Rayleigh Wavetrains in Nonlinear Elasticity

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1 Introduction

The initial formulation of the evolution equation for the leading order approximation in nonlinear elasticity in the weakly nonlinear regime goes back to [Lar83]. Moreover, Lardner identified the appropriate scaling for nonlinear effects to appear in the leading order approximation, which in our case is ε^2. This evolution equation is termed the amplitude equation. Hunter derived the analogous results for first order hyperbolic systems in his paper [Hun89]. The amplitude equation for nonlinear elasticity turns out to be a nonlocal Burgers type equation, and the argument to solve it goes back to Benzoni-Gavage. We want to stress that all of this body of work is primarily devoted to constructing *approximate* leading order solutions to equations, not the exact solution itself. So one of the main goals in geometric optics is to show that the constructed approximate solution is close to the exact solution and that the exact solution exists on a time interval independent of ε. To make the notion of close precise, one typically takes a limit of the form:

$$\lim_{\varepsilon \to 0} \varepsilon^{-\alpha} ||u^{\varepsilon}_{app} - u^{\varepsilon}||_{L^{\infty}} \tag{1}$$

where ε is a small parameter corresponding to the wavelength in $(0, 1]$, u^{ε}_{app} is the approximate solution and u^{ε} is the exact solution, and α is a positive number chosen so that both u^{ε}_{app} and u^{ε} are both $O(\varepsilon^{\alpha})$. If this limit is 0, then the difference between $u^{\varepsilon}_{app} - u^{\varepsilon}$ is a higher order remainder term.

In this paper we construct arbitrarily high order approximate Rayleigh wavetrains in the context of nonlinear elasticity. More specifically, we are analyzing a Saint Venant-Kirchhoff material on a half plane. The equation for the deformation of such a material is given by:

$$\partial_t^2 \phi + \nabla \cdot (\nabla \phi \sigma(\nabla \phi)) = 0$$

on $y > 0$, and satisfying traction boundary conditions on $y = 0$:

$$\nabla \phi \sigma(\nabla \phi) \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where f, g are smooth functions, periodic in $\frac{x-ct}{\varepsilon}$ for some c. $\sigma(\nabla \phi)$ denotes the stress, and is given by $\sigma(\nabla \phi) = \lambda \operatorname{Tr} EI + 2\mu E$, where E is the strain given by $E = \frac{1}{2}({}^t\nabla \phi \nabla \phi - I)$. λ and μ are the Lamé constants. We supplement this with the initial condition $\phi(0, x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$, i.e. there is no initial deformation.

In order to have surface waves in nonlinear elasticity, we choose $c < 1$ such that the Lopatinski matrix, which is discussed in more detail in the discussion after (34), given by:

$$\mathcal{B}_{Lop} := \begin{pmatrix} 2 - c^2 & 2\omega_2 \\ 2\omega_1 & c^2 - 2 \end{pmatrix}$$

is singular, where c is a Rayleigh frequency, $r > 2$ is a known constant determined by the Lamé constants, and $\omega_1^2 = c^2 - 1$ and $\omega_2^2 = \frac{c^2}{r} - 1$, in each case ω_j are pure imaginary numbers with positive imaginary part. The ω_j are the eigenvalues of a linear operator roughly corresponding to the linear part of the equations of nonlinear elasticity. This operator is discussed in more details in Section 4. From this, the kernel can be calculated and it is spanned by $\begin{pmatrix} \omega_2 \\ -q \end{pmatrix}$ where $q^2 = -\omega_1\omega_2$ and $q > 0$. We set $\beta := (-c, 1)$. A useful relation between q and c can be derived from the following argument. Since $\mathcal{B}_{Lop}$ is singular, its determinant is 0, and therefore we have the following equality:

$$(2 - c^2)(c^2 - 2) - 4\omega_1\omega_2 = 0$$

Upon a slight rearrangement of the terms and substituting in $q^2 = -\omega_1\omega_2$, we have that:

$$(2 - c^2)^2 = 4q^2.$$

Since $c < 1$, $2 - c^2 > 0$ and $q > 0$ shows that:

$$2 - c^2 = 2q. \tag{2}$$

These particular wavetrains are surface waves arising from the breakdown of the uniform Lopatinksi condition, which fails in a controlled manner. In our case we seek surface waves of finite energy, which arise from the failure of the Lopatinski condition. This is explained in more details in Chapter 7 of [BGS07]. One of the nice properties of wavetrains is that an arbitrary number of correctors can be constructed, and the oscillatory part of each corrector has an exponential decay in the fast variable $Y = \frac{y}{\varepsilon}$ where y is the variable normal to the boundary. As a remark, these correctors are not fully localized on the boundary in that part of them do not decay exponentially away from the boundary. This is contrasted with the pulse case where only one corrector can be constructed. The error analysis, that is showing that a nearby exact solution exists, will be carried out in a future work.

One of the most closely related works is [CW16]. In that paper, both approximate and exact pulses solutions were constructed for a Saint-Venant Kirchhoff material. In addition, they showed that the approximate solution was close, in the sense described above, to the exact solution. Their techniques can also be slightly modified to give an approximate solution in the wavetrain case and show that is close to the exact solution as well. However, the technique used in the error analysis in the [CW16] paper only work for 2 dimensional problems because the (singular) Kreiss symmetrizer has not been constructed in 3 dimensions[1]. The issues with the Kreiss symmetrizer are described in more details in [CGW14]. This paper is the first step in using an alternative approach, based on a theorem due to Guès, that requires arbitrarily high order approximate solutions. Moreover, this technique also seems likely to work in three spatial dimensions. Unfortunately, the Guès method cannot be applied to pulses because one needs a high order approximate solution which cannot be constructed for pulses. The fact that only one corrector can be constructed for pulses in this particular problem turns out to be fairly typical behavior as discussed in [CW13]. The reason only one corrector can be constructed is because, in the pulse case, there are integrals over θ in the non-compact set $\mathbb{R}$, as opposed to wavetrain case where θ is integrated over the compact set $\mathbb{T}$. For pulses, the integrals over θ induce growth in the variable θ, which makes it difficult to find decaying solutions.

Our method is similar to the one used in the first chapter of A. Marcou's thesis [Mar10]. The first chapter in her thesis is focused on first order hyperbolic conservation laws, and we modify the method used therein. There are two major differences between our problem and the conservation laws. The first is that the system of conservation laws is first order in space and time and also has a more complicated nonlinearity, whereas our model of nonlinear elasticity is a second order system with a cubic nonlinearity. In addition, her boundary conditions are of the form $Cu|_{y=0} = 0$, where C is a constant matrix, and we have nonlinear boundary conditions containing first order derivatives of the solution. The second major difference is that the right hand side for the interior equations in her case, that is the terms arising from the nonlinearities, are all in the space S. This space is given by $S = \underline{S} \oplus S^*$, where $\underline{S} = H^\infty([0,T] \times \mathbb{R} \times \mathbb{R}^+)$ is the usual Sobolev space and $S^* = H^\infty([0,T] \times \mathbb{R} \times \mathbb{T} \times \mathbb{R}^+)$ with exponential decay in the last variable. In our case, our nonlinearities are generically not in S, which introduces some complications in solving for the approximate solution. The second chapter of Marcou's thesis is also relevant to our work. There she analyzes the leading order term and its first corrector in a simplified version of nonlinear elasticity. In her case, the nonlinearity is a very simple quadratic function and we have a very lengthy cubic polynomial, which has both quadratic and cubic terms.

In order to construct our approximate solution, we suppose it has an asymptotic expansion of the form $U^\varepsilon_{app}(t,x,y) = \sum_{k=2}^N \varepsilon^k U_k(t,x,y,\frac{x-ct}{\varepsilon},\frac{y}{\varepsilon})$ with each U_k in S. Plugging in this asymptotic expansion yields a series of linear partial differential equations, denoted the "cascade". The biggest difficulty in working with the space S is that it is *not* closed under products. This is a problem because the nonlinearity in the Saint Venant-Kirchhoff material is a polynomial, and so one ends up with products of two or three elements of S. To correct this, a Taylor series approximation to the nonlinearity is implemented. After modification, we show that one can take N to be any integer greater than or equal to 2 and solve for each $U_k \in S$. The Taylor approximation introduces new errors to our approximate solution, but we show that the remainder term from this Taylor series can be "absorbed" into the preexisting error term from our asymptotic expansion. More specifically, there is a sequence of profiles $U_k \in S$, $k = 2, ..., N$, such that the approximate solution:

$$U^\varepsilon_{app} = \sum_{k=2}^N \varepsilon^k U_k$$

[1]There are issues involving eigenvalues of variable multiplicity

satisfies

$$\partial_t^2 U^\varepsilon_{app} + \nabla\cdot(L(\nabla U^\varepsilon_{app}) + Q(\nabla U^\varepsilon_{app}) + C(\nabla U^\varepsilon_{app})) = \varepsilon^{N-1} E'_N$$

on $y > 0$ and $y = 0$

$$-L_2(\nabla U^\varepsilon_{app}) - Q_2(\nabla U^\varepsilon_{app}) - C_2(\nabla U^\varepsilon_{app}) = \varepsilon^N e_N$$

for two H^∞ functions E'_N and e_N. This discussed in more details with Theorem 9.3.

2 Hypotheses

Let the space S be given by $S = \underline{S} \oplus S^*$ where $\underline{S} = H^\infty([0,T]\times\mathbb{R}\times[0,\infty))$ is the usual Sobolev space and $S^* = H^\infty([0,T]\times\mathbb{R}\times\mathbb{T}\times[0,\infty))$ has the additional restriction that:

$$||\partial^\alpha u^*(t,x,\theta,Y)||_{L^2(\mathbb{R})} \le C_\alpha e^{-\delta Y} \tag{3}$$

where α is a multi-index, C_α and δ are positive constants. This integral is taken with respect to x. As a slight remark, note that δ is independent of α. The intervals $y \in [0,\infty)$ in $\underline{S}$ and $Y \in [0,\infty)$ in S^* contain different variables. A given element $u \in S$ can be written as $u(t,x,y,\theta,Y) = \underline{u}(t,x,y) + u^*(t,x,\theta,Y)$, where $\underline{u} \in \underline{S}$ and $u^* \in S^*$. Moreover, since each $u \in S$ is periodic with respect to θ, we can further decompose u as $u(t,x,y,\theta,Y) = \underline{u}(t,x,y) + u^{0*}(t,x,Y) + \sum_{n\neq 0} u^n(t,x,Y)e^{in\theta}$. Later on in section 5 we will decompose the sum further. One of the major issues with S is that it is *not* closed under multiplication as it is currently defined, due to the fact that S^* does not contain functions that depend on y. We also make a few definitions related to products of elements of S.

Definition 2.1. *1) A function u is called mixed if $u(t,x,y,\theta,Y)$ is a linear combination of $\underline{a}(t,x,y)b^*(t,x,\theta,Y)$ for some $\underline{a} \in \underline{S}$ and $b^* \in S^*$.*
2) Let $v \in S$. The oscillatory part of v, denoted v^{osc} is given by:

$$v^{osc} = \sum_{n\neq 0} v^n(t,x,Y)e^{in\theta}$$

3) If v is a product of two elements $u_1, u_2 \in S$ then we set:

$$\underline{v}(t,x,y) := \lim_{Y\to\infty} u_1(t,x,y,\theta,Y)u_2(t,x,y,\theta,Y)$$

This is the space that [Mar10] used in her paper, and from her paper we can borrow some of its basic properties.

Proposition 2.2. *1) If $u \in S$, then $\underline{u}(t,x,y) = \lim_{Y\to\infty} u(t,x,y,\theta,Y)$.*
2) $\underline{S}$ and S^* are both closed under multiplication.
3) If $F \in S^*$, $F = \sum_{n\neq 0} F^n(t,x,Y)e^{in\theta}$ then $IF = \sum_{n\neq 0} I_n F^n e^{in\theta}$ is an element of S^* where $I_n F^n$ is of the form for $n > 0$

$$\int_Y^\infty \exp(in\lambda(Y-s))F^n(t,x,s)ds$$

or

$$\int_0^Y \exp(in\bar{\lambda}(Y-s))F^n(t,x,s)ds$$

and, if $n < 0$,

$$\int_Y^\infty \exp(in\bar{\lambda}(Y-s))F^n(t,x,s)ds$$

or

$$\int_0^Y \exp(in\lambda(Y-s))F^n(t,x,s)ds$$

where λ is a complex number with non-positive imaginary part.
4) If v is mixed, then $\underline{v}(t,x,y) = 0$.

Proof. See [Mar10] for full details of 3). □

In the nonlinearity of the Saint-Venant model, we have products of elements of S. Therefore, we are interested in exchanging mixed terms with a sequence of elements of S^* and a remainder that is not in S. To this end, let $\underline{a} \in \underline{S}$ and $b^* \in S^*$ and Taylor expand $\underline{a}$ with respect to y to find:

$$\underline{a}(t,x,y)b^*(t,x,\theta,Y) = \underline{a}(t,x,0)b^* + \partial_y \underline{a}(t,x,0)yb^* + ... + \frac{1}{n!}\partial_y^n \underline{a}(t,x,0)y^n b^* + R_n(t,x,y,\theta,Y)$$

The next step is to multiply and divide by powers of ε which turns the above equation into:

$$\underline{a}(t,x,y)b^*(t,x,\theta,Y) = \underline{a}(t,x,0)b^* + \partial_y \underline{a}(t,x,0)y\frac{\varepsilon}{\varepsilon}b^* + ... + \frac{1}{n!}\partial_y^n \underline{a}(t,x,0)y^n \frac{\varepsilon^n}{\varepsilon^n}b^* + \frac{\varepsilon^{n+1}}{\varepsilon^{n+1}}R_n(t,x,y,\theta,Y)$$

To complete this process, we use our ansatz $Y = \frac{y}{\varepsilon}$ to transform the equation into:

$$\underline{a}(t,x,y)b^*(t,x,\theta,Y) = \underline{a}(t,x,0)b^* + \partial_y \underline{a}(t,x,0)\varepsilon Y b^* + ... + \frac{1}{n!}\partial_y^n \underline{a}(t,x,0)\varepsilon^n Y^n b^* + \varepsilon^{n+1}R_n(t,x,y,\theta,Y)$$

Of course $R_n^\varepsilon \notin S$, but since we are constructing approximate solutions given by power series in ε, we can take the order of the term R_n^ε to be large enough that it is absorbed into the error terms coming from the expansion as explained in more details in 9.

Remarks 2.3. *1) In order to close S under products we could redefine S^*, in a slight abuse of notation, to be $H^\infty(t,x,y,\theta,Y)$ with the same exponential decay in Y. This approach, however, introduces new difficulties in determining the profiles. For example, two portions, $U_{k,\alpha}$ and $U_{k,h}$ discussed in Section 5, are determined by their traces on $y = Y = 0$. Therefore, extending S^* to include dependence on y makes defining $U_{k,\alpha}$ and $U_{k,h}$ on the interior somewhat ambiguous.*
2) $u^0(t,x,y,Y)$ is the Fourier mean of u, however, it is common notation in geometric optics for $\underline{u}$ to be the (Fourier) mean. In order to avoid confusion, we not refer to either $\underline{u}$ or u^0 as the mean of u.

3 Cascade of Equations

In the following discussion let ε be a small parameter. Starting from the equations of the Saint-Venant Kirchhoff model of nonlinear elasticity in two spatial dimension:

$$\partial_t^2 \phi + \nabla \cdot (\nabla\phi\sigma(\nabla\phi)) = 0 \tag{4}$$

on $y > 0$ and on $y = 0$:

$$(\nabla\phi\sigma(\nabla\phi))\hat{n} = \begin{bmatrix} f \\ g \end{bmatrix} = \varepsilon^2 G(t,x,\frac{\beta\cdot(t,x)}{\varepsilon}) \tag{5}$$

where $\hat{n} = -\hat{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, G is a smooth function periodic in $\theta = \frac{\beta\cdot(t,x)}{\varepsilon}$, and ϕ denotes the deformation of the material. $\nabla\phi$ is given by the following expression:

$$\nabla\phi = \begin{bmatrix} \partial_x\phi_1(t,x,y) & \partial_y\phi_1(t,x,y) \\ \partial_x\phi_2(t,x,y) & \partial_y\phi_2(t,x,y) \end{bmatrix}$$

and $\sigma(\nabla\phi) = \lambda\,\mathrm{Tr}(E)I + 2\mu E$ where $E = \frac{1}{2}{}^t\nabla\phi\nabla\phi - I$ where λ and μ are the Lamé constants. The Saint-Venant Kirchhoff model is this particular choice of stress, σ and strain E. We choose the initial conditions on ϕ to be $\phi(0,x,y) = (x,y)$. It is important to note that $\nabla\varphi\sigma(\nabla\phi)$ is a cubic polynomial in $\nabla\varphi$. Let $\phi(t,x,y) = U(t,x,y) + (x,y)$ where we assume that the norm of U is small. U denotes the displacement of the material. From this, it is apparent that $\nabla\phi = \nabla U + I$ where I is the 2x2 identity matrix. Plugging in this relation between ϕ and U turns (4) into:

$$\partial_t^2 U + \nabla \cdot (L(\nabla U) + Q(\nabla U) + C(\nabla U)) = 0 \tag{6}$$

on the interior where L is a linear function of ∇U, Q is a quadratic function of ∇U and C is a cubic function of ∇U. Notice that Q and C are given by 2×2 matrices, and so $\nabla \cdot Q(\nabla U) = \partial_x Q_1(\nabla U) + \partial_y Q_2(\nabla U)$ where Q_j denotes the jth column of Q. A similar expression holds for C, with C_j denoting the columns of the C matrix. On the boundary, U satisfies:

$$-L_2(\nabla U) - Q_2(\nabla U) - C_2(\nabla U) = \begin{bmatrix} f \\ g \end{bmatrix} \tag{7}$$

where $L_2(\nabla U)$, $Q_2(\nabla U)$, $C_2(\nabla U)$ denote the second column of the 2×2 matrices $L(\nabla U)$, $Q(\nabla U)$, and $C(\nabla U)$ respectively.
Suppose that U is given by the following ansatz:

$$U^\varepsilon(t,x,y) = \sum_{n=2}^{N} \varepsilon^n U_n(t,x,y,Y,\theta)|_{Y=\frac{y}{\varepsilon},\theta=\frac{x-ct}{\varepsilon}} \tag{8}$$

where each U_n is in the space S, and N is a sufficiently large positive integer. As a small remark, notice that the lowest order term in this expansion is $O(\varepsilon^2)$. Suppose we have $U^\varepsilon(t,x,y,\frac{x-ct}{\varepsilon},\frac{y}{\varepsilon})$, and we apply ∂_x. From the chain rule, this results in $\partial_x U^\varepsilon = \partial_x U^\varepsilon + \frac{1}{\varepsilon}\partial_\theta U^\varepsilon$. Applying similar logic to the other derivatives, suggests we make the following substitutions for the derivatives in (6):

$$\partial_x \to \partial_x + \frac{1}{\varepsilon}\partial_\theta \quad \partial_y \to \partial_y + \frac{1}{\varepsilon}\partial_Y \quad \partial_t \to \partial_t - \frac{c}{\varepsilon}\partial_\theta$$

Introducing this expansion for U^ε and modified derivatives into (6) and (7) and collecting powers of ε gives the following cascade of equations:

$$L_{ff}(U_k) = \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix} \tag{9}$$

on $y, Y > 0$ and on $y = Y = 0$:

$$l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix} \tag{10}$$

where $H_{k-1}, K_{k-1}, h_{k-1}, k_{k-1}$ are the terms containing the nonlinearities and all the lower order profiles. The interior equation is the coefficient of ε^{k-2} and the boundary equation the coefficient of ε^{k-1}. The L's and l's are defined below as:

$$L_{ff} := \begin{pmatrix} c^2 - r^2 & 0 \\ 0 & c^2 - 1 \end{pmatrix} \partial_{\theta\theta} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} \partial_{\theta Y} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{YY} \tag{11}$$

$$L_{fs} := -2c\partial_{t\theta} - \begin{pmatrix} 2r & 0 \\ 0 & 2 \end{pmatrix} \partial_{x\theta} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} [\partial_{xY} + \partial_{y\theta}] - \begin{pmatrix} 2 & 0 \\ 0 & 2r \end{pmatrix} \partial_{yY} \tag{12}$$

$$L_{ss} := \partial_{tt} - \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx} - \begin{pmatrix} 0 & r-1 \\ r-1 & 0 \end{pmatrix} \partial_{xy} - \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_{yy} \tag{13}$$

$$l_f := \begin{pmatrix} 0 & 1 \\ r-2 & 0 \end{pmatrix} \partial_\theta + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_Y \tag{14}$$

$$l_s := \begin{pmatrix} 0 & 1 \\ r-2 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \partial_y. \tag{15}$$

For consistency, $H_{k-1}, K_{k-1} \in S^*$ is a necessary condition since $L_{ff}(U_k) \in S^*$ because $\underline{U}_k$ is independent of the fast variables Y and θ. Observe that there is no constraint of the form $h_{k-1}, k_{k-1} \in S^*$. This is because we are only interested in the traces of the h_{k-1}, k_{k-1}'s on the boundary $y = Y = 0$. In principal, there is no reason for $H_{k-1} \in S^*$, however, this is true by our choice of $\underline{U}_{k-2}$ and some modifications that are discussed in more details later on. The key difference is that in Marcou, the terms are only quadratic in the previous profiles, and here there are both quadratic and cubic terms and have significantly more terms than the quadratic terms appearing in her work. To be more precise, (H_{k-1}, K_{k-1}), as in (9), is given by the following expression:

$$\begin{aligned}\begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix} = &- L_{fs}(U_{k-1}) - L_{ss}(U_{k-2}) + \sum_{i+j=k-2} A_{sss}(U_i, U_j) + \sum_{i+j=k-1} A_{fss}(U_i, U_j) + \sum_{i+j=k} A_{ffs}(U_i, U_j) \\ &+ \sum_{i+j=k+1} A_{fff}(U_i, U_j) + \sum_{l+m+n=k-2} B_{ssss}(U_i, U_j, U_k) + \sum_{l+m+n=k-1} B_{fsss}(U_l, U_m, U_n) \\ &+ \sum_{l+m+n=k} B_{ffss}(U_l, U_m, U_n) + \sum_{l+m+n=k+1} B_{fffs}(U_l, U_m, U_n) + \sum_{l+m+n=k+2} B_{ffff}(U_l, U_m, U_n)\end{aligned} \tag{16}$$

and (h_{k-1}, k_{k-1}), as in (10), is given by the expression:

$$\begin{aligned}\begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix} = &- l_s(U_{k-1}) - \sum_{i+j=k+1} Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_i, U_j) - \sum_{i+j=k} [Q_2(\partial_{\theta,Y}; \partial_{x,y})(U_i, U_j) + Q_2(\partial_{x,y}; \partial_{\theta,Y})(U_i, U_j)] \\ &- \sum_{i+j=k-1} Q_2(\partial_{x,y}; \partial_{x,y})(U_i, U_j) - \sum_{l+m+n=k+2} C_2(\partial_{\theta,Y}; \partial_{\theta,Y}; \partial_{\theta,Y})(U_l, U_m, U_n) - \\ &\sum_{l+m+n=k+1} [C_2(\partial_{\theta,Y}, \partial_{\theta,Y}; \partial_{x,y})(U_l, U_m, U_n) + C_2(\partial_{\theta,Y}; \partial_{x,y}; \partial_{\theta,Y})(U_l, U_m, U_n) + \\ &+ C_2(\partial_{x,y}; \partial_{\theta,Y}; \partial_{\theta,Y})(U_l, U_m, U_n)] \\ &- \sum_{l+m+n=k} [C_2(\partial_{\theta,Y}, \partial_{x,y}; \partial_{x,y})(U_l, U_m, U_n) + C_2(\partial_{x,y}; \partial_{\theta,Y}; \partial_{x,y})(U_l, U_m, U_n) + \\ &+ C_2(\partial_{x,y}; \partial_{x,y}; \partial_{\theta,Y})(U_l, U_m, U_n)] \\ &- \sum_{l+m+n=k-1} C_2(\partial_{x,y}; \partial_{x,y}; \partial_{x,y})(U_l, U_m, U_n) + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix}\end{aligned} \tag{17}$$

where the Q_j's are quadratic functions of two profiles and the C_j's are cubic functions of three profiles that are derived in the following manner. Recall that in (6) we had $\nabla \cdot Q(\nabla U) = \partial_x Q_1(\nabla U) + \partial_y Q_2(\nabla U)$ with Q_1 the first column of Q and Q_2 the second. One useful property of Q is that it is bilinear, and so we can write $Q_j(\nabla U) = Q_j(\nabla U, \nabla U) = Q_j(\partial_{x,y}; \partial_{x,y})(U, U)$ where in the third statement we placed the derivatives into a separate set of arguments. The first pair of derivatives act on the first profile in the argument, and the second derivative pair acts on the second argument. This is to make it easy to swap the slow derivatives $\partial_{x,y}$ for the fast derivatives $\partial_{\theta,Y}$. As an example, $Q_2(\partial_{x,y}; \partial_{\theta,Y})(U_i, U_j)$ is a linear combination of terms like $\partial_y u_i \partial_\theta v_j$, $\partial_x v_i \partial_Y u_j$, $\partial_y v_i \partial_\theta v_j$ for $U_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$. The cubic functions C_1, C_2 are derived in analogous manner. The A and B functions are related to the Q and C functions by the following relations:

$$A_{fff}(U_i, U_j) := \partial_\theta [Q_1(\partial_{\theta,Y}; \partial_{\theta,Y})(U_i, U_j)] + \partial_Y [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_i, U_j)] \tag{18}$$

$$\begin{aligned}A_{ffs} := &\partial_\theta Q_1(\partial_{\theta,Y}; \partial_{x,y}) + \partial_Y Q_2(\partial_{\theta,Y}, \partial_{x,y}) + \partial_\theta Q_1(\partial_{x,y}; \partial_{\theta,Y}) + \partial_Y Q_2(\partial_{x,y}, \partial_{\theta,Y}) + \\ &\partial_x Q_1(\partial_{\theta,Y}; \partial_{\theta,Y}) + \partial_y Q_2(\partial_{\theta,Y}, \partial_{\theta,Y})\end{aligned} \tag{19}$$

$$\begin{aligned}A_{fss} := &\partial_\theta Q_1(\partial_{x,y}; \partial_{x,y}) + \partial_Y Q_2(\partial_{x,y}, \partial_{x,y}) + \partial_x Q_1(\partial_{x,y}; \partial_{\theta,Y}) + \partial_y Q_2(\partial_{x,y}, \partial_{\theta,Y}) + \\ &\partial_x Q_1(\partial_{\theta,Y}; \partial_{x,y}) + \partial_y Q_2(\partial_{\theta,Y}, \partial_{x,y})\end{aligned} \tag{20}$$

$$A_{sss}(U_i, U_j) := \partial_x [Q_1(\partial_{x,y}; \partial_{x,y})(U_i, U_j)] + \partial_y [Q_2(\partial_{x,y}; \partial_{x,y})(U_i, U_j)] \tag{21}$$

$$
\begin{aligned}
B_{ffff} &:= \partial_\theta C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y}) + \partial_Y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y}) && (22)\\
B_{fffs} &:= \partial_\theta C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{x,y}) + \partial_Y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{x,y}) + \partial_\theta C_1(\partial_{\theta,Y};\partial_{x,y};\partial_{\theta,Y}) + \partial_Y C_2(\partial_{\theta,Y};\partial_{x,y};\partial_{\theta,Y})\\
&\quad + \partial_\theta C_1(\partial_{x,y};\partial_{\theta,Y};\partial_{\theta,Y}) + \partial_Y C_2(\partial_{x,y};\partial_{\theta,Y};\partial_{\theta,Y}) + \partial_x C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y}) + \partial_y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y}) && (23)\\
B_{ffss} &:= \partial_\theta C_1(\partial_{\theta,Y};\partial_{x,y};\partial_{x,y}) + \partial_Y C_2(\partial_{\theta,Y};\partial_{x,y};\partial_{x,y}) + \partial_\theta C_1(\partial_{x,y};\partial_{\theta,Y};\partial_{x,y}) + \partial_Y C_2(\partial_{x,y};\partial_{\theta,Y};\partial_{x,y})\\
&\quad + \partial_x C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{x,y}) + \partial_y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{x,y})\partial_x C_1(\partial_{\theta,Y};\partial_{x,y};\partial_{\theta,Y}) + \partial_y C_2(\partial_{\theta,Y};\partial_{x,y};\partial_{\theta,Y})\\
&\quad + \partial_x C_1(\partial_{x,y};\partial_{\theta,Y};\partial_{\theta,Y}) + \partial_y C_2(\partial_{x,y};\partial_{\theta,Y};\partial_{\theta,Y})\partial_\theta C_1(\partial_{x,y};\partial_{x,y};\partial_{\theta,Y}) + \partial_Y C_2(\partial_{x,y};\partial_{x,y};\partial_{\theta,Y}) && (24)\\
B_{fsss} &:= \partial_\theta C_1(\partial_{x,y};\partial_{x,y};\partial_{x,y}) + \partial_Y C_2(\partial_{x,y};\partial_{x,y};\partial_{x,y}) + \partial_x C_1(\partial_{\theta,Y};\partial_{x,y};\partial_{x,y}) + \partial_y C_2(\partial_{\theta,Y};\partial_{x,y};\partial_{x,y})\\
&\quad + \partial_x C_1(\partial_{x,y};\partial_{\theta,Y};\partial_{x,y}) + \partial_y C_2(;\partial_{x,y};\partial_{\theta,Y};\partial_{x,y}) + \partial_x C_1(\partial_{x,y};\partial_{x,y};\partial_{\theta,Y}) + \partial_y C_2(\partial_{x,y};\partial_{x,y};\partial_{\theta,Y}) && (25)\\
B_{ssss} &:= \partial_x C_1(\partial_{x,y};\partial_{x,y};\partial_{x,y}) + \partial_y C_2(\partial_{x,y};\partial_{x,y};\partial_{x,y}) && (26)
\end{aligned}
$$

Notice that for a given natural number k and assuming that A_{fff} and B_{fff} are coefficients of ε^k, the following inequalities hold:

$$A_{fff}(U_i, U_j) \implies i + j = k + 3 \implies 2 \le i, j \le k + 1$$

$$B_{ffff}(U_l, U_m, U_n) \implies l + m + n = k + 4 \implies 2 \le l, m, n \le k$$

this ensures that H_{k-1} *only* depend on the profiles $U_2, ..., U_{k-1}$ since H_{k-1} is the coefficient of ε^{k-2}. Moreover, from these bounds, we observe that cubic terms do not appear until the equations for U_4.

4 General Properties

Consider the following set of equations:

$$L_{ff}(U) = F \tag{27}$$

where $y, Y > 0$ and on $y = Y = 0$

$$l_f(U) = G \tag{28}$$

with $F \in S^*$. We seek solutions $U = (u, v) \in S$, and since U is expected to be in S it has a Fourier series. Using the Fourier series, we can write the interior equation, for $n \neq 0$, as:

$$\partial_{YY} u^n - in(r-1)\partial_Y v^n - n^2 u^n = f_1^n \tag{29a}$$

$$r\partial_{YY} v^n - in(r-1)\partial_Y y^n - n^2 v^n = f_2^n \tag{29b}$$

where $F^n = (f_1^n, f_2^n)$ is the nth Fourier mode of F, and similarly u^n, v^n are the nth Fourier modes of u, v respectively. On the boundary we have:

$$\partial_Y u^n - inv^n = g_1^n \tag{30a}$$

$$r\partial_Y v^n + (r-2)inv^n = g_2^n \tag{30b}$$

where g_j^n is defined analogously to the f_j^n. Introducing $\tilde{U} = (U, \partial_Y U)$ and $\tilde{F} = (0, F)$, the previous second-order system can be rewritten as the first order system:

$$\begin{pmatrix} 0 & I \\ D & B \end{pmatrix} \partial_Y \begin{pmatrix} U^n \\ \partial_Y U^n \end{pmatrix} = \begin{pmatrix} 0 \\ F^n \end{pmatrix} \tag{31}$$

where the 2×2 matrices B and D are given by:

$$B = in \begin{pmatrix} 0 & 1-r \\ \frac{1}{r} - 1 & 0 \end{pmatrix} \quad D(\beta) = n^2 \begin{pmatrix} r^2 - c^2 & 0 \\ 0 & \frac{1-c^2}{r} \end{pmatrix}.$$

Where $\beta = (-c, 1)$ as before. Note that the matrix on the interior is now 4x4 and the boundary operator is represented by a 2x4 matrix. Set $G(\beta, n) = \begin{pmatrix} 0 & I \\ D & B \end{pmatrix}$. This matrix can be diagonalized, with pure imaginary eigenvalues $\omega_1^2 = c^2 - 1$, $\omega_2^2 = \frac{c^2}{r} - 1$, $\omega_3 = \overline{\omega_1}$ and $\omega_4 = \overline{\omega_2}$ and corresponding eigenvectors:

$$R_1(n) = \begin{pmatrix} -\omega_1 \\ 1 \\ -in\omega_1^2 \\ in\omega_1 \end{pmatrix} \quad R_2(n) = \begin{pmatrix} 1 \\ \omega_2 \\ in\omega_2 \\ in\omega_2^2 \end{pmatrix} \quad R_3(n) = \overline{R_1(-n)} \quad R_4(n) = \overline{R_2(-n)}$$

To return from the first order system to the original problem, we make the following definition:

$$r_1 = \begin{pmatrix} -\omega_1 \\ 1 \end{pmatrix} \quad r_2 = \begin{pmatrix} 1 \\ \omega_2 \end{pmatrix} \text{ and } r_3 = \bar{r}_1 \quad r_4 = \bar{r}_2$$

This amounts to taking the first two components of the R_j.
Decaying solutions to the homogeneous problem, $L_{ff}(U_h) = 0$ are given by:

$$U_h = \sum_{n \neq 0} U_h^n(t, x, Y) e^{in\theta}$$

where the U_h^n are given by the formulas:

$$U_h^n = \begin{cases} \sigma_1(t, x; n) e^{in\omega_1 Y} r_1 + \sigma_2(t, x; n) e^{in\omega_2 Y} r_2 & \text{for } n > 0 \\ \sigma_3(t, x; n) e^{in\omega_3 Y} r_3 + \sigma_4(t, x; n) e^{in\omega_4 Y} r_4 & \text{for } n < 0 \end{cases} \tag{32}$$

where the σ_j are scalar functions to be determined satisfying $\sigma_3(t, x, n) = \bar{\sigma}_1(t, x, -n)$ and $\sigma_4(t, x, n) = \bar{\sigma}_2(t, x, -n)$ for $n < 0$. This condition is to ensure that U_h is a real valued function. Suppose for the moment that G^n is identically 0, this simplifies (30) to:

$$\begin{pmatrix} 0 & -in & 1 & 0 \\ (r-2)in & 0 & 0 & r \end{pmatrix} \tilde{U}^n = 0.$$

To simplify notation somewhat, the matrix corresponding to the boundary operator will be notated $C(\beta, n)$. Using the homogeneous solution for $n > 0$, we see that the boundary conditions give:

$$C(\beta, n)[\sigma_1(n) R_1 + \sigma_2(n) R_2] = \sigma_1(n) C(\beta, n) R_1 + \sigma_2(n) C(\beta, n) R_2 = [C(\beta, n) R_1, C(\beta, n) R_2] \begin{pmatrix} \sigma_1(n) \\ \sigma_2(n) \end{pmatrix} \tag{33}$$

where the t and x dependence of the σ_j have been suppressed. The 2 by 2 matrix $[C(\beta, n)R_1, C(\beta, n)R_2]$ is given by the following expression:

$$[C(\beta, n) R_1, C(\beta, n) R_2] = in \begin{pmatrix} 2 - c^2 & 2\omega_2 \\ 2\omega_1 & c^2 - 2 \end{pmatrix} = in \mathcal{B}_{Lop}. \tag{34}$$

Hence the boundary conditions in 33 can be rewritten as:

$$C(\beta, n)[\sigma_1(n) R_1 + \sigma_2(n) R_2] = in \mathcal{B}_{Lop} \begin{pmatrix} \sigma_1(n) \\ \sigma_2(n) \end{pmatrix}$$

Recall that we chose c such that $\mathcal{B}_{Lop}$ is singular. Its kernel is spanned by $\begin{pmatrix} \omega_2 \\ -q \end{pmatrix}$, where $q^2 = -\omega_1 \omega_2$ and $q > 0$. The fact that $\mathcal{B}_{Lop}$ is singular implies that there is a nontrivial decaying solution to the interior equation satisfying trivial boundary conditions; since the kernel is one dimensional, it follows that solutions of this form have the following form:

$$\begin{pmatrix} \sigma_1(t, x; n) \\ \sigma_2(t, x; n) \end{pmatrix} = \alpha(t, x; n) \begin{pmatrix} \omega_2 \\ -q \end{pmatrix} \tag{35}$$

for some scalar function α to be determined. Returning to equation (31) briefly, after diagonalizing the matrix with left eigenvectors, notice that decaying particular solutions, $\tilde{U}_P$, of the interior equation have the form:

$$\tilde{U}_P^n = \begin{cases} \int_0^Y e^{in\omega_j(Y-s)} F_j^n(t,x,s) R_j(n) ds & \text{for } j = 1,2 \text{ and } n > 0 \\ \int_\infty^Y e^{in\omega_j(Y-s)} F_j^n(t,x,s) R_j(n) ds & \text{for } j = 3,4 \text{ and } n > 0 \\ \int_\infty^Y e^{in\omega_j(Y-s)} F_j^n(t,x,s) R_j(n) ds & \text{for } j = 1,2 \text{ and } n < 0 \\ \int_0^Y e^{in\omega_j(Y-s)} F_j^n(t,x,s) R_j(n) ds & \text{for } j = 3,4 \text{ and } n < 0 \end{cases} \tag{36}$$

where $F_j^n = L_j \tilde{F}^n$ and L_j are the left eigenvectors of $G(\beta, n)$, where $G(\beta, n)$ is defined in the discussion after (31). The vectors L_j are also chosen to satisfy $L_i R_j = \delta_{ij}$.

For $n = 0$, the equation (29) simplifies to:

$$\begin{pmatrix} \partial_Y^2 u^0 \\ r\partial_Y^2 v^0 \end{pmatrix} = \begin{pmatrix} f_1^0 \\ f_2^0 \end{pmatrix} = \begin{pmatrix} \underline{f}_1 + f_1^{0*} \\ \underline{f}_2 + f_2^{0*} \end{pmatrix} = \begin{pmatrix} f_1^{0*} \\ f_2^{0*} \end{pmatrix}$$

since $F \in S^*$, which implies $\underline{F} = \underline{F}^0 = 0$, and so a particular solution of (29) is given by:

$$\begin{pmatrix} u_P^{0*} \\ v_P^{0*} \end{pmatrix} = \begin{pmatrix} -\int_Y^\infty \int_s^\infty f_1^{0*}(t,x,y,z) dz ds \\ -\frac{1}{r} \int_Y^\infty \int_s^\infty f_2^{0*}(t,x,y,z) dz ds \end{pmatrix} \tag{37}$$

Notice that the above equation only gives information about u^*, v^*. This is because $\underline{u}, \underline{v}$ are eliminated by ∂_Y, and so the above equation is insufficient to determine $\underline{u}, \underline{v}$. Moreover, we cannot solve for u^0 and v^0 in S if $\underline{f}_1, \underline{f}_2 \neq 0$, since $l_f(U) \in S^*$. This is also apparent from formula (37), because if $\underline{F} \neq 0$, then we have that u^{0*}, v^{0*} grow quadratically in Y.

5 Order of Construction

In the next few sections, we shall split the U_k's into five portions, $\underline{U}_k(t,x,y)$, $U_k^{0*}(t,x,Y)$, $U_{k,\alpha}(t,x,\theta,Y)$, $U_{k,h}(t,x,\theta,Y)$, and $U_{k,P}(t,x,\theta,Y)$, i.e. $U_k(t,x,y,\theta,Y) = \underline{U}_k + U_k^{0*} + U_{k,\alpha} + U_{k,h} + U_{k,P}$. $U_k^{0*}(t,x,Y)$ is a particular solution to $L_{ff}(U_k^{0*}) = \begin{pmatrix} H_{k-1}^0 \\ K_{k-1}^0 \end{pmatrix}$ and is given by:

$$U_k^{0*} = -\int_Y^\infty \int_s^\infty \begin{pmatrix} H_{k-1}^{0*}(t,x,y,z) \\ \frac{1}{r} K_{k-1}^{0*}(t,x,y,z) \end{pmatrix} dz ds \tag{38}$$

There is not very much flexibility to force U_k^{0*} to satisfy boundary conditions since it is a particular solution. It turns out that generically $H_{k-1}, K_{k-1} \notin S$ due to the presence of mixed terms, and so we have to modify H_{k-1}, K_{k-1} to be in S. This will be done with a procedure outlined in section 9.

$U_{k,h}(t,x,\theta,Y)$ is a sum over non-zero Fourier modes and solves the homogeneous interior equation with non-homogeneous boundary conditions, i.e. it satisfies:

$$L_{ff}(U_{k,h}) = 0$$

on $Y > 0$ and on $Y = 0$:

$$l_f(U_{k,h}) = \begin{pmatrix} h_{k-1}^{osc} \\ k_{k-1}^{osc} \end{pmatrix} - l_f(U_{k,P})$$

From this, we know that $U_{k,h}^n$ has the general form for $n > 0$ from formula (32):

$$U_{k,h}^n = \sigma_{1,k}(t,x;n) e^{in\omega_1 Y} r_1 + \sigma_{2,k}(t,x;n) e^{in\omega_2 Y} r_2 \tag{39}$$

As discussed in 4, the boundary conditions for $U_{k,h}$ is given by $in\mathcal{B}_{Lop} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$. Since $\mathcal{B}_{Lop}$ is singular, there is no a priori reason that the boundary conditions for $U_{k,h}$ can be satisfied. However, due to the choice of $U_{k-1,\alpha}$ as explained in more detail in Section 6, this equation can be solved for $U_{k,h}$ in terms of $U_{k,P}$ and the boundary forcing in the following manner. After some algebra, we found in section 4, see (34) and the surrounding discussion, that $l_f(U_{k,h})$ could be written as:

$$[l_f(U_{k,h})]^n = in\mathcal{B}_{Lop}\begin{pmatrix}\sigma_{1,k}\\ \sigma_{2,k}\end{pmatrix} = \begin{pmatrix}h^n_{k-1}\\ k^n_{k-1}\end{pmatrix} - C(\beta,n)\tilde{U}^n_{k,P} \tag{40}$$

Since $\mathcal{B}_{Lop}$ has a one dimensional kernel, it follows that $\mathcal{B}_{Lop}$ has a one dimensional image, and so the right hand side of (40) can be written as:

$$in\mathcal{B}_{Lop}\begin{pmatrix}\sigma_{1,k}(t,x;n)\\ \sigma_{2,k}(t,x;n)\end{pmatrix} = \tau_k(t,x;n)\begin{pmatrix}q\\ \omega_1\end{pmatrix} \tag{41}$$

where $\begin{pmatrix}q\\ \omega_1\end{pmatrix}$ is a basis of the image of $\mathcal{B}_{Lop}$ and τ_k is a scalar function. There is some slight ambiguity in determining $\sigma_{1,k}$ and $\sigma_{2,k}$ because the kernel of $\mathcal{B}_{Lop}$ is non-trivial. For the sake of definiteness, fix a nonzero vector $v \in \mathbb{C}^2$ such that $v \perp \ker \mathcal{B}_{Lop}$ and then set $\begin{pmatrix}\sigma_{1,k}(t,x;n)\\ \sigma_{2,k}(t,x;n)\end{pmatrix} = \sigma_k(t,x;n)v$ for some scalar function σ_k. Then we have that $\mathcal{B}_{Lop}v = C\begin{pmatrix}q\\ \omega_1\end{pmatrix}$ for some $C \in \mathbb{C}$ and $C \neq 0$ and so we have that:

$$\sigma_k(t,x;n) = \frac{\tau^n_k(t,x)}{inC} \tag{42}$$

$U_{k,\alpha}(t,x,\theta,Y)$ solves the homogeneous interior equation with homogeneous boundary conditions, that is, $U_{k,\alpha}$ satisfies:

$$L_{ff}(U_{k,\alpha}) = 0 \tag{43}$$

and on $Y > 0$ and on $Y = 0$

$$l_f(U_{k,\alpha}) = 0 \tag{44}$$

The form of $U_{k,\alpha}$ is given by:

$$U^n_{k,\alpha}(t,x,Y) = \alpha_k(t,x;n)[\omega_2 e^{in\omega_1 Y} r_1 - q e^{in\omega_2 Y} r_2] \tag{45}$$

for $n > 0$ and for $n < 0$, $U^n_{k,\alpha} = \bar{U}^{-n}_{k,\alpha}$. α_k is given by the solution to the amplitude equation given in Proposition 6.1.
Finally, $U_{k,P}(t,x,\theta,Y)$ is 0 for $n = 0$ and for $n > 0$ is given by the formula:

$$\begin{aligned} U^n_{k,P} = &\int_0^Y \frac{e^{in\omega_1(Y-s)}}{-2i\omega_1 c^2 n}[\omega_1 H^n_{k-1} - rK^n_{k-1}]r_1 + \frac{e^{in\omega_2(Y-s)}}{2i\omega_2 c^2 n}[H^n_{k-1} + r\omega_2 K^n_{k-1}]r_2 ds \\ &+ \int_\infty^Y \frac{e^{in\omega_3(Y-s)}}{2i\omega_1 c^2 n}[-\omega_1 H^n_{k-1} - rK^n_{k-1}]r_3 + \frac{e^{in\omega_4(Y-s)}}{-2i\omega_2 c^2 n}[H^n_{k-1} - r\omega_2 K^n_{k-1}]r_4 ds \end{aligned} \tag{46}$$

It can be verified from the properties of H_{k-1}, the L_j's and r_j's that $U^{-n}_{k,P} = U^{\bar{n}}_{k,P}$, which ensures that $U_{k,P}$ is a real valued function. Moreover, 3) of Proposition 2.2 guarantees that $U_{k,P}$ is in S^* if $H_{k-1}, K_{k-1} \in S^*$. It turns that generically $H_{k-1}, K_{k-1} \notin S$, this issue will be analyzed in more detail in section 9. As with U^{0*}_k, there isn't much flexibility in $U_{k,P}$. While we could choose other particular solutions, there is no guarantee that any choice will satisfy the boundary conditions. This is why we chose $U_{k,h}$ specifically to ensure that the boundary conditions can be satisfied.
The first element to determine is $U_{k,P}$ because it only depends on the previous profiles via (46). From there, the second portion to determine is $U_{k,h}$, which is given by (42). The third portion to determine is $U_{k,\alpha}$, where α_k is given by the solution to (35), and it is chosen specifically to be able to solve for $U_{k+1,h}$ in terms of $U_{k+1,P}$ and the boundary forcing. Next, we determine U^{0*}_k with the formula (38). Finally, the last element is $\underline{U}_k$ which can be solved for using the constraint $\underline{H}_{k+1} = \underline{K}_{k+1} = 0$ on $y, Y > 0$ and boundary conditions coming from

$$\int_0^\infty \begin{pmatrix}H^0_{k-1}\\ K^0_{k-1}\end{pmatrix} dY = \begin{pmatrix}h^0_{k-1}\\ k^0_{k-1}\end{pmatrix} \tag{47}$$

on $y = Y = 0$, where H^0_{k-1} is the $n = 0$ Fourier mode. To derive this, take (38) and substitute into the boundary conditions (30) and setting $n = 0$ gives:

$$[l_f(U_k)]^0 = -\begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}\partial_Y \int_Y^\infty \int_s^\infty \begin{pmatrix}H^0_{k-1}(t,x,y,z)\\ \frac{1}{r}K^0_{k-1}(t,x,y,z)\end{pmatrix} dzds|_{Y=0} = \int_0^\infty \begin{pmatrix}H^0_{k-1}(t,x,0,z)\\ K^0_{k-1}(t,x,0,z)\end{pmatrix} dz = \begin{pmatrix}h^0_{k-1}(t,x)\\ k^0_{k-1}(t,x)\end{pmatrix} \tag{48}$$

Observe that the interior equation for $\underline{U}_k$ comes from H_{k+1}, K_{k+1}, which are the first H's that contain the term $L_{ss}(U_k)$. The boundary conditions come from h_k, k_k since these contain the term $l_s(U_k)$.

Remark 5.1. *There is a little bit of flexibility in the order because U_k^{0*} is only dependent on the previous profiles, so it could be determined before $U_{k,P}$, $U_{k,h}$, or $U_{k,\alpha}$. The rest of the order is fixed because $U_{k,h}$ depends on $U_{k,P}$, $U_{k,\alpha}$ depends on both $U_{k,P}$ and $U_{k,h}$, and $\underline{U}_k$ can depend on U_k^*.*

6 The Amplitude Equations

As discussed in the previous section, we need to solve an amplitude equation in order to determine $U_{k,\alpha}$. To derive this, we use the duality relation shown in [BGC12]:

$$\int_0^\infty \mathbf{h} \cdot L^n \hat{\mathbf{w}} dY - (\mathbf{h} \cdot C^n \hat{\mathbf{w}})|_{Y=0} = \int_0^\infty L^{-n}\mathbf{h} \cdot \hat{\mathbf{w}} dY - (C^{-n}\mathbf{h} \cdot \hat{\mathbf{w}})|_{Y=0} \tag{49}$$

where L^n and C^n denote the nth mode of the Fourier transform of L_{ff} and l_f respectively, and $\mathbf{h}, \mathbf{w}$ are sufficiently smooth $L^2([0,\infty))$ vector functions. More specifically, L^n and C^n are the following differential operators:

$$L^n := n^2 \begin{pmatrix}c^2 - r^2 & 0\\ 0 & c^2 - 1\end{pmatrix} - in \begin{pmatrix}0 & r-1\\ r-1 & 0\end{pmatrix}\partial_Y - \begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}\partial_{YY}$$
$$C^n := in \begin{pmatrix}0 & 1\\ r-2 & 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}\partial_Y$$

Notice that this relation is derived by an integration by parts. Recall from section 4 that there is a non-trivial solution satisfying both $L_{ff}(U) = 0$ and $l_f(U) = 0$. Let $\hat{\mathbf{r}}(n,Y)$ be given by the following vector function:

$$\hat{\mathbf{r}}(n,Y) := \omega_2 e^{in\omega_1 Y} r_1(n) - q e^{in\omega_2 Y} r_2(n) \text{ for } n > 0 \text{ and } \hat{\mathbf{r}}(-n,Y) = \bar{\hat{\mathbf{r}}}(n,Y) \text{ for } n < 0 \tag{50}$$

It is easy to check that both $L^n\hat{\mathbf{r}}(n,Y) = 0$ and $C^n\hat{\mathbf{r}}(n,0) = 0$. In order to get the amplitude equation for α_k, we make the substitutions $\bar{\hat{\mathbf{r}}}(n,Y)$ for $\mathbf{h}$ and U^n_{k+1} for $\mathbf{w}$. This allows us to prove the analogue of proposition 2.1 in [CW16].

Proposition 6.1. (49) *is equivalent to for $k = 2$:*

$$\partial_t \alpha_2 + \frac{c_x}{c_0}\partial_x \alpha_2 + \mathcal{H}(\mathcal{B}(\alpha_2, \alpha_2)) = G_2(f_2, g_2) \tag{51}$$

where $\mathcal{H}$ denotes the Hilbert transform with respect to θ and $\mathcal{B}$ is the bilinear Fourier multiplier given by:

$$\widehat{\mathcal{B}(\alpha_2, \alpha_2)}(n) := -\frac{1}{4\pi c_0}\sum_{n' \neq 0} b(-n, n-n', n')\alpha_2(t,x;n-n')\alpha_2(t,x;n) \tag{52}$$

where c_0, c_x are constants and $b(n_1, n_2, n_3)$ is given in [CW16]. For $k \geq 3$, the amplitude equation is given by:

$$\partial_t \alpha_k + \frac{c_x}{c_0}\partial_x \alpha_k + \mathcal{H}(\mathcal{B}(\alpha_2, \alpha_k) + \mathcal{B}(\alpha_k, \alpha_2)) = G_k \tag{53}$$

In addition G_2 is a function of the boundary forcing and, G_j, $j \geq 3$ is a nonlinear function of (f_{k-1}, g_{k-1}) and $U_2, .., U_{j-1}$. We have that $c_0 := -2\tau \int_0^\infty |\hat{\mathbf{r}}(1,Y)|^2 dY$ where $\tau \neq 0$ is a fixed frequency and c_x is constant defined below.

Proof. Recall from section 3 that $L^n(U^n_{k+1}) = \begin{pmatrix} H^n_k \\ K^n_k \end{pmatrix}$ and $C^n U^n_{k+1} = \begin{pmatrix} h^n_k \\ k^n_k \end{pmatrix}$. Substituting this into (49) and using the properties of $\hat{\mathbf{r}}(n,Y)$, we see that the equations takes the following form:

$$\int_0^\infty \bar{\hat{\mathbf{r}}}(n,Y) \cdot \begin{pmatrix} H^n_k \\ K^n_k \end{pmatrix} dY - \bar{\hat{\mathbf{r}}}(n,0) \cdot \begin{pmatrix} h^n_k \\ k^n_k \end{pmatrix} = 0 \tag{54}$$

From the definitions provided in section 3, we can partially expand $H^n_k, K^n_k, h^n_k, k^n_k$ to show that the above equation is equivalent to:

$$\int_0^\infty \bar{\hat{\mathbf{r}}}(n,Y) \cdot ([L_{fs}(U_k)]^n + [A_{fff}(U_2,U_k)]^n + [A_{fff}(U_k,U_2)]^n + N_1(U_2,...,U_{k-1}))dY - \bar{\hat{\mathbf{r}}}(n,0)$$

$$\cdot([l_f(U_k)]^n - [Q_2(\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_k)]^n - [Q_2(\partial_{\theta,Y};\partial_{\theta,Y})(U_k,U_2)]^n + N_2(U_2,...,U_{k-1}) + \begin{pmatrix} f^n_k \\ g^n_k \end{pmatrix})|_{y=Y=0} = 0$$

where the N_j's are nonlinear functions dependent only on the profiles $U_2,...,U_{k-1}$ and $[f]^n$ denotes the nth Fourier mode. Using the decomposition $U_k = \underline{U}_k + U^0_k * + U_{k,\alpha_k} + U_{k,h} + U_{k,P}$, where $U^n_{k,\alpha_k} = \alpha_k(t,x,n)\hat{\mathbf{r}}(n,Y)$, and the fact that both Q_2 and A_{fff} are bilinear, we can modify the above to:

$$\int_0^\infty \bar{\hat{\mathbf{r}}}(n,Y) \cdot ([L_{fs}(U_{k,\alpha})]^n + [A_{fff}(U_2,U_{k,\alpha})]^n + [A_{fff}(U_{k,\alpha},U_2)]^n)dY -$$

$$\bar{\hat{\mathbf{r}}}(n,0) \cdot ([l_f(U_{k,\alpha})]^n - [Q_2(U_2,U_{k,\alpha})]^n - [Q_2(U_{k,\alpha},U_2)]^n + N(U_2,...,U_{k-1}) + \begin{pmatrix} f^n_k \\ g^n_k \end{pmatrix})|_{y=Y=0} = 0$$

Notice that U_2 and U_{k,α_k} have the same form, and so modifying the derivation given in [CW16] to account for the Fourier series completes the derivation. The derivatives on the Q_2 functions have been dropped for notational simplicity. We also note that c_x is given by:

$$c_x := \int_0^\infty \bar{\hat{\mathbf{r}}}(1,Y) \cdot (A_j(\eta) + A_j^T(\eta))\hat{\mathbf{r}}(1,Y)dY + 2\mathrm{Im}\int_0^\infty \bar{\hat{\mathbf{r}}}(1,Y) \cdot A_j(\nu)\partial_Y \hat{\mathbf{r}}(1,Y)dY \tag{55}$$

with the A_j defined in [CW16]. Since U_k is only acted by fast derivatives, $\underline{U}_k$ is annihilated and hence does not need to be determined. The remaining terms are in principle determined by integrals of $U_2,...,U_{k-1}$ and therefore can be absorbed into the function N. As a remark, this derivation only shows that the amplitude equation derived above is a necessary condition to be able to solve for $U_{k+1,h}$. It turns out that is also a sufficient condition as well, which is a result due to [CL55]. □

Another remark is that there is an alternative way to derive the amplitude equation. Recall that the original form of the amplitude equation was equation (49). The duality relation is convenient for deriving the amplitude equation, but it is somewhat unintuitive. Recall that $C(\beta,n)\tilde{U}^n_{k,h} = in\mathcal{B}_{Lop}\begin{pmatrix} \sigma_{1,k} \\ \sigma_{2,k} \end{pmatrix}$. A calculation shows that the cokernel of $\mathcal{B}_{Lop}$ is spanned by $\begin{pmatrix} q & \omega_2 \end{pmatrix}$. Therefore, an equivalent form of equation (51) is given by:

$$\begin{pmatrix} q & \omega_2 \end{pmatrix} \left(\begin{pmatrix} h^n_k \\ k^n_k \end{pmatrix} - C(\beta,n)\tilde{U}^n_{k+1,P} \right) = 0 \tag{56}$$

This equation, while much more intuitive than (51), makes it more difficult to determine the amplitude equation. At first glance, (56) does not seem to contain α_k, but h_k, k_k contains $l_s(U_{k-1})$ and $U_{k+1,P}$ contains an integral of $U_{k-1,P}$ and therefore both terms contain $U_{k-1,\alpha}$. The algebra to get from (56) to (51) or (53) is much more difficult than going from (51). This proposition shows that the amplitude equation is a necessary and sufficient condition to solve for $U_{k+1,h}$. Referring to [CW16] and [Hun06], we get the following proposition showing that the amplitude equation is well-posed.

Proposition 6.2. *There exists an integer $\bar{m}$ dependent only on the spatial dimension d such that for every $m \in \mathbb{N}$ with $m \geq \bar{m}$ and every $R > 0$ there exists a $T = T(m,R)$ such that if $||\alpha_0||_{H^m} < R$, then there exists a unique $\alpha \in \mathcal{C}([0,T]; H^m(\mathbb{R}^{d-1} \times \mathbb{T}; \mathbb{Z})$ to equations (51) and (53) satisfying $\alpha|_{t=0} = \alpha_0$.*

7 Analysis of U_2

The leading profile, U_2, satisfies the following equations:

$$L_{ff}(U_2) = 0 \tag{57}$$

on $y, Y > 0$ and on $y = Y = 0$:

$$l_f(U_2) = 0 \tag{58}$$

The main result of this section is the following proposition:

Proposition 7.1. *The leading order profile U_2 is given by $U_2(t, x, y, \theta, Y) = U_{2,\alpha}(t, x, \theta, Y)$.*

Proof. We follow the procedure outlined in section 5.
1) The first portion of U_2 to determine is $U_{2,P}$. From section 4 and equation (46), it is clear that $U_{2,P}$ vanishes identically.
2) Next, we determine $U_{2,h}$. In a similar fashion $U_{2,h}$ also vanishes from formula (42). To see this, observe that $\begin{pmatrix} h_1^{osc} \\ k_1^{osc} \end{pmatrix} - l_f(U_{2,P}) = 0$, and so we have that the right hand side can be written as $\sum_{n\neq 0} \tau_2(t, x, n)e^{in\theta}$ where $\tau_2(t, x, n) = 0$ for all n.
3) The next portion to determine is $U_{2,\alpha}$, where $U^n_{2,\alpha} = \alpha_2(t, x; n)(\omega_2 e^{in\omega_1 Y} r_1 - q e^{in\omega_2 Y} r_2)$ for $n > 0$ and a similar expression for $n < 0$. To do this, we need that the following condition on $y = Y = 0$ is satisfied:

$$C(\beta, n)\tilde{U}_{3,h} = \begin{pmatrix} h_2^n \\ k_2^n \end{pmatrix} - C(\beta, n)\tilde{U}^n_{3,P} \in \text{Im}C(\beta, n). \tag{59}$$

The right hand side in the equation is only dependent on $U_{2,\alpha}$, though it is important to observe that $U_{3,P}$ is unknown at this point. Recall from Section 4 that the $\tilde{U}^n_{3,h}$ for $n > 0$ is of the form $\sigma_1(t, x; n)R_1(n) + \sigma_2(t, x; n)R_2(n)$, and from the calculation in (34), $C(\beta, n)\tilde{U}^n_{3,h}$ is by $\mathcal{B}_{Lop}\begin{pmatrix} \sigma_{1,3}(t, x, n) \\ \sigma_{2,3}(t, x, n) \end{pmatrix}$ and that $\mathcal{B}_{Lop}$ is singular. Therefore, there is no a priori reason why the above equation should be solvable, but if this constraint is not satisfied it is impossible to solve for $U_{3,h}$. Recall from formulas (17), (46), and (42) that $U_{2,\alpha}$ appears on both sides of equation (59). The equation for α_2 is derived in proposition 6.1 from (56) so that we can solve for $U_{3,h}$, and the solution of that equation is given in proposition 6.2.
The next component of U_2 is U_2^{0*}, which can be calculated from equation (57) with $n = 0$:

$$\partial_{YY}\begin{pmatrix} u_2^{0*} \\ v_2^{0*} \end{pmatrix} = 0 \tag{60}$$

Since we want U_2^{0*} to decay at infinity, this means we must choose $U_2^{0*} = 0$, as the above equation only has linear functions as solutions. At this stage U_2^* is completely known, and the only remaining component to determine is $\underline{U_2}$. The constraint to solve is $\underline{H_3} = \underline{K_3} = 0$, which using (16) can be expanded as:

$$\begin{pmatrix} \underline{H_3} \\ \underline{K_3} \end{pmatrix} = \underline{A_{ffs}(U_2, U_2) + A_{fff}(U_2, U_3) + A_{fff}(U_3, U_2) + B_{ffff}(U_2, U_2, U_2) - L_{ss}(U_2) - L_{fs}(U_3)} = 0 \tag{61}$$

Now, $A_{fff}(U_2, U_3)$ is a sum of terms like $\partial_\theta[\partial_Y u_2 \partial_\theta v_3]$ and since $\partial_Y \underline{u_2} = 0$, this implies that $\partial_\theta[\partial_Y u_2 \partial_\theta v_3] \in S^*$ because $\partial_\theta v_3$ is also in S^*. Therefore, $\lim_{Y\to\infty} A_{fff}(U_2, U_3) = \underline{A_{fff}(U_2, U_3)} = 0$, similarly, $\underline{A_{fff}(U_3, U_2)} = 0$. This leaves the equation:

$$-\underline{L_{fs}(U_3)} - \underline{L_{ss}(U_2)} + \underline{A_{ffs}(U_2, U_2)} + \underline{B_f fff(U_2, U_2,)} = 0 \tag{62}$$

on $y > 0$. Since each derivative in L_{fs} contains either ∂_Y or ∂_θ it follows that:

$$L_{fs}(U_3) = L_{fs}(\underline{U}_3 + U_3^*) = L_{fs}(U_3^*) \in S^*$$

This simplifies (61) into:

$$\underline{L_{ss}(U_2)} = \underline{A_{ffs}(U_2, U_2) + B_{ffff}(U_2, U_2, U_2)} \tag{63}$$

$B_{ffff}(U_2, U_2, U_2)$ is comprised of terms like $\partial_Y[\partial_\theta u_2 \partial_Y v_2 \partial_\theta u_2] \in S^*$, so in a similar fashion to $A_{fff}(U_2, U_3)$, $B_{ffff}(U_2, U_2, U_2) \in S^*$. The remaining nonlinear term $A_{ffs}(U_2, U_2)$ is more complicated because it contains mixed terms, e.g. something of the form $\partial_Y[\partial_y v_2 \partial_\theta u_2]$. Fortunately, there is no product of two elements of $\underline{S}$ appearing in $A_{ffs}(U_2, U_2)$, so its limit as $Y \to \infty$ is 0. Thus, our final simplification of (61) is given by:

$$L_{ss}(\underline{U}_2) = 0 \tag{64}$$

The boundary conditions for $\underline{u}_2, \underline{v}_2$ come from the formula (47):

$$\int_0^\infty \begin{pmatrix} H_2^0 \\ K_2^0 \end{pmatrix}(t, x, 0, s)ds = \begin{pmatrix} h_2^0 \\ k_2^0 \end{pmatrix}(t, x, 0, 0). \tag{65}$$

Substituting in the definitions and since H_2^0, K_2^0 are independent of θ gives the following formula:

$$\int_0^\infty -[L_{fs}(U_2)]^0 + \partial_Y[Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_2)]^0 dY = -[l_s(U_2)]^0 - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_2)]^0$$

$U_2^{0*} = 0$ implies that $-L_{fs}(\underline{U}_2) = 0$ since any fast derivative eliminates the dependence on $\underline{U}_2$. Computing the integral with the fundamental theorem of calculus gives:

$$-[Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_2)]^0 = -[l_s(U_2)]^0 - [Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_2)]^0$$

which reduces to $l_s(\underline{U}_2) = 0$ on $y = 0$. $L_{ss}(\underline{U}_2) = 0$ on $y > 0$ and $l_s(\underline{U}_2) = 0$ on $y = 0$ combine to show that $\underline{U}_2 = 0$, which shows that $U_2 = U_{2,\alpha}$, completing the construction of U_2.

□

8 Analysis of U_3

The second profile, U_3, satisfies:

$$L_{ff}(U_3) = \begin{pmatrix} H_2 \\ K_2 \end{pmatrix} \tag{66}$$

on $y, Y > 0$ and on $y = Y = 0$:

$$l_f(U_3) = \begin{pmatrix} h_2 \\ k_2 \end{pmatrix}. \tag{67}$$

The construction of U_3 is similar to, but easier than, the construction of the general term. The similarity comes from the fact that the interior equation and boundary conditions are no longer homogeneous. The reason the construction of U_3 is easier than a general U_k comes from the observation that $H_2, K_2 \in S*$, whereas it is common for higher k to have $H_{k-1}, K_{k-1} \notin S$, let alone $H_{k-1}, K_{k-l} \notin S^*$.

Proposition 8.1. *There profile is a $U_3 \in S$ satisfying* (66) *on $y, Y > 0$ and* (67) *on $y = Y = 0$.*

Proof. 1) From (16), we observe that H_2, K_2 are purely functions of $U_2 \in S^*$ and so it follows that $H_2, K_2 \in S^*$. Therefore, we can use equation (46) to determine $U_{3,P}$.
2) Next, we need to determine $U_{3,h}$. From section 4, we know that $U_{3,h}^n$ has the form for $n > 0$:

$$U_{3,h}^n = \sigma_{1,3} e^{in\omega_1 Y} r_1 + \sigma_{2,3} e^{in\omega_2 Y} r_2$$

and $U_{3,h}^n$ for $n < 0$ is given by $U_{3,h}^n = \overline{U_{3,h}^{-n}}$, and we set $U_{3,h}^0 = 0$. To determine $\sigma_{1,3}$ and $\sigma_{2,3}$, we "extend" $U_{3,h}$ and $U_{3,P}$ to $\tilde{U}_{3,h} = \begin{pmatrix} U_{3,h} \\ \partial_Y U_{3,h} \end{pmatrix}$ and a similar expression for $\tilde{U}_{3,P}$. Substituting in $\tilde{U}_{3,h}^n$ and $\tilde{U}_{3,P}^n$ into the boundary condition (67) gives, for $n > 0$:

$$C(\beta, n)\tilde{U}_{3,h}^n = \begin{pmatrix} h_2^n \\ k_2^n \end{pmatrix} - C(\beta, n)\tilde{U}_{3,P}^n \tag{68}$$

At this point everything on the right hand side of (68) is known. In section 4, we calculated that $C(\beta,n)\tilde{U}^n_{3,h} = in\mathcal{B}_{Lop}\begin{pmatrix}\sigma_{1,3}\\ \sigma_{2,3}\end{pmatrix}$. In order to be able to solve this, we chose $U_{2,\alpha}$ such that $\begin{pmatrix}h_2^n\\ k_2^n\end{pmatrix} - C(\beta,n)\tilde{U}^n_{3,P} \in \mathrm{Im}C(\beta,n)$. Solving (68) with formula (42) determines the scalar functions $\sigma_{1,3}, \sigma_{2,3}$, which gives $U_{3,h}$.
3) We can determine $U_{3,\alpha}$ by solving the amplitude equation given in proposition 6.1, which completes the construction of $U_{3,\alpha}$, since the amplitude equation is only dependent on the boundary forcing and U_2.
4) Next, U_3^{0*} can computed from the formula (37), which is only dependent on U_2 and hence is known completely.
5) Finally, we need to determine $\underline{U}_3$. To start, the interior equation for $\underline{U}_3$ is given by:

$$\underline{H}_4 = \underline{K}_4 = 0 \tag{69}$$

Substituting in the definitions of H_4, K_4 provided in (16), we get the following:

$$\begin{pmatrix}\underline{H}_4\\ \underline{K}_4\end{pmatrix} = \underline{-L_{fs}(U_4) - L_{ss}(U_3) + A_{fss}(U_2,U_2) + A_{ffs}(U_2,U_3) + A_{ffs}(U_3,U_2) + A_{fff}(U_3,U_3)}$$
$$\underline{+B_{fffs}(U_2,U_2,U_2) + B_{ffff}(U_3,U_2,U_2) + B_{ffff}(U_2,U_3,U_2) + B_{ffff}(U_2,U_2,U_3)} = 0 \tag{70}$$

In L_{fs}, every derivative has either ∂_Y or ∂_θ, and so $L_{fs}(U_4) \in S^*$. Observe that every term in the nonlinearity contains a factor of U_2, and therefore every term is either mixed or in S^*. Therefore the limit as Y goes to ∞ of the nonlinear functions in (70) is 0. We can decompose $L_{ss}(U_3)$ into $L_{ss}(\underline{U}_3) + L_{ss}(U_3^*)$, with the latter term in S^*. Therefore, the interior equation ultimately simplifies to:

$$L_{ss}(\underline{U}_3) = 0 \tag{71}$$

The boundary conditions for $\underline{U}_3$ come from the formula (47):

$$\int_0^\infty \begin{pmatrix}H_3^0\\ K_3^0\end{pmatrix} = \begin{pmatrix}h_3^0\\ k_3^0\end{pmatrix} \tag{72}$$

Substituting in the definitions of H_3 ,K_3, h_3 and k_3 into (72) we get the following:

$$\begin{aligned}&\int_0^\infty [-L_{fs}(U_3) - L_{ss}(U_2) + A_{ffs}(U_2,U_2) + A_{fff}(U_2,U_3) + A_{fff}(U_3,U_2) + B_{ffff}(U_2,U_2,U_2)]^0 dY = \\ &\quad [-l_s(U_3) - Q_2(\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_3) - Q_2(\partial_{\theta,Y};\partial_{\theta,Y})(U_3,U_2) - Q_2(\partial_{\theta,Y};\partial_{x,y})(U_2,U_2) \\ &\quad - Q_2(\partial_{x,y};\partial_{\theta,Y})(U_2,U_2) - C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)]^0\end{aligned} \tag{73}$$

To simplify this lengthy expression, recall that the A's and B's are related to the Q's and the C's as described in formulas (18) and (22) respectively. The first term we look at is the cubic term $[B_{ffff}(U_2,U_2,U_2)]^0$, which expands as follows:

$$\int_0^\infty [B_{ffff}(U_2,U_2,U_2)]^0 dY = \int_0^\infty [\partial_\theta C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2) + \partial_Y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)]^0 dY \tag{74}$$

Since $C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)$ is a product of derivatives of a periodic function, it follows that it can be represented by a Fourier series. Therefore, $[\partial_\theta C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)]^0$ vanishes. In addition, $[C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y}(U_2,U_2,U_2))]^0$ is exponentially decaying in Y so the integral in (74) evaluates to:

$$\begin{aligned}&\int_0^\infty [\partial_\theta C_1(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2) + \partial_Y C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)]^0 dY = \\ &- [C_2(\partial_{\theta,Y};\partial_{\theta,Y};\partial_{\theta,Y})(U_2,U_2,U_2)]^0\end{aligned} \tag{75}$$

Notice that the right hand side of the integral is a term appearing in $\begin{pmatrix}h_3^0\\ k_3^0\end{pmatrix}$. In a similar fashion, the $A_{fff}(U_3,U_2)$ term in H_3, K_3 cancels with the $Q_2(\partial_{\theta,Y};\partial_{\theta,Y})(U_3,U_2)$ in h_3, k_3, the $A_{ffs}(U_2,U_2)$ term cancels with the $Q_2(\partial_{x,y};\partial_{\theta,Y})(U_2,U_2) + Q_2(\partial_{\theta,Y};\partial_{x,y})(U_2,U_2)$ term and so on. This reduces (73) to the following:

$$\int_0^\infty [-L_{fs}(U_3) - L_{ss}(U_2)]^0 dY = [-l_s(U_3)]^0 \tag{76}$$

Another simplification we can implement comes from L_{ss} containing no derivatives with respect to θ and $U_2^0 = 0$, which implies $[L_{ss}(U_2)]^0 = 0$. Next, $L_{fs}(U_3) = L_{fs}(U_3^*)$ and so we get the final form of the boundary conditions:

$$l_s(\underline{U}_3) = l_s(U_3^{0*}) - \int_0^\infty [L_{fs}(U_3^*)]^0 dY \tag{77}$$

Combining (71) and (77) gives a unique solution for $\underline{U}_3$, which completes the construction of U_3.

Remark 8.2. *One of Marcou's goals in the second chapter of her thesis is to show that $\underline{U}_3 \neq 0$. This conclusion is reached after quite a bit of algebra to show that the left hand side of (77) is not 0, and hence neither is $\underline{U}_3$. $\underline{U}_3 \neq 0$ is an example of "internal rectification", see [?, Marcou]or more details.*

□

9 Analysis of U_k

Ideally, we would like the general term U_k to satisfy:

$$L_{ff}(U_k) = \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix} \tag{78}$$

on $y, Y > 0$ and on $y = Y = 0$:

$$l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix} \tag{79}$$

with the functions H_{k-1}, K_{k-1} are defined in terms of $U_2, ..., U_{k-1}$ via formula (16), and h_{k-1}, k_{k-1} are defined using formula (17). However, H_{k-1}, K_{k-1} are generically *not* in S because S is not closed under products. So for instance, the term $\partial_x(\partial_y u_3 \partial_\theta v_2)$ from $A_{fss}(U_3, U_2)$ is not in S since $\underline{u}_3 \neq 0$, which implies that $H_5, K_5 \notin S$. To bypass this issue, we can use the fact that $\underline{U}_k$ is H^∞. This allows us to Taylor expand as follows:

$$\partial_x(\partial_y u_3 \partial_\theta v_2) = \partial_x[(\underline{u}_3(t,x,0) + \partial_y \underline{u}_3(t,x,0)y + ... + \frac{1}{n!}\partial_y^n \underline{u}_3(t,x,0)y^n + R_n(t,x,y))\partial_\theta v_2]$$

where $R_n(t,x,y) \notin \underline{S}$. From here, we can multiply each term by powers of $\frac{\varepsilon}{\varepsilon}$ in the expansion for $\underline{u}_3$ to rewrite the above as:

$$\partial_x(\partial_y u_3 \partial_\theta v_2) = \partial_x[(\underline{u}_3(t,x,0) + \partial_y \underline{u}_3(t,x,0)\frac{\varepsilon}{\varepsilon}y + ... + \frac{1}{n!}\partial_y^n \underline{u}_3(t,x,0)\frac{\varepsilon^n}{\varepsilon^n}y^n + \varepsilon^{n+1}R_n(t,x,y))\partial_\theta v_2]$$

From here, we can use our ansatz $Y = \frac{y}{\varepsilon}$ to simplify the above to:

$$\partial_x(\partial_y u_3 \partial_\theta v_2) = \partial_x[(\underline{u}_3(t,x,0) + \partial_y \underline{u}_3(t,x,0)\varepsilon Y + ... + \frac{1}{n!}\partial_y^n \underline{u}_3(t,x,0)\varepsilon^n Y^n + \varepsilon^{n+1}R_n(t,x,y,Y))\partial_\theta v_2]$$

As before in section 2, every term in the above expansion is in S^* except $\partial_x R_n(t,x,y)\partial_\theta v_2$, which exponentially decays with respect to Y. Recall that H_5 is a coefficient of ε^4, but this expansion has terms with $\varepsilon, \varepsilon^2, ...$ as coefficients. This means that $\partial_x[\partial_y \underline{u}_3(t,x,0)\varepsilon Y \partial_\theta v_2]$ is "absorbed" into H_6, and in a similar fashion $\partial_x[\partial_y^2 \underline{u}_3(t,x,0)\varepsilon^2 Y^2 \partial_\theta v_2]$ is absorbed into H_7 and so on. For the $R_n \notin \underline{S}$ term, we can choose n large enough that is a coefficient of say ε^{N-1}, where N is the highest profile to determine, which effectively means that R_n only appears in the error term. The upside to this argument is that we replaced a term not in S with a sequence of terms in S^*, the downside is that we have changed H_k, K_k for *every* k past the one we are currently interested in.

Remarks 9.1. *1) This Taylor series approach introduces significant complications if one tries to take the limit as $N \to \infty$, but fortunately we do not need to do this.*
2) For each R_n discussed above, we have that the trace vanishes identically on the boundary. Moreover, each term in the modified H_{k-1}'s and K_{k-1}'s that came from this Taylor expansion also vanish on the boundary. In addition $\lim_{Y\to\infty} R_n = 0$ since each R_n contains a factor of an element in S^*.

For an explicit example, we can write $H_5^{osc} = H_5^{osc,p} + H_5^{osc,m}$ where $H_5^{osc,p}$ is the part of H_5^{osc} in S^* and $H_5^{osc,m}$ represents the mixed terms. Using the Taylor expansion argument above, we can write:

$$H_5^{osc,m} = M_{5,0} + \varepsilon Y M_{5,1} + ... + \frac{\varepsilon^{N-2-4} Y^{N-2-4}}{(N-2-4)!} M_{5,N-2-4} + \varepsilon^{N-1-4} R_5^{osc} \tag{80}$$

where $M_{5,n}$ denotes $\partial_y^n H_5^{osc,m}(t,x,0,\theta,Y)$. Now the power of R_5^{osc} is $N-1-4$ since H_5 is a coefficient of ε^4, so $\varepsilon^{N-1-4} * \varepsilon^4 = \varepsilon^{N-1}$, and so the remainder term R_5^{osc} appears in the error term. Notice that $\varepsilon Y M_{5,1} \in S^*$ is a new term appearing in H_6^{osc}, $\frac{1}{2}\varepsilon^2 Y^2 M_{5,2}$ is a term in H_7^{osc} and so forth. Doing the same thing for each $5 \leq k \leq N$ gives the modified H_{k-1}^{osc}'s:

$$H_{k-1}^{\prime osc} = H_{k-1}^{osc,p} + M_{k-1,0} + Y M_{k-2,1} + \frac{1}{2!} Y^2 M_{k-3,2} + ... + \frac{1}{(k-2-4)!} Y^{k-2-4} M_{5,k-2-4} \tag{81}$$

Observe that the sum of the two numbers in the subscript of $M_{i,j}$ is $k-1$. A similar argument holds for K_{k-1}^{osc} as well as H_{k-1}^{0*} and K_{k-1}^{0*}. This procedure modifies the cascade of equations into:

$$L_{ff}(U_k) = \begin{pmatrix} H'_{k-1} \\ K'_{k-1} \end{pmatrix} (t,x,\theta,Y) \tag{82}$$

where by construction $H'_{k-1}, K'_{k-1} \in S^*$. Since we are only interested the trace of the h_k, k_k functions on $y = Y = 0$, there is no need to modify them. Therefore the same equation (10) is also satisfied. In this modified setting the pieces of our decomposition $U_k = \underline{U}_k + U_k^{0*} + U_{k,h} + U_{k,P} + U_{k,\alpha}$ now satisfy:

$$L_{ff}(U_{k,P}) = \begin{pmatrix} H_{k-1}^{\prime osc} \\ K_{k-1}^{\prime osc} \end{pmatrix} \tag{83}$$

$$L_{ff}(U_{k,h}) = 0 \qquad l_f(U_{k,h}) = \begin{pmatrix} h_{k-1}^{osc} \\ k_{k-1}^{osc} \end{pmatrix} - l_f(U_{k,P}) \tag{84}$$

$$L_{ff}(U_{k,\alpha}) = 0 \qquad l_f(U_{k,\alpha}) = 0 \tag{85}$$

$$L_{ff}(U_k^{0*}) = \begin{pmatrix} H_{k-1}^{\prime 0} \\ K_{k-1}^{\prime 0} \end{pmatrix} \tag{86}$$

$$\underline{H'_{k+1}} = \underline{K'_{k+1}} = 0 \qquad \int_0^\infty \begin{pmatrix} H_k^{\prime 0} \\ K_k^{\prime 0} \end{pmatrix} dY = \begin{pmatrix} h_k^0 \\ k_k^0 \end{pmatrix} \tag{87}$$

where the first equation is on $y, Y > 0$ and the second, if present, is on $y = Y = 0$. There are two main results to this section, the first being:

Proposition 9.2. *For each $2 \leq k \leq N$, there exists a sequence of profiles $U_k \in S$ satisfying $L_{ff}(U_k) = \begin{pmatrix} H'_{k-1} \\ K'_{k-1} \end{pmatrix}$ on $y, Y > 0$ and $l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix}$ on $y = Y = 0$*

Proof. We already showed this for U_2 and U_3 in sections 7 and 8 respectively.
Assume that $U_2, ..., U_{k-1}$ are completely known. We use the procedure given in 5.
1) $U_{k,P}$ is given by the integral formula presented in (46), and since H'_{k-1} and K'_{k-1} are known functions of $U_2, ..., U_{k-1}$, it follows that we know $U_{k,P}$ via formula (46).
2) As before, $\begin{pmatrix} h_{k-1}^{osc} \\ k_{k-1}^{osc} \end{pmatrix} - l_f(U_{k,P})$ doesn't need to be in the image of $\mathcal{B}_{Lop}$. Our choice of $U_{k-1,\alpha}$ ensures that this function lies in the image of $\mathcal{B}_{Lop}$. Therefore, it is valid to determine $U_{k,h}$ with formula (42). Therefore, we know $U_{k,h}$ as a function of the lower order profiles.
3) In order to determine $U_{k,\alpha}$, we need to ensure that the amplitude equation derived from $\begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}$ and the one derived from $\begin{pmatrix} H'_{k-1} \\ K'_{k-1} \end{pmatrix}$ have the same dependence on the amplitude α_k. This is true because the only places U_k appears in H_{k-1}, K_{k-1} are in the terms $L_{fs}(U_k)$, $A_{fff}(U_2, U_k)$, and $A_{fff}(U_k, U_2)$. None of these functions contain any terms that are mixed, and so are unchanged when exchanging H_{k-1}, K_{k-1} for

H'_{k-1}, K'_{k-1}. It should be mentioned that the amplitude equations will in general not be the same, but the only differences are in the G_k functions discussed in 6.1. Therefore we can solve for α_k.
4) U_k^{0*} is given by formula (38) using the modified H'_{k-1}, K'_{k-1} in place of the original H_{k-1}, K_{k-1}.
5) First, we analyze the interior equation for $\underline{U}_k$ given by $\underline{H}'_{k+1} = \underline{K}'_{k+1} = 0$. Recall that H'_{k+1} and K'_{k+1} are derived from H_{k+1}, K_{k+1} by replacing mixed terms with elements of S^* and adding in corresponding terms from lower order mixed terms in the procedure discussed at the beginning of this section. Moreover, since both mixed terms and elements of S^* limit to 0 as $Y \to \infty$, we have that:

$$\underline{H}_{k+1} = \underline{H}'_{k+1} \quad \underline{K}_{k+1} = \underline{K}'_{k+1} \tag{88}$$

We can partially expand the definition of H_k and K_k to get the following equation for $\underline{U}_k$:

$$\begin{aligned} &\underline{-L_{ss}(U_k) - L_{fs}(U_{k+1}) + A_{ffs}(U_k, U_2) + A_{ffs}(U_k, U_2)} + \\ &\underline{B_{ffff}(U_k, U_2, U_2) + B_{ffff}(U_2, U_k, U_2) + B_{ffff}(U_2, U_2, U_k) + N(U_2, ..., U_{k-1})} = 0 \end{aligned} \tag{89}$$

Where N is a known nonlinear function of the lower profiles. As in section 8, we have that $L_{fs}(U_{k+1}) \in S^*$ and each cubic nonlinearity is in S^* as well. In addition, every term in $A_{ffs}(U_k, U_2)$ is either in S^* or mixed because $U_2 \in S^*$. Therefore, we can write (89) as:

$$L_{ss}(\underline{U}_k) = \underline{N(U_2, ..., U_{k-1})} \tag{90}$$

Observe that at this point, every term in the right hand side is known at this point. Moreover, starting at H_8, K_8, the function $\underline{N}$ is not 0 since H_8, K_8 contains a term like $\partial_x[\partial_y \underline{u}_3 \partial_x \underline{v}_3] \neq 0$.
Next, we look at the boundary conditions given by:

$$\int_0^\infty \begin{pmatrix} H_k^{0'} \\ K_k^{0'} \end{pmatrix} dY = \begin{pmatrix} h_k^0 \\ k_k^0 \end{pmatrix} \tag{91}$$

Notice that here, the distinction between H_k, K_k and H'_k, K'_k is important. This because the terms coming from the Taylor expansion of the lower order mixed term do not integrate to 0, even though they vanish at $Y = 0$ and as $Y \to \infty$. Fortunately, the part of H_k, K_k dependent on U_k are elements of S^*, and so we can partially expand (91) to get the following:

$$\begin{aligned} &\int_0^\infty [-L_{fs}(U_k) + A_{fff}(U_k, U_2) + A_{fff}(U_2, U_k) + N_1(U_2, ..., U_{k-1})]^0 dY = \\ &-[l_s(U_k) - Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_k, U_2) - Q_2(\partial_{\theta,Y}; \partial_{\theta,Y})(U_2, U_k) + N_2(U_2, ..., U_{k-1})]^0 \end{aligned} \tag{92}$$

where the N_j's are known nonlinear functions of the lower order profiles. A similar argument to the one presented in 8 allows us to simplify the above to the final form of the boundary conditions:

$$l_s(\underline{U}_k) = -l_s(U_k^{0*}) + \int_0^\infty L_{fs}(U_k^{0*}) dY + [N(U_2, ..., U_{k-1})]^0 \tag{93}$$

Using both (90) and (93), we find that $\underline{U}_k$ is uniquely determined via a Fourier-Laplace transform and the initial condition $\underline{U}(0, x, y) = 0$. This completes the construction of U_k, and thus, it completes the inductive step. Therefore we have a unique sequence of profiles U_k in S satisfying (36) and (10). □

To motivate the second main result in this section, suppose for the moment that solutions to the unmodified problem $L_{ff}(U_k) = \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}$ on $y, Y > 0$ and on $y = Y = 0$, $l_f(U_k) = \begin{pmatrix} h_{k-1} \\ k_{k-1} \end{pmatrix}$ existed for $k = 2, ..., N$. We can form an approximate solution U^ε_{app} by $U^\varepsilon_{app}(t, x, y) = \sum_{k=2}^N \varepsilon^k U_k(t, x, y, \frac{x-ct}{\varepsilon}, \frac{y}{\varepsilon})$. Putting U^ε_{app} into the original system (6) and (7) gives, on $y > 0$:

$$\begin{aligned} &\partial_t^2 U^\varepsilon_{app} + \nabla \cdot (L(\nabla U^\varepsilon_{app}) + Q(\nabla U^\varepsilon_{app}) + C(\nabla U^\varepsilon_{app})) = \\ &\sum_{k=2}^N \varepsilon^{k-2} (L_{ff}(U_k) - \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}) + \varepsilon^{N-1} E_N \end{aligned} \tag{94}$$

where E_N is the error term arising from the approximate solution. By construction of U^{ε}_{app}, we have that (94) simplifies to:

$$\partial_t^2 U^{\varepsilon}_{app} + \nabla \cdot (L(\nabla U^{\varepsilon}_{app}) + Q(\nabla U^{\varepsilon}_{app}) + C(\nabla U^{\varepsilon}_{app})) = \varepsilon^{N-1} E_N \tag{95}$$

On $y = 0$, substituting in ∇_ε and U^{ε}_{app} and following a similar argument gives:

$$-L_2(\nabla_\varepsilon U^{\varepsilon}_{app}) - Q_2(\nabla_\varepsilon U^{\varepsilon}_{app}) - C_2(\nabla_\varepsilon U^{\varepsilon}_{app}) - \begin{bmatrix} f \\ g \end{bmatrix} = \varepsilon^N e_N \tag{96}$$

where e_N is the corresponding error term on the boundary.

In the preceding discussion, we assumed that we had U_k that solved (9) and (10), these however, do not exist in S. Fortunately, we can recover similar statements to (95) and (96).

Theorem 9.3. *Let U_k, $k = 2, ..., N$ be given by proposition 9.2. Then on $y > 0$, we have:*

$$\partial_t^2 U^{\varepsilon}_{app} + \nabla \cdot (L(\nabla U^{\varepsilon}_{app}) + Q(\nabla U^{\varepsilon}_{app}) + C(\nabla U^{\varepsilon}_{app})) = \varepsilon^{N-1} E'_N \tag{97}$$

and on $y = 0$:

$$-L_2(\nabla U^{\varepsilon}_{app}) - Q_2(\nabla U^{\varepsilon}_{app}) - C_2(\nabla U^{\varepsilon}_{app}) - \begin{bmatrix} f \\ g \end{bmatrix} = \varepsilon^N e_N \tag{98}$$

where $U^{\varepsilon}_{app}(t, x, y) = \sum_{k=2}^{N} U_k(t, x, y, \frac{x-ct}{\varepsilon}, \frac{y}{\varepsilon})$ and E'_N is given below.

Proof. First, we look at the interior equation. Plugging in U^{ε}_{app} into (6) gives:

$$\begin{aligned} &\partial_t^2 U^{\varepsilon}_{app} + \nabla \cdot (L(\nabla U^{\varepsilon}_{app}) + Q(\nabla U^{\varepsilon}_{app}) + C(\nabla U^{\varepsilon}_{app})) = \\ &\sum_{k=2}^{N} \varepsilon^{k-2}(L_{ff}(U_k) - \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}) + \varepsilon^{N-1} E_N \end{aligned} \tag{99}$$

as before. Here, the H_{k-1}, K_{k-1} appearing are the unmodified nonlinearities defined in (16). Decompose U_k as $U_k = \underline{U}_k + U_k^{0*} + U_{k,h} + U_{k,P} + U_{k,\alpha}$ as in the proof of 9.2. We have that $L_{ff}(U_{k,\alpha})$, $L_{ff}(\underline{U}_k)$, and $L_{ff}(U_{k,h})$ are identically 0 for each $k = 2, .., N$. This simplifies (99) to:

$$\sum_{k=2}^{N} \varepsilon^{k-2}(L_{ff}(U_{k,P} + U_k^{0*}) - \begin{pmatrix} H_{k-1} \\ K_{k-1} \end{pmatrix}) + \varepsilon^{N-1} E_N \tag{100}$$

Next, we look at the oscillatory part:

$$\sum_{k=2}^{N} \varepsilon^{k-2}(L_{ff}(U_{k,P}) - \begin{pmatrix} H^{osc}_{k-1} \\ K^{osc}_{k-1} \end{pmatrix}) + \varepsilon^{N-1} E^{osc}_N \tag{101}$$

By construction $U_{k,P}$ satisfies:

$$L_{ff}(U_{k,P}) = \begin{pmatrix} H^{osc,p}_{k-1} \\ K^{osc,p}_{k-1} \end{pmatrix} + M_{k-1,0} + Y M_{k-2,1} + ... + \frac{Y^{k-2-4}}{(k-2-4)!} M_{5,k-2-4} \tag{102}$$

and we also have the following formula for $H^{osc}_{k-1}, K^{osc}_{k-1}$:

$$\begin{pmatrix} H^{osc}_{k-1} \\ K^{osc}_{k-1} \end{pmatrix} = \begin{pmatrix} H^{osc,p}_{k-1} \\ K^{osc,p}_{k-1} \end{pmatrix} + M_{k-1,0} + \varepsilon Y M_{k-1,1} + \frac{1}{2!}\varepsilon^2 Y^2 M_{k-1,2} + ... + \varepsilon^{N-k-2-1} R^{osc}_{k-1} \tag{103}$$

For $k < 4$, we set $M_{k,j} = 0$ for every j. Substituting in (102) and (103) into (101) gives:

$$\begin{aligned} &\sum_{k=2}^{N} \varepsilon^{k-2}(\begin{pmatrix} H^{osc,p}_{k-1} \\ K^{osc,p}_{k-1} \end{pmatrix} + M_{k-1,0} + Y M_{k-2,1} + ... + \frac{1}{(k-2-4)!} Y^{k-2-4} M_{5,k-2-4}) - \\ &\sum_{k=2}^{N} \varepsilon^{k-2}(\begin{pmatrix} H^{osc,p}_{k-1} \\ K^{osc,p}_{k-1} \end{pmatrix} + M_{k-1,0} + \varepsilon Y M_{k-1,1} + \frac{1}{2!}\varepsilon^2 Y^2 M_{k-1,2} + ... + \frac{\varepsilon^{N-k-2-2} Y^{N-k-2-2}}{(N-k-2-2)!} M_{k-1,N-2} + \varepsilon^{N-k-2-1} R^{osc}_{k-1}) \\ &+ \varepsilon^{N-1} E^{osc}_N \end{aligned} \tag{104}$$

We can collect the remainder terms, $\sum_{k=6}^{N} \varepsilon^{N-1} R_{k-1}^{osc}$, and add them into the error term yielding:

$$E_N'^{osc} := E_N^{osc} - \sum_{k=6}^{N} R_{k-1}^{osc} \tag{105}$$

This sums starts at $k = 6$ because H_5, K_5 were the first H_{k-1}, K_{k-1} to be modified. Next, we collect the terms involving $M_{5,j}$, after factoring out ε^4:

$$\sum_{k=0}^{N-4} \varepsilon^k (\frac{1}{k!} Y^k M_{5,k-2}) - \sum_{k=0}^{N-4} \varepsilon^k \frac{1}{k!} Y^k M_{5,k-2} = 0 \tag{106}$$

Doing this for each $M_{k,j}$ gives the result stated in (97). Similar analysis holds for $\sum_{k=2}^{N} \varepsilon^{k-2} (L_{ff}(U_k^{0*}) - \begin{pmatrix} H_{k-1}^0 \\ K_{k-1}^0 \end{pmatrix})$.

On the boundary, we have $l_f(U_{k,h}) = \begin{pmatrix} h_{k-1}^{osc} \\ k_{k-1}^{osc} \end{pmatrix} - l_f(U_{k,P})$ and $l_f(U_k^{0*}) = \begin{pmatrix} h_{k-1}^0 \\ k_{k-1}^0 \end{pmatrix}$, so the result follows from the discussion preceding this proposition. As an aside, the error terms are not in S, but fortunately this is not necessary. $\square$

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