015%	3.849%	0	0.003%	0.508%	19
V_2	X_2	+/-	1	0.014%	3.810%	5	0.003%	0.460%	19
V_2	X_3	+	1	0.273%	16.624%	1	0.048%	3.988%	19
V_2	X_3	-	2	0.278%	17.007%	0	0.045%	4.149%	19
V_2	X_3	+/-	1	0.275%	16.835%	1	0.047%	4.082%	19
V_2	X_4	+	1	0.247%	11.876%	1	0.040%	3.927%	19
V_2	X_4	-	1	0.258%	12.083%	1	0.041%	4.258%	19
V_2	X_4	+/-	1	0.249%	11.720%	1	0.040%	4.024%	19
V_3	X_1	+	1	0.699%	10.446%	1	0.168%	4.417%	19
V_3	X_1	-	1	0.707%	10.814%	1	0.170%	4.404%	20
V_3	X_1	+/-	1	0.703%	10.907%	0	0.171%	4.360%	19
V_3	X_2	+	0	0.006%	0.758%	1	0.002%	0.084%	19
V_3	X_2	-	0	0.006%	0.826%	0	0.002%	0.069%	19
V_3	X_2	+/-	1	0.006%	0.797%	0	0.002%	0.072%	18
V_3	X_3	+	1	0.186%	13.269%	1	0.028%	1.714%	19
V_3	X_3	-	9	0.188%	13.239%	8	0.029%	1.979%	19
V_3	X_3	+/-	2	0.187%	13.163%	0	0.029%	1.858%	19
V_3	X_4	+	1	0.221%	12.382%	1	0.032%	3.095%	19
V_3	X_4	-	1	0.227%	12.602%	2	0.031%	3.065%	19
V_3	X_4	+/-	2	0.223%	12.576%	0	0.032%	3.232%	19

Table 4.9: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 10$, $\alpha = 10$, $M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	1	0.675%	6.698%	0	0.251%	3.274%	20
V_1	X_1	-	1	0.680%	6.852%	1	0.252%	3.181%	19
V_1	X_1	+/-	0	0.676%	6.670%	1	0.251%	3.295%	19
V_1	X_2	+	1	0.004%	0.583%	0	0.003%	0.110%	19
V_1	X_2	-	0	0.004%	0.619%	1	0.003%	0.098%	19
V_1	X_2	+/-	1	0.004%	0.605%	0	0.003%	0.114%	19
V_1	X_3	+	1	0.221%	6.591%	1	0.067%	3.251%	19
V_1	X_3	-	1	0.220%	6.693%	1	0.065%	3.413%	18
V_1	X_3	+/-	1	0.221%	6.694%	1	0.065%	3.282%	19
V_1	X_4	+	1	0.236%	8.183%	1	0.068%	1.282%	19
V_1	X_4	-	1	0.240%	8.222%	1	0.068%	1.238%	20
V_1	X_4	+/-	1	0.236%	8.173%	1	0.068%	1.287%	19
V_2	X_1	+	1	0.913%	9.047%	1	0.308%	3.338%	19
V_2	X_1	-	1	0.941%	8.998%	0	0.314%	3.379%	19
V_2	X_1	+/-	1	0.918%	9.807%	1	0.311%	3.414%	22
V_2	X_2	+	0	0.036%	2.274%	1	0.017%	0.343%	21
V_2	X_2	-	0	0.035%	2.357%	1	0.016%	0.355%	19
V_2	X_2	+/-	1	0.036%	2.328%	0	0.016%	0.370%	27
V_2	X_3	+	2	0.337%	16.701%	1	0.098%	3.360%	19
V_2	X_3	-	1	0.345%	17.187%	1	0.098%	3.699%	19
V_2	X_3	+/-	1	0.342%	16.983%	1	0.100%	3.563%	19
V_2	X_4	+	1	0.327%	8.023%	1	0.096%	1.692%	19
V_2	X_4	-	1	0.335%	8.042%	2	0.099%	2.055%	24
V_2	X_4	+/-	1	0.330%	7.950%	1	0.100%	1.713%	21
V_3	X_1	+	1	0.793%	8.796%	1	0.272%	3.078%	26
V_3	X_1	-	0	0.805%	8.784%	1	0.275%	3.059%	20
V_3	X_1	+/-	1	0.794%	8.799%	1	0.271%	3.079%	18
V_3	X_2	+	1	0.018%	0.605%	0	0.010%	0.279%	20
V_3	X_2	-	1	0.018%	0.867%	0	0.009%	0.255%	21
V_3	X_2	+/-	1	0.018%	0.788%	0	0.009%	0.283%	19
V_3	X_3	+	2	0.260%	13.199%	0	0.079%	2.262%	19
V_3	X_3	-	1	0.260%	13.343%	1	0.079%	2.209%	19
V_3	X_3	+/-	1	0.258%	13.259%	1	0.077%	2.249%	19
V_3	X_4	+	1	0.288%	7.925%	1	0.081%	2.133%	18
V_3	X_4	-	1	0.291%	7.960%	1	0.081%	2.510%	19
V_3	X_4	+/-	1	0.289%	7.971%	1	0.081%	2.214%	18

Table 4.10: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 10,\ \alpha = 10,\ M = 1000$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	3	0.649%	3.050%	2	0.294%	1.062%	22
V_1	X_1	-	2	0.655%	3.101%	3	0.292%	1.063%	21
V_1	X_1	+/-	3	0.651%	3.056%	2	0.296%	1.134%	25
V_1	X_2	+	1	0.016%	0.111%	1	0.011%	0.069%	22
V_1	X_2	-	2	0.016%	0.117%	1	0.012%	0.068%	23
V_1	X_2	+/-	2	0.016%	0.112%	1	0.012%	0.065%	23
V_1	X_3	+	2	0.281%	1.722%	3	0.148%	0.800%	23
V_1	X_3	-	2	0.285%	1.976%	3	0.149%	0.803%	23
V_1	X_3	+/-	2	0.282%	1.812%	3	0.148%	0.800%	24
V_1	X_4	+	2	0.277%	2.675%	3	0.148%	0.782%	23
V_1	X_4	-	2	0.279%	2.777%	3	0.147%	0.814%	23
V_1	X_4	+/-	2	0.277%	2.636%	3	0.148%	0.775%	23
V_2	X_1	+	2	0.780%	3.219%	2	0.339%	1.607%	26
V_2	X_1	-	4	0.789%	3.186%	3	0.343%	1.623%	33
V_2	X_1	+/-	3	0.788%	3.062%	2	0.343%	1.798%	24
V_2	X_2	+	2	0.076%	0.695%	2	0.041%	0.268%	23
V_2	X_2	-	3	0.078%	0.615%	2	0.042%	0.321%	23
V_2	X_2	+/-	3	0.077%	0.581%	2	0.041%	0.302%	24
V_2	X_3	+	3	0.356%	2.986%	5	0.178%	1.071%	23
V_2	X_3	-	3	0.363%	3.087%	3	0.178%	0.941%	23
V_2	X_3	+/-	3	0.355%	3.010%	3	0.179%	1.075%	23
V_2	X_4	+	3	0.361%	3.592%	3	0.173%	0.958%	24
V_2	X_4	-	3	0.366%	3.961%	3	0.176%	1.023%	24
V_2	X_4	+/-	3	0.364%	3.578%	3	0.174%	1.013%	23
V_3	X_1	+	3	0.736%	3.986%	3	0.315%	1.320%	23
V_3	X_1	-	3	0.739%	3.975%	2	0.318%	1.650%	24
V_3	X_1	+/-	3	0.741%	3.989%	2	0.314%	1.608%	24
V_3	X_2	+	2	0.048%	0.489%	2	0.028%	0.261%	23
V_3	X_2	-	2	0.049%	0.512%	1	0.029%	0.240%	24
V_3	X_2	+/-	2	0.049%	0.487%	2	0.028%	0.251%	24
V_3	X_3	+	2	0.330%	2.161%	3	0.166%	0.936%	24
V_3	X_3	-	2	0.332%	2.331%	4	0.166%	0.795%	23
V_3	X_3	+/-	2	0.330%	2.303%	3	0.167%	0.790%	23
V_3	X_4	+	2	0.311%	2.597%	3	0.161%	0.935%	24
V_3	X_4	-	2	0.312%	2.891%	3	0.162%	0.913%	24
V_3	X_4	+/-	2	0.312%	2.627%	3	0.162%	0.974%	24

Table 4.11: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 16,\ \alpha = 1,\ M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	2	0.550%	2.059%	2	0.236%	0.876%	24
V_1	X_1	-	2	0.548%	2.356%	2	0.236%	0.875%	24
V_1	X_1	+/-	1	0.549%	1.744%	2	0.236%	1.004%	24
V_1	X_2	+	1	0.041%	0.178%	1	0.035%	0.160%	24
V_1	X_2	-	1	0.043%	0.175%	0	0.036%	0.147%	24
V_1	X_2	+/-	1	0.042%	0.172%	1	0.035%	0.158%	23
V_1	X_3	+	2	0.339%	1.519%	2	0.175%	0.586%	24
V_1	X_3	-	3	0.337%	1.439%	2	0.177%	0.622%	23
V_1	X_3	+/-	3	0.337%	1.531%	2	0.176%	0.615%	23
V_1	X_4	+	3	0.338%	1.838%	2	0.176%	0.582%	24
V_1	X_4	-	2	0.337%	1.854%	2	0.178%	0.776%	24
V_1	X_4	+/-	2	0.338%	1.826%	2	0.177%	0.611%	24
V_2	X_1	+	2	0.633%	3.019%	2	0.268%	1.423%	23
V_2	X_1	-	3	0.639%	2.781%	1	0.270%	1.326%	24
V_2	X_1	+/-	2	0.637%	2.743%	2	0.269%	1.449%	24
V_2	X_2	+	2	0.131%	0.671%	2	0.089%	0.477%	23
V_2	X_2	-	2	0.137%	0.679%	1	0.095%	0.460%	24
V_2	X_2	+/-	2	0.134%	0.666%	1	0.092%	0.422%	24
V_2	X_3	+	3	0.409%	1.833%	2	0.197%	1.128%	23
V_2	X_3	-	3	0.409%	1.898%	2	0.198%	1.316%	23
V_2	X_3	+/-	3	0.409%	1.842%	2	0.198%	0.870%	24
V_2	X_4	+	3	0.405%	2.175%	2	0.192%	0.771%	23
V_2	X_4	-	3	0.408%	2.217%	2	0.193%	1.134%	24
V_2	X_4	+/-	2	0.405%	2.082%	2	0.192%	0.844%	24
V_3	X_1	+	2	0.590%	2.387%	2	0.256%	1.054%	24
V_3	X_1	-	2	0.596%	2.352%	2	0.257%	1.074%	24
V_3	X_1	+/-	2	0.595%	2.395%	2	0.255%	1.111%	24
V_3	X_2	+	1	0.093%	0.666%	1	0.068%	0.355%	24
V_3	X_2	-	1	0.096%	0.629%	2	0.071%	0.361%	24
V_3	X_2	+/-	1	0.095%	0.625%	1	0.069%	0.358%	24
V_3	X_3	+	2	0.395%	1.626%	3	0.188%	0.931%	23
V_3	X_3	-	3	0.395%	1.681%	2	0.189%	0.893%	23
V_3	X_3	+/-	3	0.394%	1.739%	2	0.188%	0.738%	24
V_3	X_4	+	2	0.360%	1.549%	2	0.186%	0.801%	23
V_3	X_4	-	3	0.362%	1.753%	2	0.188%	0.801%	23
V_3	X_4	+/-	3	0.362%	1.671%	2	0.188%	0.844%	24

Table 4.12: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 16$, $\alpha = 1$, $M = 1000$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	2	0.690%	7.302%	2	0.254%	2.510%	24
V_1	X_1	-	3	0.692%	7.308%	2	0.252%	2.527%	23
V_1	X_1	+/-	3	0.691%	7.304%	2	0.254%	2.525%	24
V_1	X_2	+	1	0.003%	0.112%	0	0.002%	0.032%	24
V_1	X_2	-	1	0.003%	0.116%	1	0.002%	0.034%	23
V_1	X_2	+/-	1	0.003%	0.123%	1	0.002%	0.032%	23
V_1	X_3	+	3	0.215%	3.861%	3	0.054%	1.944%	24
V_1	X_3	-	3	0.216%	4.119%	3	0.054%	1.942%	24
V_1	X_3	+/-	4	0.215%	3.924%	2	0.054%	1.942%	24
V_1	X_4	+	3	0.222%	4.878%	3	0.053%	1.663%	24
V_1	X_4	-	3	0.221%	4.893%	3	0.052%	1.670%	24
V_1	X_4	+/-	4	0.223%	4.926%	3	0.052%	1.642%	24
V_2	X_1	+	3	0.968%	8.062%	3	0.332%	3.614%	24
V_2	X_1	-	4	0.989%	8.429%	2	0.340%	3.743%	24
V_2	X_1	+/-	3	0.978%	8.191%	3	0.335%	3.516%	24
V_2	X_2	+	1	0.027%	1.394%	2	0.010%	0.728%	24
V_2	X_2	-	2	0.027%	1.515%	1	0.010%	0.686%	24
V_2	X_2	+/-	2	0.028%	1.427%	1	0.010%	0.703%	24
V_2	X_3	+	3	0.305%	5.824%	3	0.079%	2.533%	24
V_2	X_3	-	5	0.305%	6.676%	3	0.079%	2.542%	24
V_2	X_3	+/-	3	0.307%	5.805%	3	0.080%	2.591%	24
V_2	X_4	+	3	0.327%	6.809%	3	0.077%	2.622%	24
V_2	X_4	-	3	0.329%	8.030%	3	0.078%	2.507%	24
V_2	X_4	+/-	3	0.327%	6.908%	3	0.078%	2.642%	24
V_3	X_1	+	3	0.838%	7.925%	2	0.292%	3.918%	24
V_3	X_1	-	3	0.841%	7.970%	2	0.298%	4.047%	24
V_3	X_1	+/-	3	0.838%	7.946%	2	0.296%	3.907%	24
V_3	X_2	+	1	0.014%	1.411%	1	0.006%	0.184%	24
V_3	X_2	-	1	0.014%	1.498%	1	0.005%	0.174%	24
V_3	X_2	+/-	1	0.015%	1.439%	1	0.006%	0.181%	24
V_3	X_3	+	3	0.248%	4.533%	4	0.066%	2.199%	24
V_3	X_3	-	4	0.249%	4.561%	3	0.065%	2.175%	24
V_3	X_3	+/-	4	0.248%	4.599%	2	0.066%	2.175%	24
V_3	X_4	+	3	0.291%	5.891%	3	0.063%	3.438%	24
V_3	X_4	-	4	0.294%	6.133%	2	0.063%	3.569%	24
V_3	X_4	+/-	4	0.291%	5.855%	2	0.064%	3.421%	27

Table 4.13: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 16,\ \alpha = 10,\ M = 500$

			MRP_2			I-MRP			Opt
σ	x	ρ	time (sec)	e_{ave}	e_{max}	time (sec)	e_{ave}	e_{max}	time (sec)
V_1	X_1	+	2	0.848%	6.140%	1	0.391%	2.636%	25
V_1	X_1	-	2	0.848%	6.096%	1	0.392%	2.377%	24
V_1	X_1	+/-	2	0.851%	6.122%	1	0.391%	2.172%	24
V_1	X_2	+	1	0.011%	0.128%	1	0.009%	0.104%	23
V_1	X_2	-	1	0.012%	0.172%	1	0.010%	0.083%	24
V_1	X_2	+/-	0	0.012%	0.131%	0	0.009%	0.093%	24
V_1	X_3	+	2	0.352%	3.715%	2	0.158%	1.773%	23
V_1	X_3	-	3	0.354%	3.911%	2	0.158%	1.749%	23
V_1	X_3	+/-	2	0.352%	3.622%	2	0.158%	1.775%	24
V_1	X_4	+	2	0.355%	4.389%	2	0.158%	1.142%	24
V_1	X_4	-	2	0.356%	4.638%	2	0.158%	1.226%	24
V_1	X_4	+/-	2	0.355%	4.411%	2	0.157%	1.110%	23
V_2	X_1	+	3	1.126%	6.478%	1	0.488%	3.429%	24
V_2	X_1	-	2	1.138%	6.365%	2	0.492%	3.681%	23
V_2	X_1	+/-	2	1.134%	6.466%	2	0.489%	3.389%	24
V_2	X_2	+	1	0.080%	0.882%	1	0.047%	0.715%	24
V_2	X_2	-	1	0.082%	0.992%	1	0.048%	0.647%	23
V_2	X_2	+/-	2	0.080%	0.942%	1	0.048%	0.711%	23
V_2	X_3	+	3	0.487%	5.408%	2	0.213%	2.235%	23
V_2	X_3	-	3	0.488%	5.516%	2	0.210%	2.328%	24
V_2	X_3	+/-	2	0.490%	5.620%	2	0.213%	2.235%	24
V_2	X_4	+	2	0.501%	6.480%	2	0.206%	2.216%	24
V_2	X_4	-	3	0.507%	7.391%	2	0.205%	2.099%	23
V_2	X_4	+/-	3	0.504%	6.535%	2	0.206%	2.198%	23
V_3	X_1	+	2	0.985%	5.891%	2	0.445%	3.905%	24
V_3	X_1	-	2	0.995%	5.852%	1	0.445%	3.451%	24
V_3	X_1	+/-	2	0.986%	5.844%	1	0.447%	3.539%	24
V_3	X_2	+	1	0.045%	0.908%	1	0.030%	0.453%	23
V_3	X_2	-	1	0.046%	0.969%	1	0.030%	0.466%	23
V_3	X_2	+/-	1	0.046%	0.924%	1	0.030%	0.445%	24
V_3	X_3	+	2	0.428%	3.933%	2	0.191%	1.646%	24
V_3	X_3	-	2	0.430%	4.106%	2	0.190%	1.666%	24
V_3	X_3	+/-	2	0.428%	3.922%	2	0.190%	1.660%	24
V_3	X_4	+	2	0.433%	5.262%	2	0.178%	2.195%	24
V_3	X_4	-	2	0.438%	5.823%	2	0.176%	2.081%	23
V_3	X_4	+/-	3	0.434%	5.222%	2	0.178%	2.138%	23

Table 4.14: Performance of MRP algorithm 2 and I-MRP algorithm. $n = 16,\ \alpha = 10,\ M = 1000$

As shown in Tables 4.3 to 4.14, MRP_2 and I-MRP algorithm perform very well in terms of accuracy and efficiency. Algorithm I-MRP performs better than MRP_2 in accuracy, and the speed of algorithm I-MRP is almost as fast as MRP_2. Therefore, we use I-MRP algorithm to solve stage 2 problems when we investigate the stage 1 problem using the following parameter settings.

1) Set n=4.

2) Form the reallocation cost matrix by generating the reallocation costs based on $k_{ij} \sim$ Uniform (0,200) $i, j = 1, ..., n,\ i \neq j$.

3) Form the unit cost of resource based on $c_i/C \sim$ Uniform(0.4,0.6), $C = 1000,\ 1500$. $i = 1, ..., n$.

4) Form the slope vector by generating each slope based on $\alpha_i \sim$ Uniform(0.5,0.7), $i = 1, ..., n$.

5) Form the mean demand vector by generating each mean demand based on $\mu_i \sim$ Uniform(600,1000).

6) Form the standard deviation vector as follows:

   (a) $\sigma_i = 10\%\mu_i\ i = 1, ..., n$.

   (b) $\sigma_i = 30\%\mu_i\ i = 1, ..., n$.

7) Form the correlation coefficient matrix in three forms as follows:

   (a) Generate $\rho_{ij} \sim$ Uniform (0,1) (All demands are positively correlated).

   (b) Generate $\rho_{ij} \sim$ Uniform (-1,0) (All demands are negatively correlated).

   (c) Generate $\rho_{ij} \sim$ Uniform (-1,1) (Demands can be negatively or positively correlated).

Based on this experimental setup, we have a total of 12 different scenarios. We replicate each scenario with different random number seeds 10 times, which results in a total of

120 experiments. In Tables 4.15 to 4.26 show the optimal resource capacities and optimal expected objective values. The results show that there can be significant gap between the optimal expected objective value with reallocation and the one without reallocation, and, (1) The larger the unit resource cost the larger the gap; (2) The larger the standard deviation of demand the larger the gap; (3) The smaller the correlation between demands the larger the gap.

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
166.1	343.9	178.2	194.7	320103	0.0	360.4	543.4	0.0	336030	4.98%
268.2	273.2	255.3	295.2	515006	0.0	275.5	450.6	365.5	522863	1.53%
205.2	289.7	343.1	158.8	452592	0.0	0.0	343.3	661.7	467165	3.22%
307.4	245.8	162.4	200.7	342646	304.1	242.8	158.7	209.3	345108	0.72%
183.0	390.1	236.7	205.7	504302	0.0	886.8	0.0	190.4	559055	10.86%
325.7	168.6	276.0	285.2	481432	823.7	0.0	0.0	250.8	496212	3.07%
248.7	266.6	244.3	181.3	397333	0.0	764.9	0.0	184.7	413913	4.17%
304.0	342.1	134.4	322.4	584091	291.4	310.8	164.0	333.1	587056	0.51%
292.2	248.4	199.5	199.5	409181	288.9	461.1	0.0	209.8	427285	4.42%
247.9	322.5	234.1	307.0	544665	0.0	0.0	569.5	580.5	584456	7.31%

Table 4.15: Optimal solution of P_1 with $\sigma = 0.1\mu,\ \rho = -1,\ C = 1000$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
166.1	343.9	178.2	194.7	320103	0.0	357.5	545.9	0.0	335915	4.94%
268.2	273.2	255.3	295.2	515006	0.0	274.6	453.3	363.9	522744	1.50%
205.2	289.7	343.1	158.8	452592	0.0	0.0	343.5	660.9	467290	3.25%
307.4	245.8	162.4	200.7	342646	304.1	242.4	159.0	209.3	345102	0.72%
183.0	390.1	236.7	205.7	504302	0.0	886.6	0.0	189.0	558506	10.75%
325.7	168.6	276.0	285.2	481432	822.9	0.0	0.0	251.6	496132	3.05%
248.7	266.6	244.3	181.3	397333	0.0	765.9	0.0	184.0	414058	4.21%
304.0	342.1	134.4	322.4	584091	292.8	315.5	158.9	333.0	586842	0.47%
292.2	248.4	199.5	199.5	409181	288.9	460.8	0.0	210.0	427265	4.42%
247.9	322.5	234.1	307.0	544665	0.0	0.0	568.9	581.1	584768	7.36%

Table 4.16: Optimal solution of P_1 with $\sigma = 0.1\mu,\ \rho = 0,\ C = 1000$

4.4.2 Impact of System Parameters on the Stage 1 Model

In this section, we provide an algorithm to solve the stage 1 problem, and investigate the impact of various system parameters such as the slope of the demand function, unit investment cost, mean, variance and correlation of the random market sizes on the optimal expected profit and the optimal resource capacity levels.

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
166.1	343.9	178.2	194.7	320103	0.0	358.0	546.2	0.0	335712	4.88%
268.2	273.2	255.3	295.2	515006	0.0	274.4	459.0	358.8	522386	1.43%
205.2	289.7	343.1	158.8	452592	0.0	0.0	343.7	660.1	467087	3.20%
307.4	245.8	162.4	200.7	342646	304.9	241.1	159.0	210.1	344888	0.65%
183.0	390.1	236.7	205.7	504302	0.0	885.8	0.0	191.0	558137	10.68%
325.7	168.6	276.0	285.2	481432	822.2	0.0	0.0	252.4	496102	3.05%
248.7	266.6	244.3	181.3	397333	0.0	765.5	0.0	184.5	413414	4.05%
304.0	342.1	134.4	322.4	584091	293.6	317.6	157.3	332.4	586598	0.43%
292.2	248.4	199.5	199.5	409181	288.7	461.3	0.0	209.8	427187	4.40%
247.9	322.5	234.1	307.0	544665	0.0	0.0	568.9	581.4	584473	7.31%

Table 4.17: Optimal solution of P_1 with $\sigma = 0.1\mu$, $\rho = 1$, $C = 1000$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
98.8	265.6	108.1	103.2	150733	0.0	277.0	339.1	0.0	168228	11.61%
196.5	210.9	199.2	222.3	297818	0.0	210.0	219.3	410.6	311464	4.58%
129.1	226.8	274.8	83.1	255053	0.0	0.0	276.5	458.2	275217	7.91%
213.4	157.6	71.8	128.6	140753	211.3	161.8	60.8	135.8	143399	1.88%
103.7	336.7	143.3	123.4	293655	0.0	736.3	0.0	59.4	355412	21.03%
257.0	65.7	204.5	216.8	262973	820.1	0.0	0.0	0.0	301543	14.67%
178.5	203.4	161.9	113.2	199108	0.0	568.1	0.0	114.8	223782	12.39%
218.3	281.4	50.0	256.5	351071	201.4	257.6	74.7	271.2	354189	0.89%
238.7	190.6	126.7	135.9	233460	235.1	346.6	0.0	142.5	252407	8.12%
157.9	250.4	179.0	239.4	308023	0.0	0.0	454.1	437.0	356673	15.79%

Table 4.18: Optimal solution of P_1 with $\sigma = 0.1\mu$, $\rho = -1$, $C = 1500$

Let x_i^{NR} denote the optimal capacity of resource i when there is no reallocation between the resources. $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is the expected objective function value of P_2 given a random market size vector $\overrightarrow{\Gamma}$ and a capacity vector $\overrightarrow{x}$.

We compute the optimal resource capacities by the following algorithm:

(1) Let $x_i = x_i^{NR}$ $\forall i$ and $l = 1$.

(2) Fixing other capacities, compute the capacity of resource l that maximizes the objective function of P_1, which is given by $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x})) - \sum_{i=1}^{n} c_i x_i$. The computation of $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is based on Monte Carlo simulation as explained in detail below. A new value of x_i, x_i^n, which maximizes the objective function of P_1, is obtained based on binary search in interval $[0, \sum_{i=1}^{n} x_i^{NR}]$. Note that the optimal value of x_i can not be larger than $\sum_{i=1}^{n} x_i^{NR}$.

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
98.8	265.6	108.1	103.2	150733	0.0	274.1	342.1	0.0	168062	11.50%
196.5	210.9	199.2	222.3	297818	0.0	209.0	218.2	412.4	311349	4.54%
129.1	226.8	274.8	83.1	255053	0.0	0.0	275.8	458.9	275354	7.96%
213.4	157.6	71.8	128.6	140753	208.6	162.8	62.2	135.6	143360	1.85%
103.7	336.7	143.3	123.4	293655	0.0	747.2	0.0	42.6	355012	20.89%
257.0	65.7	204.5	216.8	262973	820.9	0.0	0.0	0.0	301427	14.62%
178.5	203.4	161.9	113.2	199108	0.0	567.9	0.0	114.5	223927	12.47%
218.3	281.4	50.0	256.5	351071	203.2	260.4	70.5	271.1	353876	0.80%
238.7	190.6	126.7	135.9	233460	235.2	346.5	0.0	142.6	252384	8.11%
157.9	250.4	179.0	239.4	308023	0.0	0.0	452.8	438.1	356955	15.89%

Table 4.19: Optimal solution of P_1 with $\sigma = 0.1\mu,\ \rho = 0,\ C = 1500$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
98.8	265.6	108.1	103.2	150733	0.0	274.0	341.3	0.0	167853	11.36%
196.5	210.9	199.2	222.3	297818	0.0	210.0	217.0	412.7	310955	4.41%
129.1	226.8	274.8	83.1	255053	0.0	0.0	275.8	459.4	275137	7.87%
213.4	157.6	71.8	128.6	140753	210.0	161.2	62.0	136.5	143129	1.69%
103.7	336.7	143.3	123.4	293655	0.0	781.1	0.0	1.2	354829	20.83%
257.0	65.7	204.5	216.8	262973	820.1	0.0	0.0	0.0	301378	14.60%
178.5	203.4	161.9	113.2	199108	0.0	568.9	0.0	114.8	223260	12.13%
218.3	281.4	50.0	256.5	351071	205.3	258.4	72.3	268.4	353620	0.73%
238.7	190.6	126.7	135.9	233460	234.8	346.6	0.0	142.8	252287	8.06%
157.9	250.4	179.0	239.4	308023	0.0	0.0	452.6	438.4	356627	15.78%

Table 4.20: Optimal solution of P_1 with $\sigma = 0.1\mu,\ \rho = 1,\ C = 1500$

(3) If $|\overrightarrow{x} - \overrightarrow{x^n}| < \varepsilon$, return $\overrightarrow{x^n}$ as the optimal solution. Otherwise, let $i = i+1$. If $i > n$, $i = 1$. Go to step (2).

In step 2, $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is obtained by Monte Carlo simulation. We generate M independent realizations of the market size $\overrightarrow{\Gamma}$. For each realization i, $i = 1, 2, ..., M$, and a capacity vector $\overrightarrow{x}$, we compute $\Phi^*(\overrightarrow{\gamma}, \overrightarrow{x})$ based on I-MRP algorithm. Then, $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$ is approximated by the average over all realizations, i.e., $\sum_{i=1}^{M} \Phi^*(\overrightarrow{\gamma}, \overrightarrow{x})/M$. In order to generate a realization of the demand vector $\overrightarrow{\Gamma}$, we first generate a vector $\overrightarrow{Z}$ with size n, where $E(z_1) = E(z_2) = ... = E(z_n) = 0$, $Var(z_1) = Var(z_2) = ... = Var(z_n) = 1$ and $z_1, z_2, ..., z_n$ are independent. Suppose that Σ is the covariance matrix for the demands, and $\Sigma = A^T A$ after conducting the Cholesky decomposition where A is an upper triangular matrix. Let $\overrightarrow{\mu}$ denote the mean vector for the market sizes of the demands. Then $\overrightarrow{\Gamma} = \overrightarrow{\mu} + A\overrightarrow{Z}$ is the correlated market size vector, which has mean $\overrightarrow{\mu}$ and covariance matrix Σ. We use 40,000 replications to compute $E(\Phi^*(\overrightarrow{\Gamma}, \overrightarrow{x}))$. All of our standard errors are within 0.5%.

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
167.1	344.6	182.8	196.9	321338	3.3	397.3	487.1	0.0	360452	12.17%
285.7	279.8	256.6	303.9	521933	0.0	280.7	335.9	474.7	576006	10.36%
206.2	310.2	343.3	159.8	459156	74.1	0.0	354.2	570.7	502630	9.47%
315.8	253.2	163.2	201.0	345660	297.8	231.8	146.7	237.9	377722	9.28%
186.1	401.2	239.2	211.0	509033	0.0	878.9	0.0	161.6	607819	19.41%
326.4	169.1	276.6	290.3	482801	926.3	0.0	0.0	124.4	515519	6.78%
267.6	289.1	245.8	181.8	408188	0.0	751.7	0.0	199.2	478063	17.12%
306.3	344.8	140.6	327.5	587396	262.1	212.3	246.5	380.1	624860	6.38%
292.3	249.8	214.5	202.6	412129	277.3	433.0	0.0	250.9	455890	10.62%
250.4	328.5	234.6	321.0	550399	0.0	0.0	579.1	569.1	625793	13.70%

Table 4.21: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = -1,\ C = 1000$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
167.1	344.6	182.8	196.9	321338	0.0	391.3	512.2	0.0	358624	11.60%
285.7	279.8	256.6	303.9	521933	0.0	284.5	346.1	463.1	573594	9.90%
206.2	310.2	343.3	159.8	459156	40.0	0.0	350.9	611.5	503989	9.76%
315.8	253.2	163.2	201.0	345660	294.4	235.8	148.9	237.3	377110	9.10%
186.1	401.2	239.2	211.0	509033	0.0	899.5	0.0	167.2	603250	18.51%
326.4	169.1	276.6	290.3	482801	877.4	0.0	0.0	183.3	514082	6.48%
267.6	289.1	245.8	181.8	408188	0.0	758.0	0.0	196.9	478664	17.27%
306.3	344.8	140.6	327.5	587396	258.2	260.8	219.0	368.8	620784	5.68%
292.3	249.8	214.5	202.6	412129	277.7	432.4	0.0	251.4	455911	10.62%
250.4	328.5	234.6	321.0	550399	0.0	0.0	582.8	568.5	626724	13.87%

Table 4.22: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = 0,\ C = 1000$

First, we consider a simple system setting to investigate the sensitivity of the optimal expected profit and the optimal capacity levels with respect to various system parameters. We consider a system with 3-resources. Our main goal is to investigate the impact of flexibility on the performance of the system. To this end, we choose the unit reallocation costs as $k_{ii} = 0,\ i = 1,2,3,\ k_{12} = k_{13} = k_{21} = k_{23} = 10,000$, and $k_{3j} = 0,\ j = 1,2,3$. Since $k_{12},\ k_{13},\ k_{21}$ and k_{23} are very large numbers (in comparison to other reallocation costs), resources 1 and 2 behave as dedicated resources serving their own markets. Since k_{31} and k_{32} are 0, resource 3 serves the demand for resources 1 and 2 with no reallocation cost. Therefore, it behaves as the flexible resource. We assign $\mu_3 = 0$ and $\sigma_3 = 0$ so that there is no demand for resource 3. This setting was analyzed analytically by Bish and Wang (2003). Here, our goal is to investigate it further numerically.

The values for the remaining parameters are given in Table 4.27.

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
166.2	344.0	181.0	195.6	319841	0.0	399.2	496.6	0.0	355173	11.05%
283.2	277.8	255.5	301.6	518791	0.0	281.3	354.7	455.3	566832	9.26%
205.2	307.7	342.8	158.8	456955	42.4	0.0	351.5	605.3	500239	9.47%
313.1	250.8	162.1	200.6	344237	297.8	228.9	146.5	240.1	373978	8.64%
184.3	399.6	237.1	209.0	506096	0.0	914.4	0.0	147.9	596211	17.81%
325.9	168.6	276.1	288.9	480699	928.1	0.0	0.0	136.0	511277	6.36%
265.1	286.6	244.6	181.4	405908	0.0	753.2	0.0	200.0	472138	16.32%
304.8	343.5	138.7	326.0	584318	263.6	241.2	234.0	364.3	614906	5.23%
291.9	248.7	212.2	200.9	410891	276.5	433.5	0.0	249.5	453618	10.40%
248.9	327.0	234.2	318.6	546651	0.0	0.0	580.2	568.6	619484	13.32%

Table 4.23: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = 1,\ C = 1000$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
99.1	265.9	108.1	103.5	150443	0.0	309.7	308.7	0.0	190754	26.79%
201.2	211.9	199.3	224.1	298183	0.0	203.5	260.7	376.7	361539	21.25%
128.7	233.5	274.7	83.2	256642	0.0	0.0	291.2	446.2	308937	20.38%
214.4	158.1	71.6	128.5	140995	210.6	177.4	28.0	160.5	175318	24.34%
103.5	339.5	143.4	123.5	293993	0.0	799.7	0.0	0.0	406822	38.38%
256.4	65.7	204.3	217.2	262048	817.3	0.0	0.0	0.0	320351	22.25%
183.8	211.1	162.2	113.0	201910	0.0	568.8	0.0	123.5	285686	41.49%
218.4	281.4	50.5	256.8	350453	147.7	231.0	125.6	318.5	387779	10.65%
238.4	190.5	130.8	135.8	233514	223.1	334.3	0.0	170.0	279995	19.91%
157.4	251.2	178.9	242.8	307885	0.0	0.0	475.6	414.6	395400	28.42%

Table 4.24: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = -1,\ C = 1500$

Impact of α_i: We have already shown that the optimal objective values of P_1 and P_2 decrease in $\alpha_i,\ \ i = 1,2,...,n$. Through numerical analysis, we further investigate the impact of α_i on optimal objective value of P_1 and the optimal investment level.

Figure 4.5 shows that the optimal objective of P_1 decreases from 5500 to about 800 as α_1 increases from 0.5 to 4.

Figure 4.6 illustrates how the optimal resource capacities change in α_1. The optimal capacity of resource 1 (the solid curve) decreases as α_1 increases from 0.5 to 1.75, and the curve is approximately a straight line with a negative slope. When $\alpha_1 > 0.75$, the optimal capacity of resource 1 is zero. Intuitively, as α_1 increases, the selling price of resource 1 at market 1 has to be decreased to keep up the sales amount. Therefore, resource 1 becomes less profitable. As a result, the investment in resource 1 decreases. When the value of α_1 exceeds some threshold, it is not optimal to invest in resource 1. When $\alpha_1 \leq 1.75$, the optimal capacities for resources 2 and 3 do not change significantly. The reason is that the

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
99.1	265.9	108.1	103.5	150443	0.0	306.9	309.3	0.0	188516	25.31%
201.2	211.9	199.3	224.1	298183	0.0	203.2	255.8	382.0	359043	20.41%
128.7	233.5	274.7	83.2	256642	0.0	0.0	287.3	452.6	310141	20.85%
214.4	158.1	71.6	128.5	140995	205.0	176.9	31.0	162.5	174670	23.88%
103.5	339.5	143.4	123.5	293993	0.0	804.1	0.0	0.0	401455	36.55%
256.4	65.7	204.3	217.2	262048	818.3	0.0	0.0	0.0	318539	21.56%
183.8	211.1	162.2	113.0	201910	0.0	569.0	0.0	122.2	286280	41.79%
218.4	281.4	50.5	256.8	350453	155.4	227.8	117.5	321.7	383473	9.42%
238.4	190.5	130.8	135.8	233514	223.5	333.5	0.0	170.1	279937	19.88%
157.4	251.2	178.9	242.8	307885	0.0	0.0	474.0	417.6	396238	28.70%

Table 4.25: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = 0,\ C = 1500$

x_1^{NA*}	x_2^{NA*}	x_3^{NA*}	x_4^{NA*}	obj^{NA*}	x_1^*	x_2^*	x_3^*	x_4^*	obj^*	diff
99.1	265.9	108.1	103.5	150443	0.0	304.7	314.0	0.0	186684	24.09%
201.2	211.9	199.3	224.1	298183	0.0	205.5	261.1	373.8	354436	18.87%
128.7	233.5	274.7	83.2	256642	0.0	0.0	292.1	443.6	308228	20.10%
214.4	158.1	71.6	128.5	140995	210.1	173.0	28.4	163.5	172308	22.21%
103.5	339.5	143.4	123.5	293993	0.0	805.8	0.0	0.0	396256	34.78%
256.4	65.7	204.3	217.2	262048	815.6	0.0	0.0	0.0	317642	21.22%
183.8	211.1	162.2	113.0	201910	0.0	562.7	0.0	125.4	280797	39.07%
218.4	281.4	50.5	256.8	350453	155.8	228.2	121.1	315.6	380369	8.54%
238.4	190.5	130.8	135.8	233514	222.4	333.9	0.0	170.9	278355	19.20%
157.4	251.2	178.9	242.8	307885	0.0	0.0	471.2	419.2	391964	27.31%

Table 4.26: Optimal solution of P_1 with $\sigma = 0.3\mu,\ \rho = 1,\ C = 1500$

impact of the change of α_1 is absorbed by the change in the optimal capacity of resource 1, which is most directly related. When $\alpha_1 > 1.75$, the optimal capacity of resource 1 is always zero. Optimal capacity of resource 3 (dotted curve) decreases significantly in α_1. As α_1 increases, demand from market 1 becomes less profitable, and less capacity is needed from the flexible resource 3, which is used to cover the demand from market 1. Since the optimal capacity of flexible resource 3 decreases, the optimal capacity of resource 2 (dashed curve) increases.

Impact of c_i: We have already shown that the optimal objective function values of P_1 and P_2 decrease in c_i, and the optimal capacity of resource i decreases in c_i, $i = 1,2,...,n$.

Figure 4.7 shows that the optimal expected profit is a convex nonincreasing function of c_1. Figure 4.8 shows that the optimal capacity of resource 1 (solid curve) decreases as c_1 increases. When the unit cost of resource 1 is low, the investment level in resource 1 is high. As c_1 increases, the investment level in resource 1 decreases rapidly, and the

Resource	c_i	α_i	μ_i	σ_i
1	55	1.21	120	50
2	60	1.6	165	80
3	65	1.5	0	0

Table 4.27: Parameter values for the three-resource system

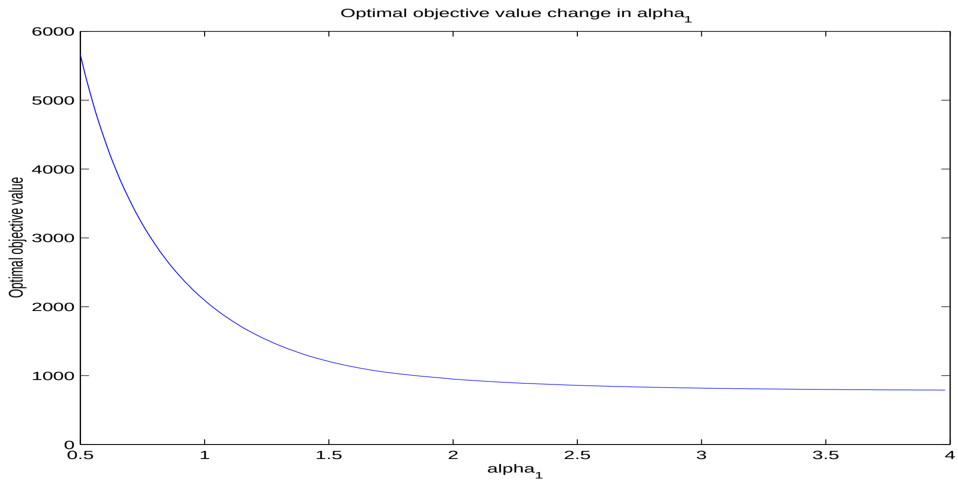

Figure 4.5: Sensitivity of the optimal expected profit with respect to α_1

optimal capacity of flexible resource 3 (dotted curve) increases. In the mean time, the optimal capacity of resource 2 (dashed curve) decreases as the optimal capacity of resource 3 increases. When c_1 exceeds some threshold, it is not optimal to invest in resource 1. All the demand for market 1 is satisfied by resource 3, and further increasing c_1 does not affect the optimal expected profit and the optimal resource capacities anymore.

Impact of μ_i: Figure 4.9 shows that the optimal expected profit increases from more than 800 to about 8000 as μ_1 increases from 0 to 250.

Figure 4.10 illustrates how the optimal resource capacities change in μ_1. The optimal capacity of resource 1 (solid curve) is zero when μ_1 is less than some threshold. Intuitively, when μ_1 is small, the demand for resource 1 can be covered by the flexible resource 3, and it does not worth to invest in dedicated capacity of resource 1. As μ_1 increases, the expected profit increases with the increased demand in market 1. As a result, the investment in resource 3 (dotted curve) increases. In the mean time, the optimal capacity of resource 2 (dashed curve) decreases. When the value of μ_1 exceeds some threshold, it is optimal to invest in dedicated but cheaper capacity of resource 1. As μ_1 increases, the optimal capacity

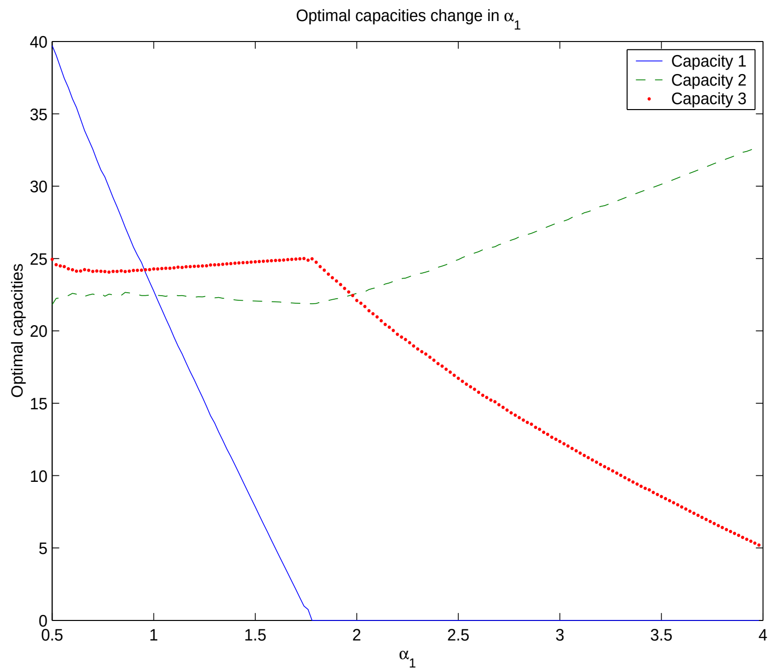

Figure 4.6: Sensitivity of the optimal resource capacities with respect to α_1

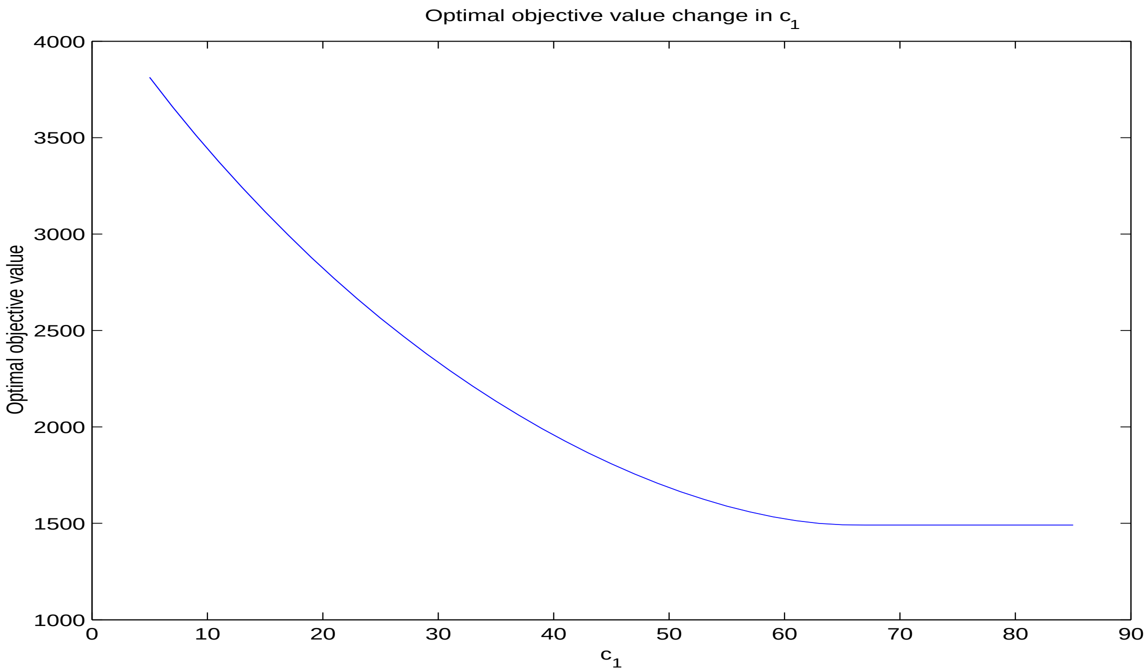

Figure 4.7: Sensitivity of the optimal expected profit with respect to c_1

of resource 1 increases, and the curve is approximately a straight line with a positive slope. On the other hand, the optimal capacities for resources 2 and 3 remain constant because the

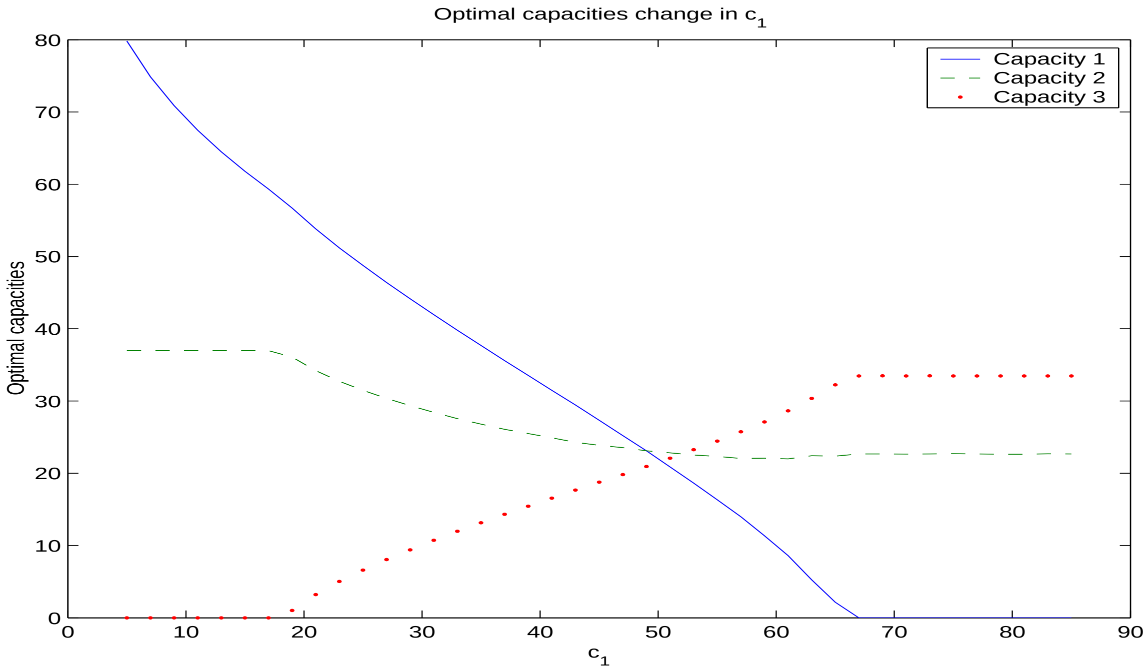

Figure 4.8: Sensitivity of the optimal resource capacities with respect to c_1

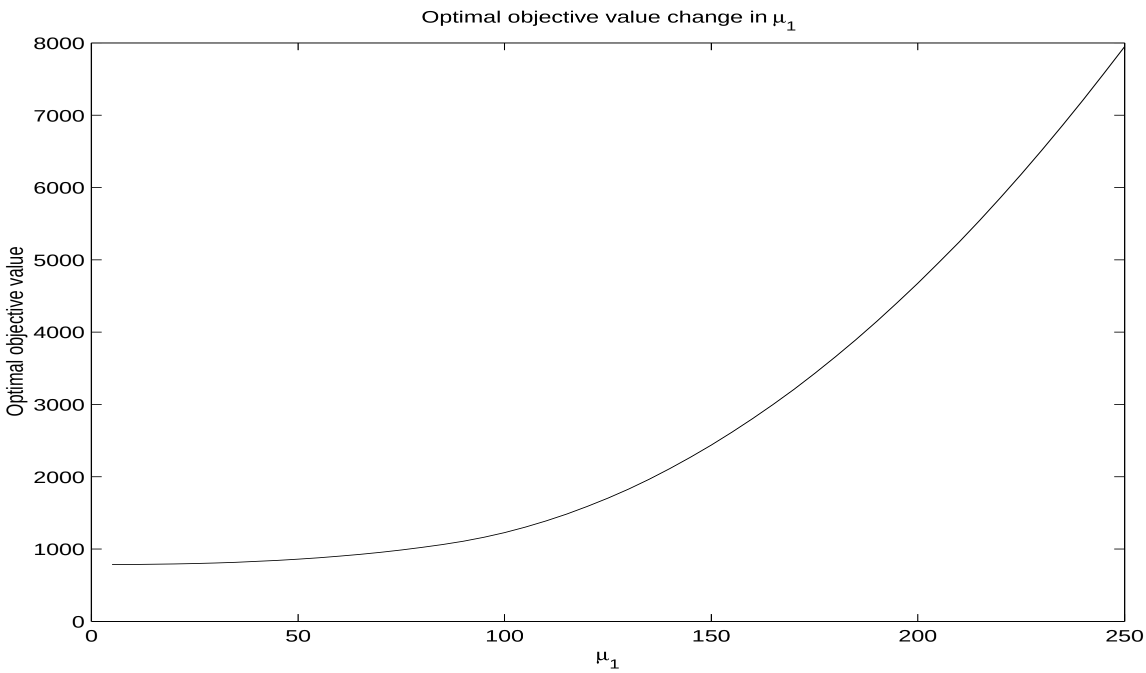

Figure 4.9: Sensitivity of the optimal expected profit with respect to μ_1

impact of the change in μ_1 is completely absorbed by the change in the optimal capacity of resource 1.

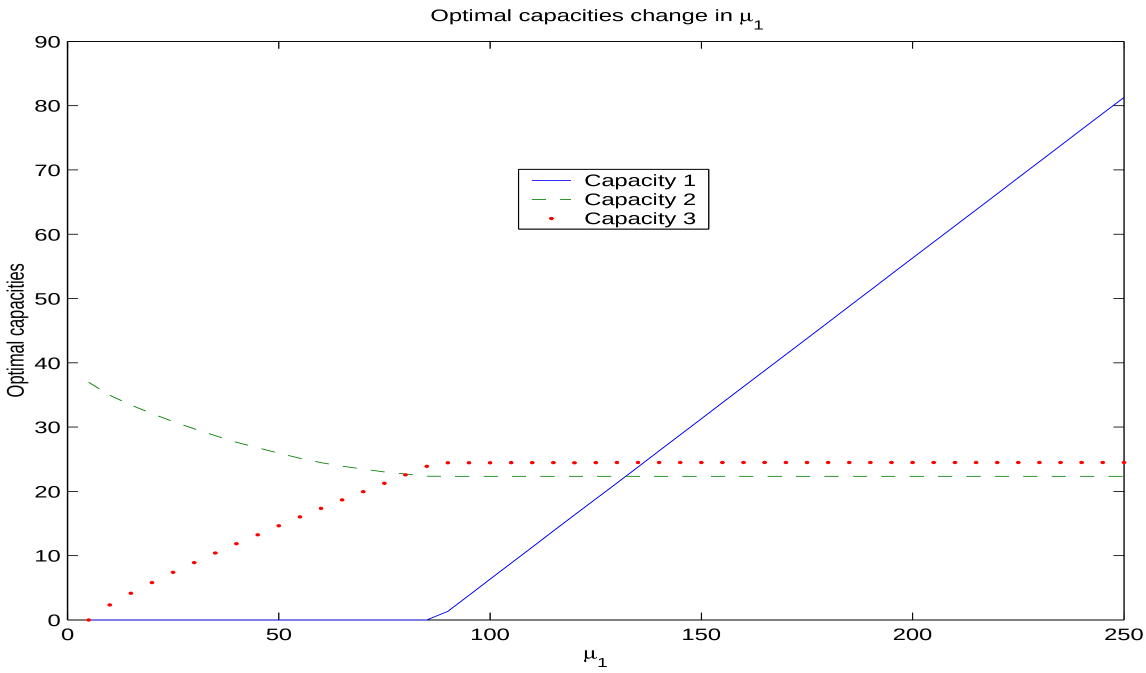

Figure 4.10: Sensitivity of the optimal resource capacities with respect to μ_1

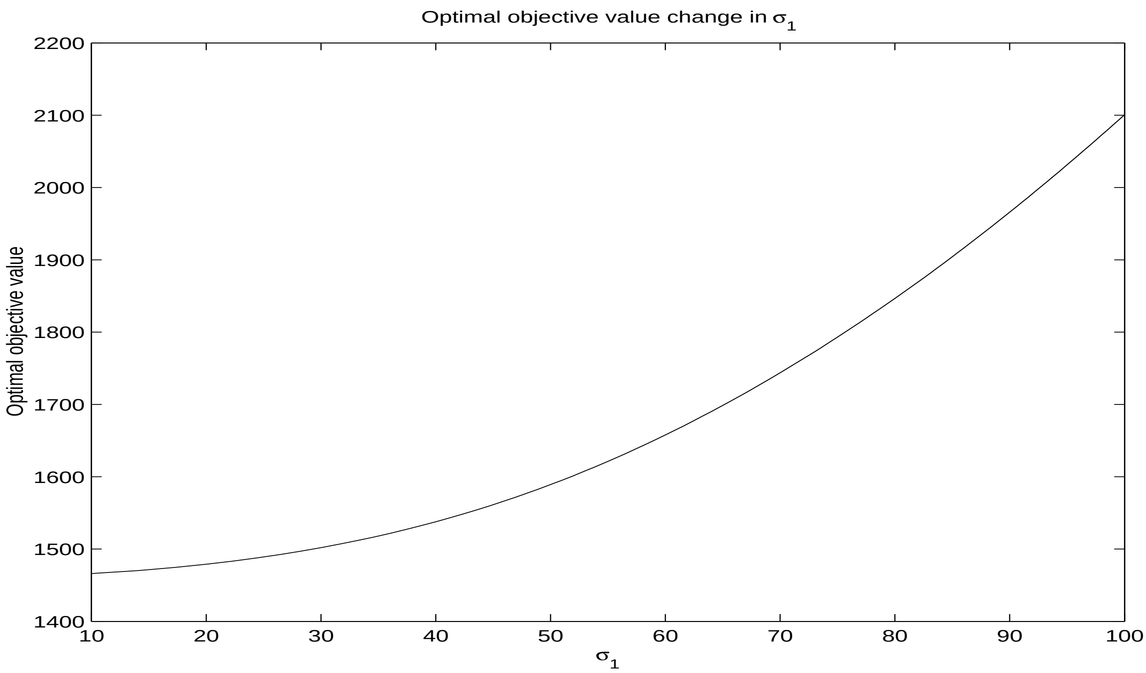

Figure 4.11: Sensitivity of the optimal expected profit with respect to σ_1

Impact of σ_i: Figure 4.11 presents the optimal expected profit as a function of the standard deviation of demand for resource 1. Keeping other parameters fixed, when the standard deviation of the demand for resource 1 increases from 10 to 100, the optimal expected profit increases.

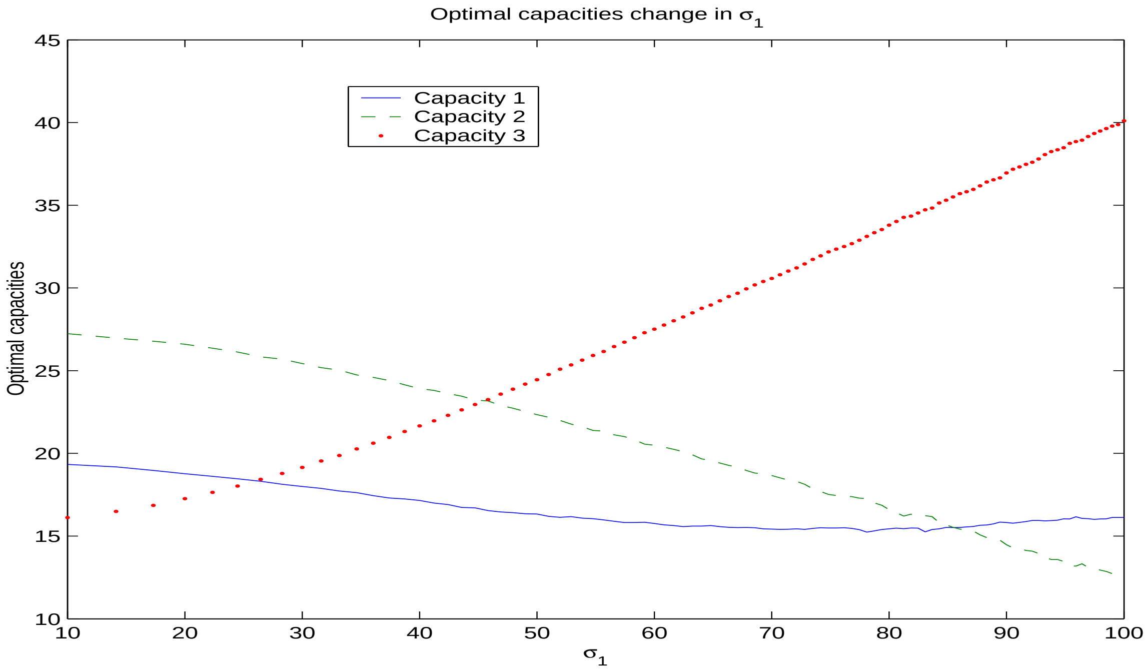

Figure 4.12: Sensitivity of the optimal resource capacities with respect to σ_1

As shown by the solid curve in Figure 4.12, the optimal capacity of resource 1 first decreases, and then increases with respect to σ_1. The optimal capacity of flexible resource 3 (dotted curve) increases as σ_1 increases. Intuitively, the increase in demand variability requests more flexibility in the system, and the investment level in the flexible resource increases. In the mean time, the optimal capacity of resource 2 (dashed curve) decreases.

Impact of correlation: We vary the correlation in between markets 1 and 2 (i.e., ρ_{12}) and observe how the optimal expected profit and the optimal resource capacities change. As ρ_{12} increases, the expected profit decreases. Figure 4.13 illustrates the optimal resource capacities as a function of the correlation coefficient between demands 1 and 2. As the correlation increases, from -1 to 1, the optimal capacity of flexible resource 3 decreases. The optimal capacities of resources 1 and 2 increase. As ρ_{12} increases, the diversity of the environment decreases. As a result, the investment in flexible resource 3 decreases and the investment in the dedicated resources increases.

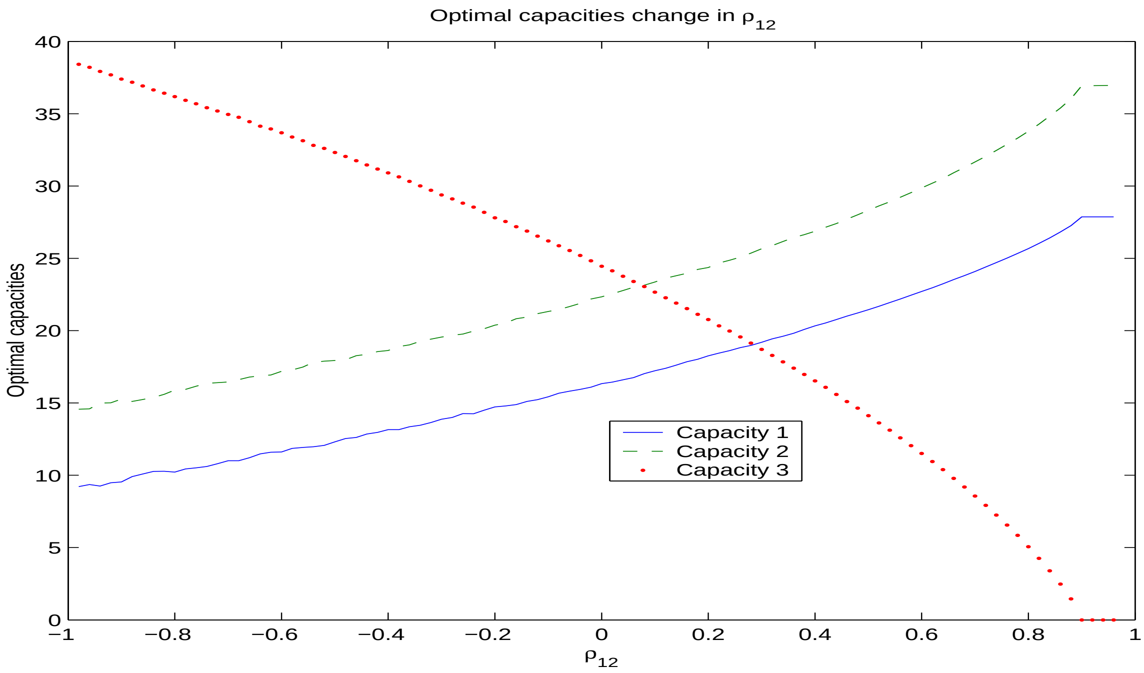

Figure 4.13: Sensitivity of the optimal resource capacities with respect to ρ_{12}

Next, we further investigate the impact of the correlation for different systems. We consider a 3-resource system with the following base values for the parameters:

1. $\alpha_1 = \alpha_2 = \alpha_3 = 0.4$.

2. $c_1 = c_2 = c_3 = 900$.

3. $\mu_1 = \mu_2 = \mu_3 = 500$.

4. $\sigma_1 = \sigma_2 = \sigma_3 = 100$.

5. $k_{12} = k_{13} = 50,\ k_{21} = k_{23} = 60, k_{31} = k_{32} = 70$.

Figure 4.14 shows the optimal expected profit as a function of ρ_{23} when $\rho_{12} = \rho_{13} = 0.5$. Note that since the correlation matrix needs to be positive definite, ρ_{23} takes values from -0.5 to 1 as shown in Figure 4.14. The straight line in Figure 4.14 represents the optimal expected profit when there is no reallocation in the system. Keeping other parameters fixed, when the correlation between demands 2 and 3 increases from -1 to 1, the optimal profit decreases. The decrease of the optimal expected profit can be explained by the decrease of

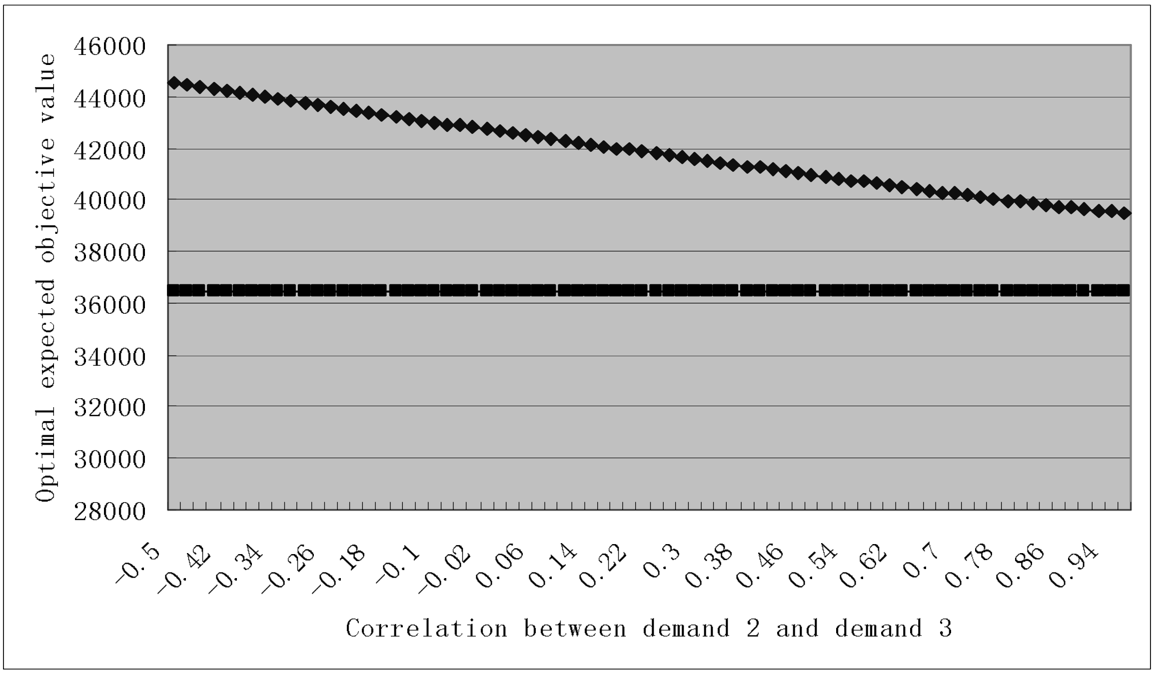

Figure 4.14: Optimal expected profit as a function of ρ_{23} when $\rho_{12} = \rho_{13} = 0.5$

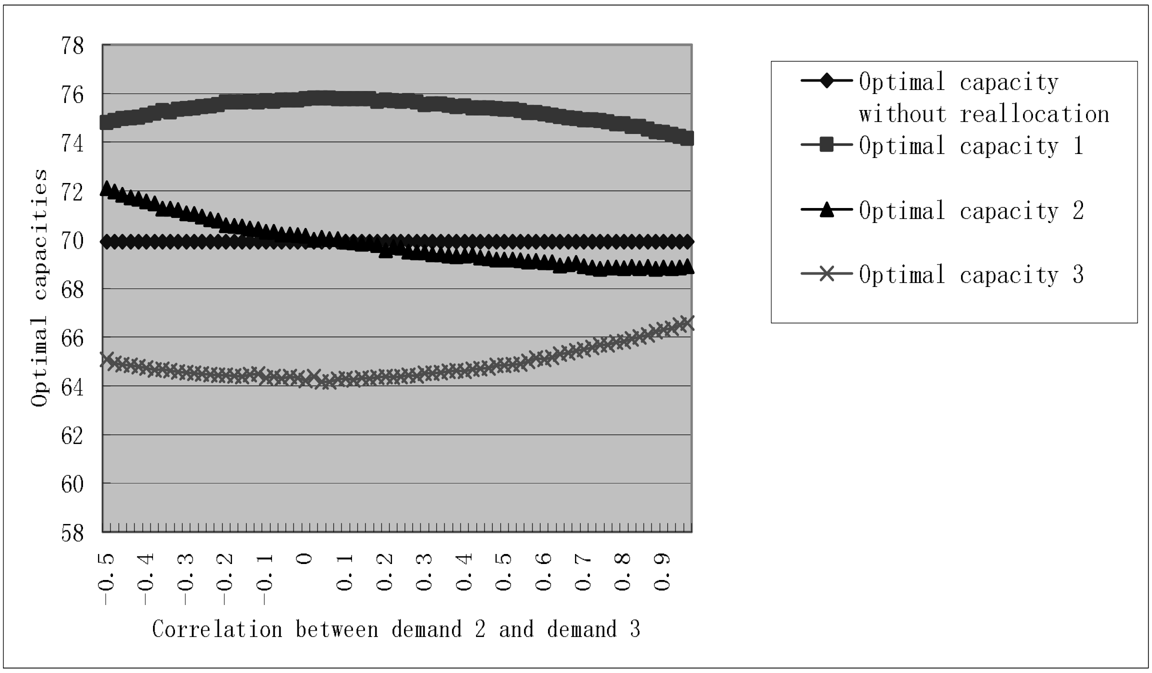

Figure 4.15: Optimal resource capacities as a function of ρ_{23} when $\rho_{12} = \rho_{13} = 0.5$

the diversity of the system which makes the reallocation less profitable. We also observe that the reallocation has the most benefit when the market sizes are negatively correlated.

Figure 4.15 illustrates how the optimal resource capacities change in ρ_{23}. The straight line in Figure 4.15 represents the optimal resource capacities when there is no reallocation in the system. The optimal capacity of resource 1 first increases, and then decreases as ρ_{23} increases from -0.5 to 1, while the optimal capacity of resource 2 keeps decreasing and the optimal capacity of resource 3 first decrease then increases.

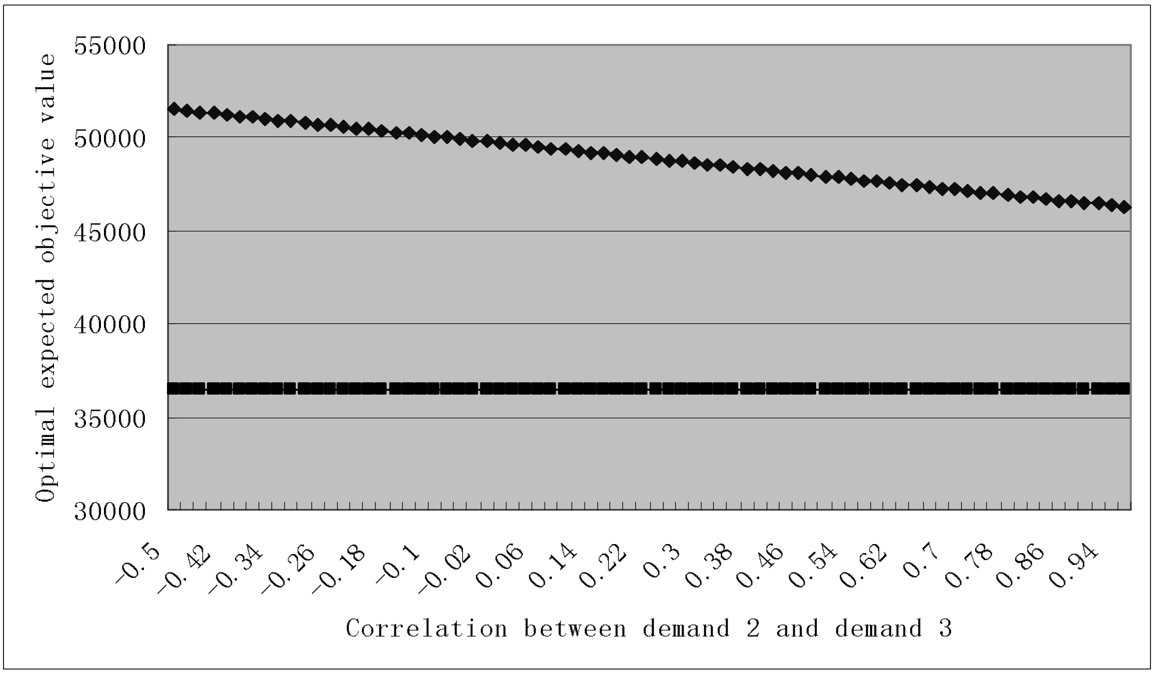

Figure 4.16: Optimal expected profit as a function of ρ_{23} when $\rho_{12} = \rho_{13} = -0.5$

Figure 4.16 shows the optimal expected profit as a function of ρ_{23} when $\rho_{12} = \rho_{13} = -0.5$. Keeping other parameters fixed, when the correlation between demands 2 and 3 increases from -0.5 to 1, the optimal expected decreases. Figure 4.17 illustrates how the optimal resource capacities change in ρ_{23}. The optimal capacity of resource 1 decreases. The optimal capacity of resource 2 first decreases, and then increases. The optimal capacity of resource 3 keeps increasing.

Figure 4.18 shows the optimal expected profit as a function of ρ_{23} when $\rho_{12} = 0.5$, and $\rho_{13} = -0.5$. Keeping other parameters fixed, when the correlation between demands 2 and 3 increases from -1 to 0.5, the optimal profit decreases. Figure 4.19 illustrates how the optimal resource capacities change in ρ_{23}. The optimal capacities of resource 1 and 2 decrease. The optimal capacity of resource 3 first decreases, and then increases.

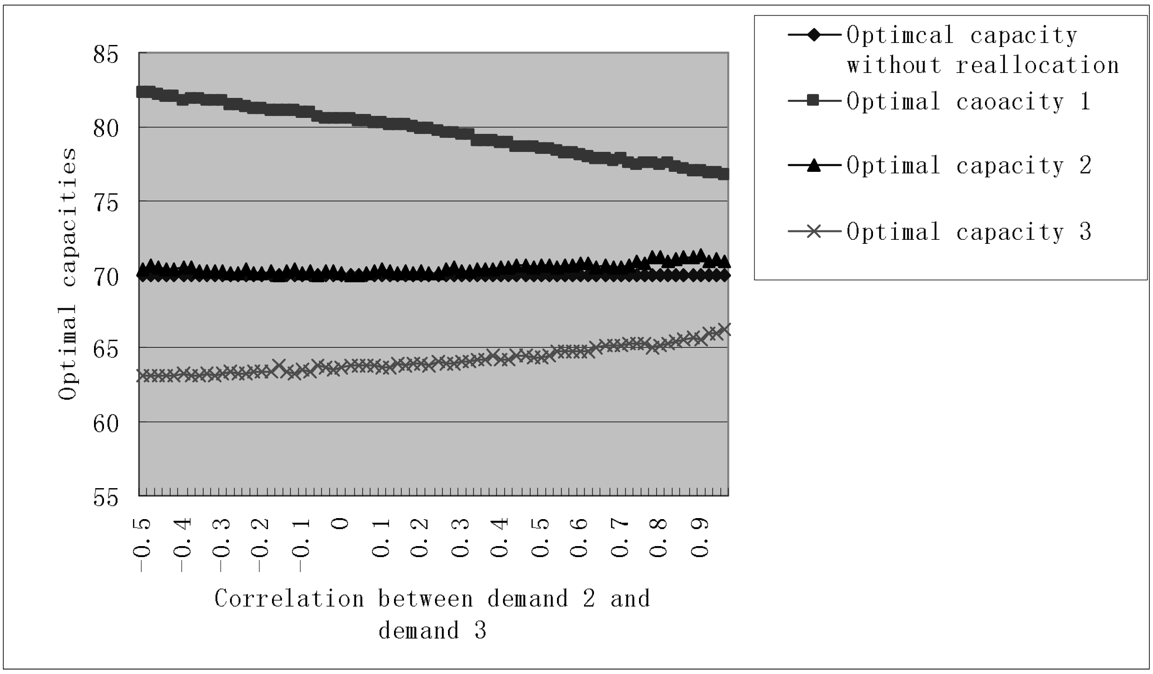

Figure 4.17: Optimal resource capacities as a function of ρ_{23} when $\rho_{12} = \rho_{13} = -0.5$

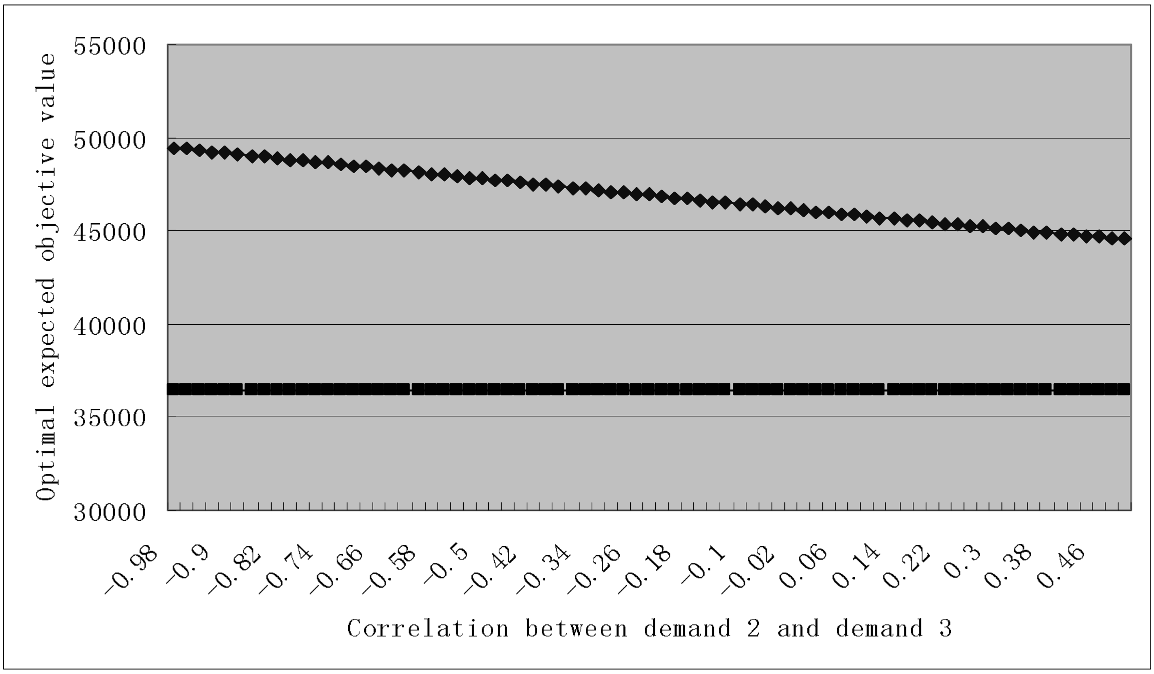

Figure 4.18: Optimal expected profit as a function of ρ_{23} when $\rho_{12} = 0.5,\ \rho_{13} = -0.5$

4.5 Conclusion

We investigated the optimal capacity investment strategies under operational flexibility in this chapter. In our model, investment decision in multiple resources is made before the de-

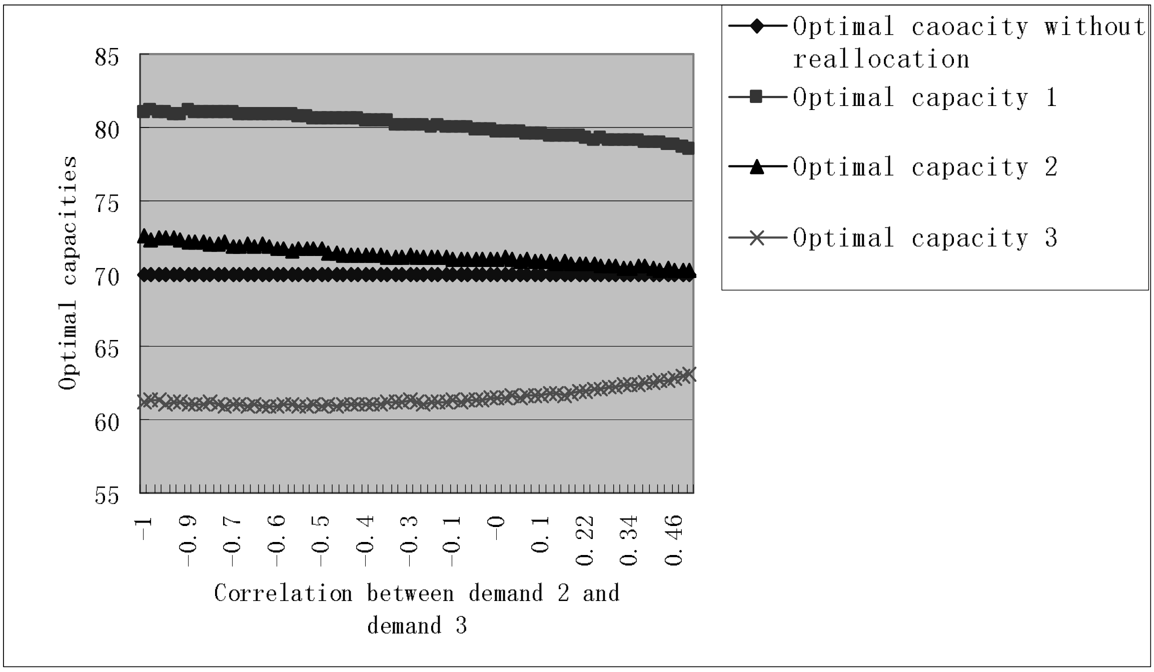

Figure 4.19: Optimal resource capacities as a function of ρ_{23} when $\rho_{12} = 0.5,\ \rho_{13} = -0.5$

mand is known in accuracy. Then, the resource capacities are allocated and priced to satisfy the realized demand. We formulated this problem as a two-stage stochastic programming model. We characterized the structural properties of the stage 2 problem, and showed that it can be solved by using a partitioning method when the size of problem is small. For the larger size problems, we proposed three heuristics to solve it efficiently. Based on the results of the stage 2 problem, we also showed some useful properties of the optimal solution. The numerical results indicated that a significant increase in expected profit can be obtained when the reallocation is allowed in the system. The reallocation is more desirable when the diversity of the system is high.

Chapter 5

Extensions to the Multi-Resource Model

In this chapter, we consider two extensions of the multi-resource model discussed in Chapter 4. In section 5.1, we relax the assumption that each facility has its own market, and investigate a model in which q facilities are used to satisfy demands from m demand markets where $m \neq q$. We discuss how this model can be transformed to the model studied in Chapter 4. In section 5.2, we consider a multi-resource and multi-period model. In this model, the second stage problem has multiple periods during which resource capacities are utilized flexibly to meet random demands from multiple market segments.

5.1 Asymmetric Resource-Market Segment Models

5.1.1 Model Formulation

In this section, we consider a different capacity investment problem which can be easily formulated in a different way from (P_1, P_2). However, after conducting appropriate transition, the formulation of the new model can be transformed to (P_1, P_2) which is relatively easy to analyze.

We consider a firm that manages the capacity investment decisions of q resources in order to satisfy random demands from m different market segments. An illustration of the system is given in Figure 5.1. Here, different market segments may correspond to the customer groups who are willing to pay different prices for the same product or they

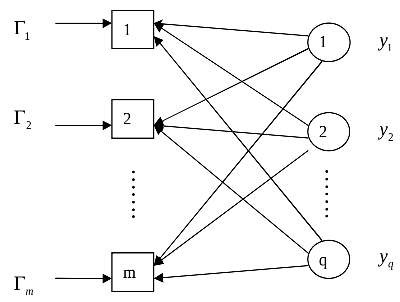

Figure 5.1: An illustration of the resource allocation model

may correspond to the markets for different products. In addition, a resource represents a flexible manufacturing facility or an inventory stock-point for a product at a given location. We assume that the potential market size of demand for market segment i is a nonnegative random variable $\Gamma_i,\ i = 1,...,m$, and each unit is sold at a price p_i in market i. Random market demand D_i for resource i can be controlled by the selling price p_i according to linear function

$$D_i = \Gamma_i - \alpha_i p_i$$

where α_i is the slope of the demand-curve and Γ_i is the intercept. We assume that $\alpha_i > 0$ $\forall i$. When a demand from market segment i is satisfied by using a unit of resource j, a nonnegative unit production/procurement fee of w_{ji} is incurred, $i = 1,2,...,m,\ j = 1,2,...,q$. Each unit of resource j costs $c_j,\ j = 1,2,...,q$.

The capacity investment decision for the q resources is made long before the market potential demand for each resource is realized. The term "capacity investment" decision either corresponds to the quantity of inventory to purchase or the production quantity that should be set at the beginning of a period. Once the market size of demand $\overrightarrow{\Gamma} = (\Gamma_1, \Gamma_2, ..., \Gamma_m)$ is observed, the optimal operation strategy is conducted accordingly, i.e., setting the optimal prices $\overrightarrow{p} = (p_1, p_2, ..., p_m)$ for each market segment and allocating resource capacities optimally to satisfy the demands from m market segments. We denote a realization of $\overrightarrow{\Gamma} = (\Gamma_1, \Gamma_2, ..., \Gamma_m)$ by $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_m)$. The objective is to find the optimal initial capacities of resources $\overrightarrow{y} = (y_1, y_2, ..., y_q)$ in order to maximize the expected profit. Let b_{ji} denote the amount of resource j allocated to satisfy demand from market segment i and B

denote the allocation amount matrix (i.e., $[b_{ji}]$) $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,q$. Then, the model can be formulated as a two-stage optimization problem. Stage 1 problem P_3 makes the investment decisions as follows:

Stage 1 (P_3):

$$\max_{\overrightarrow{y}}[E[\Phi^*(\overrightarrow{y},\overrightarrow{\Gamma})] - \sum_{i=1}^{q} c_i y_i]$$

subject to:

$$y_1,\ y_2,\ ...,\ y_q \geq 0$$

$E[\Phi^*(\overrightarrow{y},\overrightarrow{\Gamma})]$ is the expected revenue for a resource capacity vector $\overrightarrow{y}$, where $\Phi^*(\overrightarrow{y},\overrightarrow{\gamma})$ is the optimal expected profit of the stage 2 problem (P_4), which decides the optimal prices and allocates the resource capacities optimally to fulfill the demand based on an observed market potential vector $\overrightarrow{\gamma}$.

Stage 2 (P_4):

$$\Phi^*(\overrightarrow{y},\overrightarrow{\gamma}) = \max_{B,\overrightarrow{p}} \sum_{i=1}^{m} p_i(\gamma_i - \alpha_i p_i) - \sum_{i=1}^{m}\sum_{j=1}^{q} w_{ji} b_{ji} \tag{5.1}$$

$$s.t.\quad \gamma_i - \alpha_i p_i = \sum_{j=1}^{q} b_{ji} \quad i = 1,2,...,m \tag{5.2}$$

$$\sum_{i=1}^{m} b_{ji} \leq y_j \quad j = 1,2,...,q \tag{5.3}$$

$$b_{ji} \geq 0 \quad j = 1,2,...q,\ i = 1,2,...,m \tag{5.4}$$

$$p_i \geq 0 \quad i = 1,...,m \tag{5.5}$$

The stage 2 model (P_4) maximizes the profit given the resource capacities $\overrightarrow{y}$ and the realized market sizes $\overrightarrow{\gamma}$. In (P_4), constraint (5.2) ensures that the satisfied demand is equal to the total capacity allocated to the market. Constraint (5.3) ensures that the total amount of resource j allocated to the market segments does not exceed the total available capacity $y_j,\ j = 1,...,q$. Constraints (5.4) and (5.5) are the nonnegativity constraints on the resource capacity allocations and prices, respectively. P_4 is a concave problem.

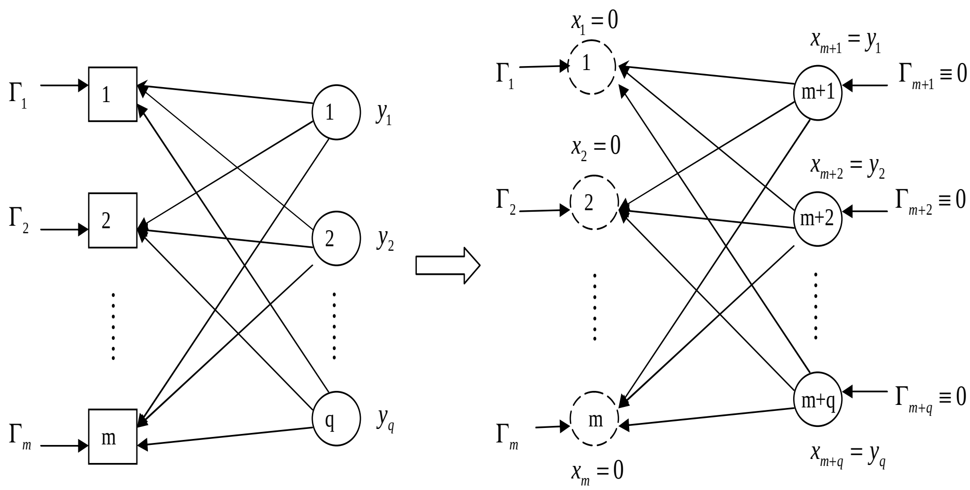

Figure 5.2: Transformation from P_4 to P_2

5.1.2 Transformation of the Model

In this section, we transform the stage 2 model presented above (i.e., P_4) to the stage 2 model discussed in Chapter 4 (i.e., P_2).

Let us set $n = m + q$, and define a vector $\overrightarrow{x} = (x_1, x_2, ..., x_n)$, where $x_i = 0$ when $1 \le i \le m$ and $x_i = y_{i-m}$ when $m+1 \le i \le n$. The demand vector is expanded as $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_m, \gamma_{m+1}, \gamma_{m+2}, ..., \gamma_n)$, where $\gamma_i = 0$ when $i \ge m+1$. The unit reallocation cost between i and j is defined for $i, j = 1, 2, ..., n$ as

$$k_{ji} = \begin{cases} w_{ji} & \text{if } m+1 \le j \le n, 1 \le i \le m; \\ \infty & \text{else.} \end{cases}$$

Figure 5.2 illustrates the transformation. Using parameters ($\overrightarrow{x}$, $\overrightarrow{\gamma}$, k_{ji}, $i, j = 1, 2, ..., n$) and decision variables ($z_{ji}, i = 1, 2, ..., n, j = 1, 2, ..., n$ and p_i, $i = 1, 2, ..., n$) as the input of P_2, let $z^*_{ji}, i = 1, 2, ..., n, j = 1, 2, ..., n$ and p^*_i, $i = 1, 2, ..., n$ be the corresponding optimal solution. The following proposition shows that solving P_2 with the transformed input parameters, we obtain the optimal solution to P_4.

Proposition 21. $(p^*_i,\ i = 1, 2, ..., m,\ z^*_{ji}, i = 1, 2, ..., m, j = m+1, m+2, ..., n)$ is an optimal solution of P_4.

Proof: Based on the definition of k_{ji}, we have $z^*_{ji} \geq 0$ if $m+1 \leq j \leq n, 1 \leq i \leq m$, and $z^*_{ji} = 0$ otherwise.

Since $x_i = \begin{cases} 0 & \text{if } 1 \leq i \leq m; \\ \zeta_i & \text{if } m+1 \leq i \leq n. \end{cases}$ and $\gamma_i = 0$ when $i \geq m+1$, based on constraint (4.2), we have

$$\gamma_i - \alpha_i p_i \leq \sum_{j=m+1}^{n} z_{ji} \quad \text{if } 1 \leq i \leq m$$

$$0 \leq y_i - \sum_{j=1}^{m} z_{ij}, \; p^*_i = 0 \quad \text{if } m+1 \leq i \leq n$$

As a result, P_2 can be written as follows:

$$\begin{aligned}
\Phi^*(\overrightarrow{x}, \overrightarrow{\gamma}) \quad = \quad & \max_{Z, \overrightarrow{p}} \sum_{i=1}^{m} p_i(\gamma_i - \alpha_i p_i) - \sum_{j=m+1}^{n} \sum_{i=1}^{m} w_{ji} z_{ji} \\
s.t. \quad & \gamma_i - \alpha_i p_i \leq \sum_{j=m+1}^{n} z_{ji} \quad i = 1, 2, ..., m \\
& \sum_{i=1}^{m} z_{ji} \leq y_j \quad j = m+1, m+2, ..., n \\
& \gamma_i - \alpha_i p_i \geq 0 \quad i = 1, 2, ..., m \\
& z_{ji} \geq 0 \quad j = m+1, m+2, ...n, \; i = 1, 2, ..., m \\
& p_i \geq 0 \quad i = 1, ..., n
\end{aligned}$$

Note that the above model is the same as P_4 except that the first constraint is an inequality instead of an equality. However, a solution satisfying $\gamma_i - \alpha_i p_i < \sum_{j=m+1}^{n} z_{ji}, i = 1, 2, ..., m$ can not be an optimal solution of the above model because decreasing $\sum_{j=m+1}^{n} z_{ji}$ results in a new feasible solution, which has a larger profit. Therefore, the optimal solution of P_2 must satisfy $\gamma_i - \alpha_i p_i = \sum_{j=m+1}^{n} z_{ji}, i = 1, 2, ..., m$. Consequently, $p^*_i, \; i = 1, 2, ..., m$, obtained from solving the above model is also the optimal selling price of resource i of P_4 and $b^*_{j-m,i} = z^*_{ji}$ when $m+1 \leq j \leq n, 1 \leq i \leq m$. ■

To conclude, the optimal solution of P_4 can always be obtained by transforming the input parameters and solving P_2.

5.1.3 Heuristic Procedure for Solving the Stage 2 Model

In this section, we discuss the implementation of the I-MRP heuristic developed in Chapter 4 for solving P_4. As illustrated in Figure 5.2 and discussed in section 5.1.2, P_4 can be transformed into P_2 with the following properties:

1. Resource $i,\ 1 \leq i \leq m$, can only be a consumer and has zero initial capacity.

2. Resource $i,\ m+1 \leq i \leq m+q$, can only be a supplier.

3. If there exist zero-capacity sets when the algorithm stops, the number of the resources in each zero-capacity set is equal to 1 because the suppliers are not connected (i.e., unit reallocation cost from one supplier to the other is infinity).

As a result of the transformation, both MRP_1 and MRP_2 algorithms can be used for solving P_2. Due to the above special properties, the I-MRP algorithm can be simplified as follows:

1. If $m < 2$, stop the algorithm.

2. Starting from the output of the MRP_1(MRP_2), find all the zero-capacity sets, denoted by $S^i_{zero},\ i = 1, 2, ..., a$. If $a < 1$, stop the algorithm. Otherwise, let $j := 1$.

3. If $j = a+1$, stop the algorithm. Otherwise, choose zero-capacity set S^j_{zero}, where $|S^j_{zero}| = 1$ and denote the only resource in S^j_{zero} as j_{zero}.

4. Calculate the marginal reallocation profit from g to l, for all $1 \leq g \leq m,\ 1 \leq l \leq m,\ g \neq l$ as follows:

$$MRP^{j_{zero}}_{gl} = [(\frac{\gamma_l - 2y_l}{\alpha_l})^+ - (\frac{\gamma_g - 2y_g}{\alpha_g})^+ + k_{j_{zero}g} - k_{j_{zero}l}]\mathbb{1}(y_g > 0)\mathbb{1}(z_{j_{zero}g} > 0)$$

5. Choose the pair of resources, say g^*, l^*, with the largest marginal reallocation profit.

6. If $MRP^{j_{zero}}_{g^*l^*} < \varepsilon$, let $j := j+1$ and return to step 3. Otherwise, let $\Delta_{g^*l^*}$ denote the adjustment of the reallocation amount from g^* to l^* through j_{zero} and conduct the reallocation as follows:

(a) If $(\frac{\gamma_{l^*}-2y_{l^*}-2\min(y_{g^*},z_{j^*_{zero}g^*})}{\alpha_{l^*}})^+ - (\frac{\gamma_{g^*}-2y_{g^*}+2\min(y_{g^*},z_{j^*_{zero}g^*})}{\alpha_{g^*}})^+ + k_{j_{zero}g} - k_{j_{zero}l} > 0$,

$\Delta_{g^*l^*} = \min(y_{g^*}, z_{j^*_{zero}g^*})$.

(b) Else

i. If $\frac{\gamma_{l^*}-2y_{l^*}-2y_{g^*}}{\alpha_{l^*}} - \frac{\gamma_{g^*}}{\alpha_{g^*}} + k_{j_{zero}g} - k_{j_{zero}l} > 0$, $\Delta_{g^*l^*} = y_{g^*}$.

ii. Else if $\gamma_{g^*} + \gamma_{l^*} - 2y_{g^*} - 2y_{l^*} - \alpha_{g^*}(k_{j_{zero}g} - k_{j_{zero}l}) \geq 0$,

$$\Delta_{g^*l^*} = \frac{\alpha_{g^*}\alpha_{l^*}}{2(\alpha_{g^*}+\alpha_{l^*})}(\frac{\gamma_{l^*}-2y_{l^*}}{\alpha_{l^*}} - \frac{\gamma_{g^*}-2y_{g^*}}{\alpha_{g^*}} + k_{j_{zero}g} - k_{j_{zero}l}).$$

iii. Else, $\Delta_{g^*l^*} = -\frac{\gamma_{g^*}-2y_{g^*}-\alpha_{g^*}(k_{j_{zero}g}-k_{j_{zero}l})}{2}$.

7. Update y_{g^*}, y_{l^*}, $z_{j^*_{zero}g^*}$ and $z_{j^*_{zero}l^*}$ with $y_{g^*} - \Delta_{g^*l^*}$, $y_{l^*} + \Delta_{g^*l^*}$, $z_{j^*_{zero}g^*} - \Delta_{g^*l^*}$ and $z_{j^*_{zero}l^*} + \Delta_{g^*l^*}$, respectively. Go to step 4.

If I-MRP algorithm finds at least one profitable reallocation, MRP_1(MRP_2) should be run again. I-MRP and MRP_1(MRP_2) algorithm are run consecutively in this manner until no improvement is obtained in the profit. This process will converge to a feasible solution as shown in Proposition 18.

5.2 Multi-period Pricing Models

In this section, we consider a similar capacity investment problem as in Chapter 4 except that in this case the second stage problem has multiple periods during which resources are utilized flexibly to meet random demands from multiple market segments.

5.2.1 Model Formulation

Suppose that, in the second stage, the resources will be sold in T periods, and the demands in all periods $\gamma_i^1, \gamma_i^2, ..., \gamma_i^T$, $\forall i$ can be observed or predicted precisely before the operational decisions are made. MP_1, the first stage problem, determines the optimal capacities to maximize the expected profit given the random demands for T periods.

Stage 1 (MP_1):

$$\max_{\overrightarrow{x}} \Pi(\overrightarrow{x}) = E[\Phi^*(\overrightarrow{x}, \overrightarrow{\Gamma}^1, \overrightarrow{\Gamma}^2, ..., \overrightarrow{\Gamma}^T)] - \sum_{i=1}^{n} c_i x_i$$

subject to:

$$x_1,\ x_2,\ ...,\ x_n \geq 0$$

$\Phi^*(\overrightarrow{x}, \overrightarrow{\gamma}^1, \overrightarrow{\gamma}^2, ..., \overrightarrow{\gamma}^T)$ is the optimal objective value of the stage 2 problem (MP_2) which allocates the resource capacities and decides the optimal prices of each period optimally to fulfill the demand based on the observed market sizes $\overrightarrow{\gamma}^1, \overrightarrow{\gamma}^2, ..., \overrightarrow{\gamma}^T$. Market demand d_i^t for resource i at the t^{th} period can be controlled by the selling price p_i^t according to linear function $d_i^t = \gamma_i^t - \alpha_i^t p_i^t$ where α_i^t is the slope of the demand-curve and γ_i^t is the intercept. We assume that $\alpha_i^t > 0\ \forall i, t$.

Stage 2 (MP_2):

$$\Phi^*(\overrightarrow{x}, \overrightarrow{\gamma}) = \max_{Z, \overrightarrow{p}} \sum_{i=1}^{n} \sum_{t=1}^{T} p_i^t(\gamma_i - \alpha_i^t p_i^t) - \sum_{i} \sum_{j \neq i} k_{ij} z_{ij} \tag{5.6}$$

$$s.t. \quad \sum_{t=1}^{T} (\gamma_i^t - \alpha_i^t p_i^t) \leq x_i + \sum_{j \neq i} z_{ji} - \sum_{j \neq i} z_{ij} \quad i = 1, ...n \tag{5.7}$$

$$z_{ij} \geq 0 \quad i = 1, ..., n,\ j = 1, ..., n,\ j \neq i \tag{5.8}$$

$$\gamma_i^t - \alpha_i^t p_i^t \geq 0 \quad i = 1, ..., n,\ t = 1, ..., T. \tag{5.9}$$

$$p_i^t \geq 0 \quad i = 1, ..., n,\ t = 1, ..., T \tag{5.10}$$

In (MP_2), constraint (5.7) ensures that the sum of the demands during the T periods for resource i does not exceed the total available capacity.

MP_1 has similar properties to P_1, where MP_2 has a more complicated structure than P_2. Therefore, we focus on analyzing the properties of MP_2 and provide an efficient heuristic method to solve it.

5.2.2 Properties of MP_2

Similar to P_2, MP_2 is a concave problem. Let λ_i, u_{ij}, β_i^t, $i=1,2,...,n$, $j\neq i$, $t=1,...,T$ be the Lagrange multipliers of (5.7), (5.8), and (5.9) respectively, and we use $p_i^{t*}, z_{ij}^*, \lambda_i^*$, u_{ij}^*, β_i^{t*} to denote the corresponding optimal values of the decision variables and the Lagrange multipliers. The optimal solution satisfies the following K-K-T conditions:

$$p_i^{t*} = \frac{\gamma_i^t}{2\alpha_i^t} + \frac{\lambda_i^* - \beta_i^{t*}}{2} \quad \forall i,\ \forall t \tag{5.11}$$

$$\lambda_j^* - \lambda_i^* = k_{ij} - u_{ij}^* \quad \forall j \neq i \tag{5.12}$$

$$\lambda_i^* \left(\sum_{t=1}^{T}(-\gamma_i^t + \alpha_i^t p_i^{t*}) + x_i + \sum_{j\neq i} z_{ji}^* - \sum_{j\neq i} z_{ij}^*\right) = 0 \quad \forall i \tag{5.13}$$

$$u_{ij}^* z_{ij}^* = 0 \quad \forall j \neq i \tag{5.14}$$

$$\beta_i^{t*}(\gamma_i^t - \alpha_i^t p_i^{t*}) = 0 \quad \forall i,\ \forall t \tag{5.15}$$

$$\beta_i^* \geq 0, \lambda_i^* \geq 0, u_{ij}^* \geq 0 \quad \forall i, j \tag{5.16}$$

Note that we omit the last constraint (5.10) in MP_2 because p_i is always nonnegative in an optimal solution since the demand for resource i is nonnegative. Based on the K-K-T conditions, Lemma 7 still holds.

Proposition 22. In an optimal solution of MP_2, let $y_i^* = x_i + \sum_{j\neq i} z_{ji}^* - \sum_{j\neq i} z_{ij}^*$.

1. If $y_i^* = 0$, $p_i^{t*} = \frac{\gamma_i^t}{\alpha_i^t}$, $\forall t$.

2. If $y_i^* \geq \frac{\sum_{t=1}^T \gamma_i^t}{2}$, $p_i^{t*} = \frac{\gamma_i^t}{2\alpha_i^t}$, $\forall t$.

3. If $0 < y_i^* < \frac{\sum_{t=1}^T \gamma_i^t}{2}$, let us order the periods based on the values of $\frac{\gamma_i^t}{\alpha_i^t}$, $t=1,...,T$ from the largest to the smallest, and obtain a set of indices $\{t_1, t_2, ..., t_T\}$, i.e.,

$$\frac{\gamma_i^{t_1}}{\alpha_i^{t_1}} \geq \frac{\gamma_i^{t_2}}{\alpha_i^{t_2}} \geq ... \geq \frac{\gamma_i^{t_T}}{\alpha_i^{t_T}}.$$

An unique integer d which satisfies $1 \leq d \leq T$ can be found, and

$$p_i^{t_l *} = \begin{cases} \frac{\gamma_i^{t_l}}{2\alpha_i^{t_l}} + \frac{\sum_{j=1}^d \gamma_i^{t_j} - 2y_i^*}{2\sum_{j=1}^d \alpha_i^{t_j}} & \text{if } l = 1,2,...,d; \\ \frac{\gamma_i^{t_l}}{\alpha_i^{t_l}} & \text{if } l = d+1,...,T. \end{cases}$$

Proof:

In an optimal solution, all decision variables and Lagrange multipliers are nonnegative. We first show that

$$\lambda_i^* - \beta_i^{t*} \geq 0 \; \forall i, t \tag{5.17}$$

If $\lambda_i^* - \beta_i^{t*} < 0$, $\beta_i^{t*} > 0$. Based on (5.15), $p_i^{t*} = \frac{\gamma_i^t}{\alpha_i^t}$, and based on condition (5.11), $\lambda_i^* - \beta_i^{t*} = \frac{\gamma_i^t}{\alpha_i^t} \geq 0$, which is a contradiction.

1. If $y_i^* = 0$, based on constraints (5.7) and (5.9), $p_i^{t*} = \frac{\gamma_i^t}{\alpha_i^t}$, $\forall t$.

2. Suppose $y_i^* \geq \frac{\sum_{t=1}^T \gamma_i^t}{2}$. If $\exists t$, $p_i^{t*} \neq \frac{\gamma_i^t}{2\alpha_i^t}$. Based on condition (5.11), $\lambda_i^* \neq \beta_i^{t*}$. Based on (5.17), $\frac{\lambda_i^* - \beta_i^{t*}}{2} > 0 \Rightarrow \lambda_i^* > 0$. When $\lambda_i^* > 0$, based on condition (5.13),

$$\sum_{t=1}^T (\gamma_i^t - \alpha_i^t p_i^{t*}) = y_i^*. \tag{5.18}$$

   Let us plug (5.11) into (5.18). We have, $\sum_{t=1}^T \gamma_i^t - 2y_i^* = \sum_{t=1}^T \alpha_i^t(\lambda_i^* - \beta_i^{t*})$. Since $y_i^* \geq \frac{\sum_{t=1}^T \gamma_i^t}{2}$, $\sum_{t=1}^T \alpha_i^t(\lambda_i^* - \beta_i^{t*}) \leq 0$. Based on (5.17), $\lambda_i^* = 0$. It is a contradiction. Therefore, $p_i^{t*} = \frac{\gamma_i^t}{2\alpha_i^t}$, $\forall t$. Note that in this case $\lambda_i^* = 0$.

3. If $0 < y_i^* < \frac{\sum_{t=1}^T \gamma_i^t}{2}$. First, we show that $\exists t$, $\beta_i^{t*} = 0$. If $\beta_i^{t*} > 0$, $\forall t$, based on (5.15) and (5.17), we have $p_i^{t*} = \frac{\gamma_i^t}{\alpha_i^t}$, and $\lambda_i > 0$. Based on (5.13), $y_i^* = 0$, which is a contradiction. Now we show, for the ordered indexes $\{t_1, t_2, ..., t_T\}$, we have

   (a) if $\beta_i^{t_l*} = 0$, $\beta_i^{t_j*} = 0$, $\;j = 1, 2, ..., l$

   (b) if $\beta_i^{t_l*} > 0$, $\beta_i^{t_j*} > 0$, $\;j = l+1, l+2, ..., T$.

   When $\beta_i^{t_l*} = 0$, based on (5.11), $p_i^{t_l*} = \frac{\gamma_i^{t_l}}{2\alpha_i^{t_l}} + \frac{\lambda_i^*}{2}$. Since $\gamma_i^l - \alpha_i^{t_l} p_i^{t_l*} \geq 0$, $\lambda_i^* \leq \frac{\gamma_i^{t_l}}{\alpha_i^{t_l}}$.

   If $\beta_i^{t_j*} > 0$, $j \in \{1, 2, ..., l-1\}$, based on (5.15), we have $p_i^{t_j*} = \frac{\gamma_i^{t_j}}{\alpha_i^{t_j}}$ and based on (5.11),

$$\lambda_i^* = \frac{\gamma_i^{t_j}}{\alpha_i^{t_j}} + \beta_i^{t_j*} > \frac{\gamma_i^{t_l}}{\alpha_i^{t_l}},$$

   which is a contradiction. Therefore, $\beta_i^{t_j*} = 0$, $\;j = 1, 2, ..., l$

When $\beta_i^{t_l*} > 0$, if $\exists\beta_i^{t_j*} = 0,\ j \in \{l+1, l+2, ..., T\}$, we have $\beta_i^{t_l*} = 0$ because $l < j$, which is a contradiction. Therefore, $\beta_i^{t_j*} > 0,\ \ j = l+1, l+2, ..., T$.

Let d be an positive integer number which satisfies $\beta_i^{t_j*} = 0$ if $l \leq d$ and $\beta_i^{t_j*} > 0$ if $j > d$. If $d = T$, there does not exist any integer $1 \leq l \leq T$ satisfies $\beta_i^{t_j*} > 0$. Since we have shown $\exists t,\ \beta_i^{t*} = 0$, so, $d \geq 1$.

Next, we show that $\lambda_i^* > 0$. If $\lambda_i = 0$, based on (5.17), $\beta_i^{t*} = 0,\ \forall t$. Based on (5.11), we have $p_i^{t_j*} = \frac{\gamma_i^{t_j}}{2\alpha_i^{t_j}}$ and based on (5.7), $\frac{\sum_{t=1}^{T}\gamma_i^t}{2} \leq y_i^*$. This is a contradiction. Therefore, $\lambda_i^* > 0$.

Since $\lambda_i^* > 0$, based on (5.13),

$$\begin{aligned}
y_i^* &= \sum_{t=1}^{T}(\gamma_i^t - \alpha_i^t p_i^{t*}) \\
&= \sum_{j=1}^{d}(\gamma_i^{t_j} - \alpha_i^{t_j} p_i^{t_j*}) + \sum_{j=d+1}^{T}(\gamma_i^{t_j} - \alpha_i^{t_j} p_i^{t_j*}) \\
&= \sum_{j=1}^{d}(\frac{\gamma_i^{t_j} - \alpha_i^{t_j}\lambda_i^*}{2}).
\end{aligned}$$

We have

$$\lambda_i^* = \frac{\sum_{j=1}^{d}\gamma_i^{t_j} - 2y_i^*}{\sum_{j=1}^{d}\alpha_i^{t_j}},$$

and

$$p_i^{t_l*} = \begin{cases} \frac{\gamma_i^{t_l}}{2\alpha_i^{t_l}} + \frac{\sum_{j=1}^{d}\gamma_i^{t_j} - 2y_i^*}{2\sum_{j=1}^{d}\alpha_i^{t_j}} & \text{if } l = 1, 2, ..., d; \\ \frac{\gamma_i^{t_l}}{\alpha_i^{t_l}} & \text{if } l = d+1, ..., T. \end{cases} \quad \blacksquare$$

Note the procedure to find the value of d is similar to the procedure to determine set T presented in Appendix A.

Proposition 22 demonstrates how the optimal selling prices are related to the optimal reallocation amounts in the optimal solution of MP_2. If we can find the optimal reallocation amounts, then the solution of MP_2 is achieved.

5.2.3 Heuristic for Solving MP_2

In this section, we propose a heuristic algorithm to solve MP_2. The idea is the same as the heuristics developed in Chapter 4, i.e., make the reallocation if there is positive reallocation profit until we cannot find any profitable reallocation. The heuristic algorithm is given as follows:

Algorithm 3:

Let ε be a small positive real number.

1. Start with a solution without reallocation among the resources. $\overrightarrow{y} = \overrightarrow{x}$, $z_{ij} = 0, i, j = 1, 2, ..., n$. Order the periods based on the values of $\frac{\gamma_i^t}{\alpha_i^t}$, $t = 1, ..., T$ from the largest to the smallest for every resource $i, i = 1, 2, ..., n$, and obtain n sets of indices $\{t_1^i, t_2^i, ..., t_T^i\}$, i.e.,

$$\frac{\gamma_i^{t_1^i}}{\alpha_i^{t_1^i}} \geq \frac{\gamma_i^{t_2^i}}{\alpha_i^{t_2^i}} \geq ... \geq \frac{\gamma_i^{t_T^i}}{\alpha_i^{t_T^i}}, \quad i = 1, 2, ..., n.$$

2. Compute the marginal profit of resource i as follows:

   (a) If $y_i \geq \frac{\sum_{t=1}^T \gamma_i^t}{2}$, $\lambda_i = 0$.

   (b) If $0 < y_i < \frac{\sum_{t=1}^T \gamma_i^t}{2}$, let $d = 1$.

      i. If $d = T$ or

$$d < t \ \ and \ \ \frac{\gamma_i^{t_d^i}}{\alpha_i^{t_d^i}} > \frac{\sum_{j=1}^d \gamma_i^{t_j^i} - 2y_i}{\sum_{j=1}^d \alpha_i^{t_j^i}} \geq \frac{\gamma_i^{t_{d+1}^i}}{\alpha_i^{t_{d+1}^i}},$$

      let $\lambda_i = \frac{\sum_{j=1}^d \gamma_i^{t_j^i} - 2y_i}{\sum_{j=1}^d \alpha_i^{t_j^i}}$. Record d and go to step 3.

      ii. Otherwise, let $d = d + 1$ and go to step $2(b)(i)$.

   (c) If $y_i^* = 0$, let $\lambda_i = \frac{\gamma_i^{t_1}}{\alpha_i^{t_1}}$.

3. Compute the reallocation profit from resource i to resource j, denoted by MRP_{ij}, for all $i \neq j$ as follows:

   (a) If $z_{ji} = 0$, $MRP_{ij} = [\lambda_j - \lambda_i - k_{ij}]\mathbb{1}(y_i > 0)$.

(b) If $z_{ji} > 0$, $MRP_{ij} = -[\lambda_i - \lambda_j - k_{ji}]\mathbb{1}(y_i > 0)$.

4. Choose the pair of the resources, say i^*, j^*, with the largest marginal reallocation profit.

5. If $MRP_{i^*j^*} \leq \varepsilon$, stop. Compute the output as follows:

   (a) $\widehat{z}^*_{ij} = z_{ij},\ i,j = 1,2,...,n$

   (b) i. If $y^*_i = 0$, $\widehat{p}^{t*}_i = \frac{\gamma^t_i}{\alpha^t_i}$, $\forall t$.

   ii. If $y^*_i \geq \frac{\sum_{t=1}^T \gamma^t_i}{2}$, $\widehat{p}^{t*}_i = \frac{\gamma^t_i}{2\alpha^t_i}$, $\forall t$.

   iii. If $0 < y^*_i < \frac{\sum_{t=1}^T \gamma^t_i}{2}$,

$$\widehat{p}^{t_l*}_i = \begin{cases} \frac{\gamma^{t_l}_i}{2\alpha^{t_l}_i} + \frac{\sum_{j=1}^d \gamma^{t_j}_i - 2y^*_i}{2\sum_{j=1}^d \alpha^{t_j}_i} & \text{if } l = 1,2,...,d; \\ \frac{\gamma^{t_l}_i}{\alpha^{t_l}_i} & \text{if } l = d+1,...,T. \end{cases}$$

   (c) $\widehat{T}^* = \sum_{i=1}^n \sum_{t=1}^T \widehat{p}^{t*}_i(\gamma_i - \alpha_i \widehat{p}^{t*}_i) + \sum_{i=1}^n \sum_{j\neq i} k_{ij}\widehat{z}^*_{ij}$

   Otherwise, go to the next step.

6. Reallocate the resource from i^* to j^* based on the following (binary search):

   Let $\Delta_{i^*j^*}$ denote the adjustment of the reallocation amount from i^* to j^*.

   (a) If $z_{j^*i^*} = 0$, and after reallocating amount of y_{i^*} from i^* to j^*, $MRP_{i^*j^*} > 0$, let $\Delta_{i^*j^*} = y_{i^*}$. Otherwise, search and obtain $\Delta_{i^*j^*} \in (0, y_{i^*})$, which makes $MRP_{i^*j^*} = 0$ after reallocating $\Delta_{i^*j^*}$ from i^* to j^*.

   (b) If $z_{j^*i^*} > 0$,

   i. If after reallocating $\min(y_{i^*}, z_{j^*i^*})$ from i^* to j^*, $MRP_{i^*j^*} > 0$, let $\Delta_{i^*j^*} = \min(y_{i^*}, z_{j^*i^*})$.

   ii. Else if after reallocating $\max(y_{i^*}, z_{j^*i^*})$ from i^* to j^*, $MRP_{i^*j^*} > 0$, let $\Delta_{i^*j^*} = \max(y_{i^*}, z_{j^*i^*})$.

   iii. Else, search and obtain $\Delta_{i^*j^*} \in (0, y_{i^*})$, which makes $MRP_{i^*j^*} = 0$ after reallocating $\Delta_{i^*j^*}$ from i^* to j^*.

Update y_{i^*}, y_{j^*} and $z_{i^*j^*}$ with $y_{i^*} - \Delta_{i^*j^*}$, $y_{j^*} + \Delta_{i^*j^*}$ and $z_{i^*j^*} + \Delta_{i^*j^*}$ respectively, and go to step 2.

Step 1 of the above algorithm orders the periods for each resource. In step 2, marginal shadow prices are computed based on Proposition 22. In step 3, we choose the pair of resources i^* and j^* which have the maximum marginal reallocation profit. Step 4 stops the algorithm if the approximated optimal solution is obtained. In step 5, we calculate the amount of the adjustment of reallocation between the pair of the resources to eliminate the reallocation profit between them by binary search. The complexity of the algorithm is $O(n^3 T \ln^2 M)$.

Similar to MRP_1 and MRP_2 algorithms discussed in Chapter 4, this algorithm converges to an feasible solution. The proof is the same as the proof for MRP_1 given in Proposition 17.

5.3 Conclusion and Future work

In this chapter, we investigated two extensions of the multi-resource model discussed in Chapter 4. The first model contains q facilities which are used to satisfy demands from m demand markets where $m \neq q$. We show that this model can be transformed to the model studied in chapter 4 and solved by similar MRP_1, MRP_2 and I-MRP heuristics. Next, we considered a multi-resource and multi-period model. We presented the properties of the stage 2 problem of this model, and proposed a heuristic method to solve it.

A more realistic version of this multi-period problem is that, at the beginning of the second stage, only the demand of the first period can be observed precisely. The pricing and reallocation decisions need to be made based on the partial information. As time goes on, the demand information of other periods becomes observable and the decisions on pricing and reallocation are made accordingly. This decision process involves dynamic programming and huge computational complexity. Finding an appropriate model and heuristic methods to solve the problem efficiently remains as a future research question.

Appendix A

Lemma 3. *If a,b,c,d are real numbers satisfying*

$$c > 0,\ d > 0,\ \frac{a}{c} \geq \frac{b}{d},$$

then

$$\frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d}.$$

Proof:

$$c > 0,\ d > 0$$

$$\Longrightarrow \frac{a}{c} \geq \frac{b}{d} \Longleftrightarrow ad \geq bc$$

$$\Longrightarrow ad + ac \geq bc + ac,\ \ ad + bd \geq bc + bd$$

$$\Longrightarrow \frac{a}{c} \geq \frac{a+b}{c+d} \geq \frac{b}{d} \qquad \blacksquare$$

Procedure to determine set T

When

$$\sum_{j\in S}(\gamma_j - \alpha_j \sum_{(f,h)\in S^{arc}_{lj}} v_{fh}k_{fh})^+ > \sum_{j\in S} 2x_j, \quad \lambda_l > 0.$$

Let us order resources based on the values of $\frac{\gamma_j}{\alpha_j} - \sum_{(f,h)\in S^{arc}_{lj}} v_{fh}k_{fh}, \quad j \in S$ from the largest to the smallest, and obtain a set of indices $c(i) \in S$, i.e., resource $c(i)$ has the ith position in the ordered sequence. If $\exists i \in S$ with $\beta_{c(i)} = 0$, according to (4.14),

$$\frac{\gamma_{c(1)}}{\alpha_{c(1)}} - \sum_{(f,h)\in S^{arc}_{lc(1)}} v_{fh}k_{fh} - \lambda_l \geq \frac{\gamma_{c(2)}}{\alpha_{c(2)}} - \sum_{(f,h)\in S^{arc}_{lc(2)}} v_{fh}k_{fh} - \lambda_l$$

$$\geq \ldots \geq \frac{\gamma_{c(i)}}{\alpha_{c(i)}} - \sum_{(f,h)\in S^{arc}_{lc(i)}} v_{fh}k_{fh} - \lambda_l \geq 0$$

$$\Longrightarrow \quad \beta_{c(1)} = \beta_{c(2)} = \ldots\beta_{c(i-1)} = 0$$

Let d denote the position of the last resource i with $\beta_i = 0$ in the sequence $c(1), c(2), ..., c(n_s)$, i.e., $\beta_{c(1)} = \beta_{c(2)} = ... = \beta_{c(d)} = 0$, and $\beta_{c(d+1)} = \beta_{c(d+2)} = ...\beta_{c(n_s)} > 0$.

Then $T \neq \emptyset$ and $\beta_{c(1)} = 0$.

Now, we discuss the following two cases: $d = n_s$ and $1 \leq d < n_s$.

If $d = n_s$, it means $T = S$, and

$$\lambda_l = \frac{\sum_{j \in S}(\gamma_j - \alpha_j \sum_{(f,h) \in S^{arc}_{lj}} v_{fh} k_{fh}) - 2\sum_{j \in S} x_j}{\sum_{j \in S} \alpha_j}.$$

According to (4.14),

$$0 \leq \frac{\sum_{j \in S}(\gamma_j - \alpha_j \sum_{(f,h) \in S^{arc}_{lj}} v_{fh} k_{fh}) - 2\sum_{j \in S} x_j}{\sum_{j \in S} \alpha_j} \leq \frac{\gamma_{c(n_s)}}{\alpha_{c(n_s)}} - \sum_{(f,h) \in S^{arc}_{lc(n_s)}} v_{fh} k_{fh}$$

must be satisfied.

If $1 \leq d < n_s$,

$$\lambda_l = \frac{\sum_{j=1}^{d}(\gamma_{c(j)} - \alpha_{c(j)} \sum_{(f,h) \in S^{arc}_{lc(j)}} v_{fh} k_{fh}) - 2\sum_{j \in S} x_j}{\sum_{j=1}^{d} \alpha_{c(j)}} \tag{5.19}$$

and

$$\frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h) \in S^{arc}_{lc(d+1)}} v_{fh} k_{fh} < \frac{\sum_{j=1}^{d}(\gamma_{c(j)} - \alpha_{c(j)} \sum_{(f,h) \in S^{arc}_{lc(j)}} v_{fh} k_{fh}) - 2\sum_{j \in S} x_j}{\sum_{j=1}^{d} \alpha_{c(j)}}$$

$$\leq \frac{\gamma_{c(d)}}{\alpha_{c(d)}} - \sum_{(f,h) \in S^{arc}_{lc(d)}} v_{fh} k_{fh} \tag{5.20}$$

must be satisfied.

The value of d can be determined by increasing d from 1 to n_s until (5.20) is satisfied. Now, we show that d can be found in this way and it is unique. In equation (5.19), let us increase d from 1 to n_s. At some point, λ_l becomes positive because

$$\sum_{j \in S}(\gamma_j - \alpha_j \sum_{(f,h) \in S^{arc}_{lj}} v_{fh} k_{fh})^+ > \sum_{j \in S} 2x_j.$$

If at this point,

$$\lambda_l > \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh},$$

then we stop increasing d. Otherwise we increase d until

$$\lambda_l > \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh}.$$

If

$$\lambda_l \leq \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh},$$

even when d is increased to $n_s - 1$, then $\beta_j = 0 \ \ \forall j \in S$. This follows from Lemma (3), which implies that if

$$\frac{\sum_{j=1}^{n_s-1}(\gamma_{c(j)} - \alpha_{c(j)}\sum_{(f,h)\in S^{arc}_{lc(j)}} v_{fh}k_{fh}) - 2\sum_{j\in S} x_j}{\sum_{j=1}^{n_s-1}\alpha_{c(j)}} \leq \frac{\gamma_{c(n_s)}}{\alpha_{c(n_s)}} - \sum_{(f,h)\in S^{arc}_{lc(n_s)}} v_{fh}k_{fh}$$

then

$$\frac{\sum_{j=1}^{n_s}(\gamma_{c(j)} - \alpha_{c(j)}\sum_{(f,h)\in S^{arc}_{lc(j)}} v_{fh}k_{fh}) - 2\sum_{j\in S} x_j}{\sum_{j=1}^{n_s}\alpha_{c(j)}} \leq \frac{\gamma_{c(n_s)}}{\alpha_{c(n_s)}} - \sum_{(f,h)\in S^{arc}_{lc(n_s)}} v_{fh}k_{fh}$$

Consider the case that d is increased until

$$\lambda_l > \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh}.$$

Since

$$\lambda_l \leq \frac{\gamma_{c(d)}}{\alpha_{c(d)}} - \sum_{(f,h)\in S^{arc}_{lc(d)}} v_{fh}k_{fh},$$

condition (5.20) is satisfied, and the value of d is computed. Moreover d is unique, because if we continue increasing d to $d+1$, since

$$\frac{\sum_{j=1}^{d}(\gamma_{c(j)} - \alpha_{c(j)}\sum_{(f,h)\in S^{arc}_{lc(j)}} v_{fh}k_{fh}) - 2\sum_{j\in S} x_j}{\sum_{j=1}^{d}\alpha_{c(j)}} > \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh},$$

according to Lemma (3), we have

$$\frac{\sum_{j=1}^{d+1}(\gamma_{c(j)} - \alpha_{c(j)}\sum_{(f,h)\in S^{arc}_{lc(j)}} v_{fh}k_{fh}) - 2\sum_{j\in S} x_j}{\sum_{j=1}^{d+1}\alpha_{c(j)}} > \frac{\gamma_{c(d+1)}}{\alpha_{c(d+1)}} - \sum_{(f,h)\in S^{arc}_{lc(d+1)}} v_{fh}k_{fh}.$$

Hence, condition (5.20) cannot be satisfied. By this argument, we can show that condition (5.20) cannot be satisfied by increasing d after the first point which satisfies condition (5.20). ■

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